Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2010
ASSESSMENT TASK 4

Mathematics
Extension 2

General Instructions

- Working time – 50 minutes
- Write in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number:

(To be used by the exam markers only.)

Question No	1	2	3	4	5	Total	Total
Mark	$\frac{}{5}$	$\frac{}{6}$	$\frac{}{7}$	$\frac{}{9}$	$\frac{}{11}$	$\frac{}{38}$	$\frac{}{100}$

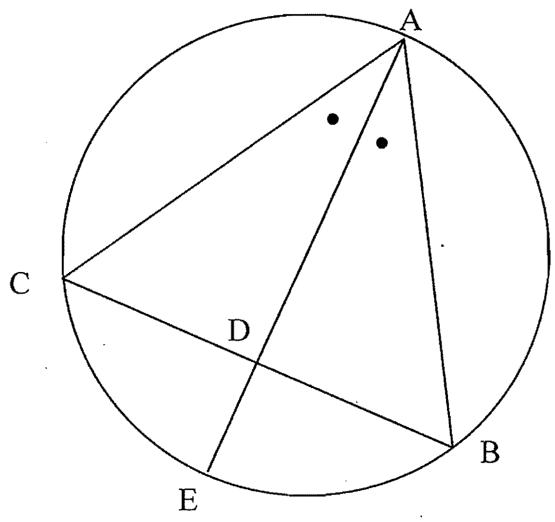
Question 1 (5 marks) **Marks**

A sequence U_n is defined as $U_{n+2} = 6U_{n+1} - 5U_n$. **5**

$U_1 = 2$ and $U_2 = 6$

Use mathematical induction to prove that $U_n = 5^{n-1} + 1$

Question 2 (6 marks)



In the diagram above AD, extended to E is the bisector of $\angle BAC$.

Copy this diagram into your answer book.

a) Prove that ΔABE and ΔADC are similar **2**

b) Show that $AB \times AC = AD \times AE$ **1**

c) Prove that $AD^2 = AB \times AC - BD \times DC$ **3**

Question 3 (7 marks)

a) Find the term independent of x in $\left(x^2 + \dfrac{1}{2x}\right)^{18}$ **3**

b) Show that $\displaystyle\sum_{r=0}^{n} \frac{{}^{n}C_{r}}{r+1} = \frac{2^{n+1} - 1}{n+1}$ **4**

Question 4 (9 marks)

A mass of m kilograms falls from a stationary balloon at a height above the ground. It experiences air resistance during its fall equal to mkv^2, where v is its speed in metres per second and k is a positive constant.

Let x be the distance in metres of the mass from the balloon measured positively as it falls.

a) Show that the equation of motion of the mass is $\ddot{x} = g - kv^2$, where g is the acceleration due to gravity. **1**

b) Show that $v^2 = \dfrac{g}{k}\left(1 - e^{-2kx}\right)$ **4**

c) Find the terminal velocity of the mass. **2**

d) Find the distance fallen when the mass reaches half of its terminal velocity. **2**

TURN OVER FOR QUESTION 5

Question 5 (11 marks)

