NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2011 HSC Course Assessment Task 4

General instructions

- Working time – 50 min.
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ○ 12M4A – Mr Fletcher/Mrs Collins
- ○ 12M4B – Mr Ireland
- ○ 12M4C – Mrs Collins/Mr Rezcallah /Mr Lin

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	$\overline{13}$	$\overline{13}$	$\overline{13}$	$\overline{39}$	

Question 1 (13 Marks) Commence a NEW page. **Marks**

(a) Show that $\left(\dfrac{a+b+c}{3}\right)^2 \leq \dfrac{a^2+b^2+c^2}{3}$. **3**

(b) The function $f(x)$ is defined as

$$f(x) = x - \log_e\left(x^2+1\right)$$

for $x \geq 0$.

i. Show that $x > \log_e\left(x^2+1\right)$ for $x > 0$. **2**

ii. By evaluating $\displaystyle\int_0^1 x\,dx$ and $\displaystyle\int_0^1 \log_e\left(x^2+1\right)\,dx$, show that $5 > 2\log_e 2 + \pi$. **3**

(c) i. A sequence of numbers t_n where $n \geq 1$ is defined as **1**

$$t_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

Show that $t_n + \dfrac{1}{2n} = \dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n-1}$.

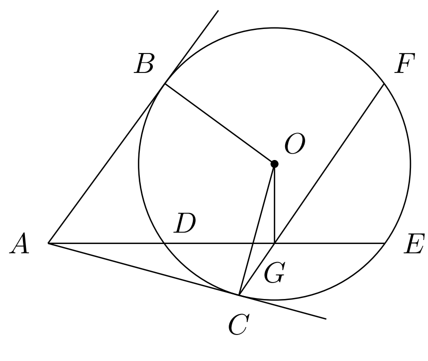
ii. Another sequence S_n, where $n \geq 1$, is defined as **4**

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

Use mathematical induction to show that $S_n = t_n$.

Question 2 (13 Marks) Commence a NEW page. **Marks**

(a) In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.



i. Copy the diagram on to your answer sheet and prove that $ABOC$ and $AOGC$ are cyclic quadrilaterals. **2**

ii. Explain why $\angle OGF = \angle OAC$. **1**

iii. Show that $\angle BAO = \angle OAC$. **1**

iv. Hence or otherwise, prove that $BF \parallel AE$. **3**

Question 2 continues overleaf . . .

Question 2 continued from previous page...

(b) A particle is projected from a point O with speed $80\,\mathrm{ms}^{-1}$ at an angle of elevation α, where $\tan\alpha = \frac{5}{12}$.

Two seconds later, a second particle is projected from O with initial speed U at an angle β. It collides with the first particle one second after leaving O. Take $g = 10\,\mathrm{ms}^{-2}$.

i. Show that t seconds after the first particle is projected, the second particle is at **2**

$$\Big(U(t-2)\cos\beta, U(t-2)\sin\beta - 5(t-2)^2\Big)$$

ii. Find the value of $\tan\beta$. **3**

iii. Find the initial velocity of the second particle. **1**

Question 3 (13 Marks) Commence a NEW page. **Marks**

A ball of mass 0.5 kg is projected vertically upwards from ground level at $10\,\mathrm{ms}^{-1}$.

During its motion, the ball is subject to gravity and it also experiences air resistance of magnitude $\frac{1}{20}v^2$, where v is the velocity of the ball after t seconds. Take $g = 10\,\mathrm{ms}^{-2}$.

(a) Show that $\dfrac{1}{2}\ddot{x} = -\dfrac{1}{2}\times 10 - \dfrac{v^2}{20}$. **1**

(b) Show that the ball reaches a maximum height of $5\log_e 2$ metres above the ground. **4**

(c) Show that the ball takes $\frac{\pi}{4}$ seconds to reach its maximum height. **3**

(d) The ball, having reached its maximum height, falls back towards its starting point. As it falls, it once again is subject to gravity and air resistance of magnitude $\frac{1}{20}v^2$. **5**

Find the speed of the ball when it returns to its starting point, and hence show that the ball has not attained its terminal velocity before it returns to the ground.