Suppose $a > 0$, $b > 0$ and $c > 0$

a) Prove $a^2 + b^2 \geq 2ab$ **2**

b) Hence prove $a^2 + b^2 + c^2 \geq ab + bc + ca$ **2**

c) Given that

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

prove that $a^3 + b^3 + c^3 \geq 3abc$ **2**

d) If x, y, z are > 0,

Make a suitable substitution into part iii) to show that

$$x + y + z \geq 3(xyz)^{\frac{1}{3}}$$ **1**

e) Suppose $(1 + x)(1 + y)(1 + z) = 8$, prove that $xyz \leq 1$ **3**

END OF EXAMINATION

Q1

5 marks

Step 1

$n=1 \Rightarrow u_1 = 5^{1-1}+1 = 2$ True 1

$n=2 \Rightarrow U_2 = 5^{2-1}+1 = 6$ True 1

Step 2

Assume true for $1, 2, 3, \ldots n$ 1

ie $U_n = 5^{n-1}+1$

Consider $U_{n+1} = 6U_n - 5u_{n-1}$

$= 6(5^{n-1}+1) - 5(5^{n-2}+1)$

$= 6\times 5^{n-1} + 6 - 5^{n-1} - 5$ 1

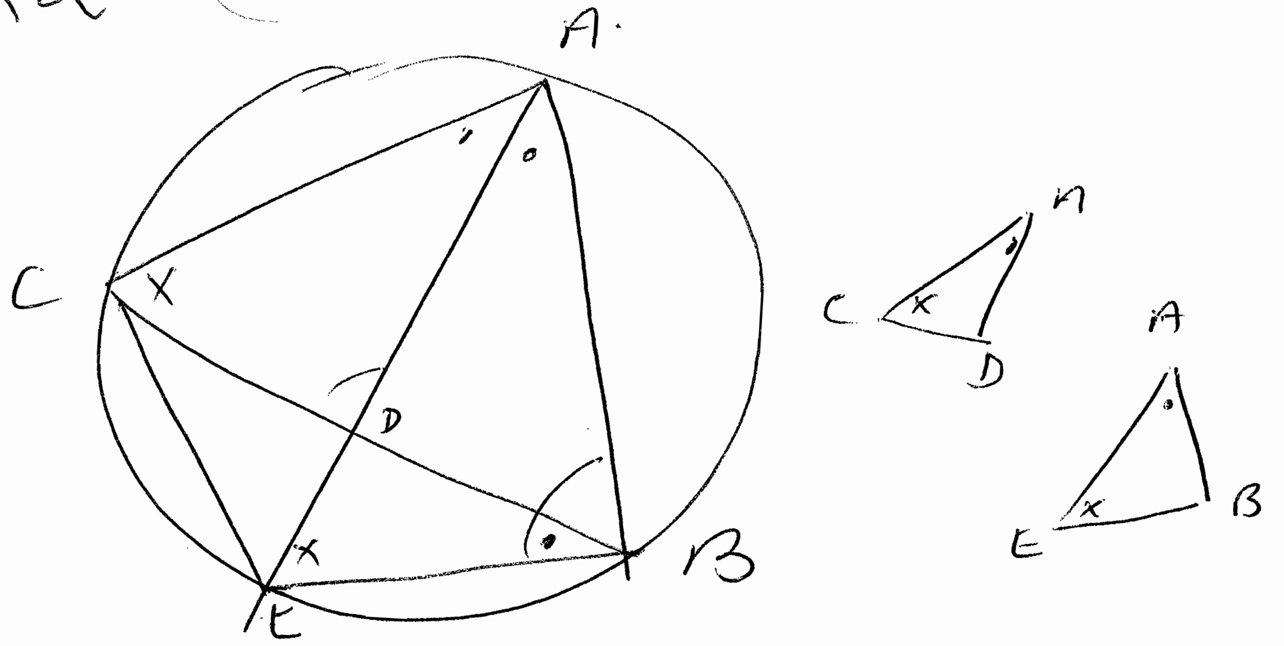
$= 5\times 5^{n-1}+1$

$= 5^n + 1$

∴ If true for U_n it is true for U_{n+1}

∴ As it is true for $n = 1, 2, 3, \ldots$ it is true for for all n by Mathematical Induction.

Q2 (6 marks)



i) In Δs ΔABE and ΔADC

$\angle EAB = \angle DAC$ (given)

$\angle AEB = \angle ACD$ (angle standing on the same arc AB) 2

$\therefore$ $\Delta ABE \parallel\!\mid \Delta ADC$ (AAA)

ii) $\frac{AB}{AE} = \frac{AD}{AC}$ (Sides in proportion)

$\therefore$ $AB \times AC = AD \times AE$ ① 1

iii) Prove $AD^2 = AB \times AC - BD \times DC$ ②

Now $AD \times DE = BD \times DC$ ③ (intersecting chords)

Consider $AB \times AC - BD \times DC$

Subs ①+③

$AD \times AE - AD \times DE$ 3

$AD(AD + DE) - AD \times DE$

$AD^2 + AD \times DE - AD \times DE$

$= AD^2$.

3c) $T_{k+1} = {}^{18}C_{k} ([illegible])^{[illegible]} \cdot \frac{1}{([illegible])^{k}}$

$$\Rightarrow \frac{x^{[illegible]}}{x^{k}} = 1$$

ie $x^{36-3k} = 1$

$36 - 3k = 0$

$k = 12$

$$\therefore T_{13} = {}^{18}C_{12} \times \frac{1}{2^{12}}$$

$$= \frac{18!}{12!\,6!} \times \frac{1}{2^{12}}$$

$$= \frac{4641}{1024}$$

3 b) $(1+x)^n = \sum {}^nC_r x^r$

$$LHS = \frac{{}^nC_0}{1} + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_r}{r+1} + \cdots \frac{{}^nC_n}{n+1}$$

Consider $\int (1+x)^n \, dx$

$$= \int 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots {}^nC_r x^r + \cdots \, dx$$

$$= \frac{(1+x)^{n+1}}{n+1} + C = \sum_{r=0}^{n} \frac{{}^nC_r \, x^{r+1}}{r+1} + \cdots$$

$x = 0 \Rightarrow$

$$\frac{1}{n+1} + C = 0 \quad \text{ie} \quad C = \frac{-1}{n+1}$$

$$\therefore \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \frac{{}^nC_r \, x^{r+1}}{r+1}$$

let $x = 1$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \sum_{r=0}^{n} \frac{{}^nC_r}{r+1}$$

Q4.

a) $m\ddot{x} = mg - mkv^2$ $\qquad \downarrow mg \qquad \uparrow mkv^2$

$\therefore \ddot{x} = g - kv^2$

b) $\therefore \ddot{x} = v\frac{dv}{dx} = g - kv^2$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$\int dx = \int \frac{v\,dv}{g - kv^2}$$

$$x + c = -\frac{1}{2k}\log(g - kv^2)$$

$$x = 0,\ v = 0,\ \Rightarrow\ c = -\frac{1}{2k}\log g$$

$$x - \frac{1}{2k}\log g = -\frac{1}{2k}\log(g - kv^2)$$

$$x = \frac{1}{2k}\log\left(\frac{g}{g - kv^2}\right)$$

$$\log\left(\frac{g}{g - kv^2}\right) = 2kx$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$\frac{g - kv^2}{g} = e^{-2kx}$$

$$g - kv^2 = ge^{-2kx}$$

$$kv^2 = g - ge^{-2kx}$$

$$v^2 = \frac{g}{k}(1 - e^{-2kx})$$

4 c) $\ddot{x} = 0 = g - kv^2$

$v^2 = \frac{g}{k}$

$\therefore v = \sqrt{\frac{g}{k}}$

d) $v = \frac{1}{2}\sqrt{\frac{g}{k}}$

$v^2 = \cancel{\frac{4g}{k}} \quad \frac{g}{4k}$

$\frac{\cancel{g}}{4\cancel{k}} = \frac{\cancel{g}}{\cancel{k}}\left(1 - e^{-2kx}\right)$

$\frac{1}{4} = 1 - e^{-2kx}$

$e^{-2kx} = \frac{3}{4}$

$-2kx = \ln\left(\frac{3}{4}\right)$

$x = \frac{1}{2k}\ln\left(\frac{4}{3}\right)$

Q8

a) $(a-b)^2 = a^2 - 2ab + b^2 \geqslant 0$

$a^2 + b^2 \geqslant 2ab.$

b) $a^2 + b^2 \geqslant 2ab$ (1)

$b^2 + c^2 \geqslant 2bc$ (2)

$c^2 + a^2 \geqslant 2ac$ (3)

(1) + (2) + (3)

$2a^2 + 2b^2 + 2c^2 \geqslant 2ab + 2bc + 2ac$

$\therefore a^2 + b^2 + c^2 \geqslant ab + bc + ac$

c) Since $a, b, c, > 0$

it follows $(a+b+c) > 0$

and from b) $a^2 + b^2 + c^2 > ab + bc + ca.$

$\therefore (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \geqslant 0$

$\therefore a^3 + b^3 + c^3 - 3abc \geqslant 0$

$a^3 + b^3 + c^3 \geqslant 3abc$

d) $a^3 = x \quad b^3 = y \quad c^3 = z$

$a = x^{1/3} \quad b = y^{1/3} \quad c = z^{1/3}$

$\therefore x + y + z = 3(xyz)^{1/3}$

d) $8 = (1+x)(1+y)(1+z)$

$= (1+y+x+xy)(1+z)$

$= 1+z+y+yz+x+xz+xy+xyz$

$= 1+x+y+xy+xz+yz+xyz$

Now $x+y+z \geq 3(xyz)^{1/3}$

$xy+xz+yz \geq 3(x^2y^2z^2)^{1/3} = 3x^{2/3}y^{2/3}z^{2/3}$

$\geq$

$8 \geq 1+3(xyz)^{1/3}+3(xyz)^{2/3}+xyz$

$\cancel{\text{let } u = (xyz)^3}$

$\cancel{8 \geq 1+3u+3u^2+u^3}$

$\cancel{\geq (1+u)^3}$

$\cancel{2 \geq 1+u}$

$\cancel{u \leq 1}$

$\cancel{xyz \leq 1}$

let $u = xyz^{1/3}$

$8 \geq 1+3u+3u^2+u^3$

$8 \geq (1+u)^3$

$2 \geq 1+u$

$u \leq 1$

$xyz^{1/3} \leq 1$

$xyz \leq 1$