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

NOTE: $\ln x = \log_e x,\ x > 0$

EXTENSION 2 ASSESSMENT NO 4. SOLUTIONS

Q1 (a) $a^2+b^2=(a-b)^2+2ab$

$\therefore a^2+b^2 \geq 2ab$

Similarly $b^2+c^2 \geq 2bc$

$c^2+a^2 \geq 2ca$

i.e. $2(a^2+b^2+c^2) \geq 2ab+2bc+2ca \quad \cdots (*)$ 1 mark

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$ 1 mark

$\leq 3(a^2+b^2+c^2)$ from $(*)$

$\therefore \left(\frac{a+b+c}{3}\right)^2 \leq \frac{a^2+b^2+c^2}{3}$ 1 mark

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(b) $f(x) = x - \log_e(x^2+1)$ where $x \geq 0$

(i) $f'(x) = 1 - \frac{2x}{x^2+1} = \frac{(x-1)^2}{x^2+1}$

$f'(x) > 0$ for $x \neq 1$ 1 mark

At $x=1$, $f(1) = 1 - \ln 2 > 0$ 1 mark

At $x=0$, $f(0) = 0$

$f(x)$ is a continuous function

$\therefore f(x) > x$ for $x > 0$

$\therefore x > \log_e(x^2+1)$ for $x > 0$

(ii) $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ 1 mark

$\int_0^1 \log_e(x^2+1)\,dx = \left[x\log_e(x^2+1)\right]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx$

$= \log_e 2 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$

$= \log_e 2 - 2\left[x - \tan^{-1}x\right]_0^1$ 1 mark

$= \log_e 2 - 2\left(1 - \frac{\pi}{4}\right)$

$= \log_e 2 - 2 + \frac{\pi}{2}$

From (i) $\frac{1}{2} > \log_e 2 - 2 + \frac{\pi}{2}$ 1 mark

$\therefore 5 > 2\log_e 2 + \pi$

(c) (i) $t_n + \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} + \frac{1}{2n}$

$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{2}{2n}$ — 1 mark

$= \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}$

(ii) When $n=1$, $S_n = 1 - \frac{1}{2} = \frac{1}{2}$

$t_n = \frac{1}{2}$ — 1 mark

$S_n = t_n$ $\therefore$ true for $n=1$

Assume true for $n=k$

i.e. $S_k = t_k$

i.e. $S_k = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k-1} + \frac{1}{2k}$

$S_{k+1} = 1 - \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{2k+1} - \frac{1}{2k+2}$

$= S_k + \frac{1}{2k+1} - \frac{1}{2k+2}$

$= \frac{1}{k+1} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2}$ — 1 mark

from assumption

$t_{k+1} = \frac{1}{k+2} + \cdots \frac{1}{2k+1} + \frac{1}{2k+2}$ — 1 mark

$t_{k+1} + \frac{1}{2k+2} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k+1}$ from (i)

$\therefore t_{k+1} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k+1} - \frac{1}{2k+2}$ — 1 mark

$= S_{k+1}$

$\therefore$ true for $n=k+1$ if true for $n=k$

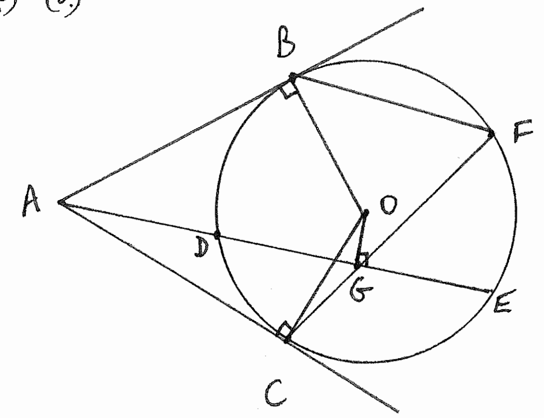
But already proved true for $n=1$

$\therefore$ true for $n=2,3,4,\ldots$ etc

i.e. true for all positive integers

Question 2

(a) (i)



$\angle ABO = \angle ACO = 90°$ (tangent $\perp$ radius)
i.e. $\angle ABO + \angle ACO = 180°$
$\therefore ABOC$ is cyclic (opposite $\angle$s supplementary)
Also, $\angle AGO = 90°$ (line from centre to midpoint of chord is $\perp$ chord)

1 mark correct reasoning

$\therefore \angle ACO = \angle AGO$
$\therefore AOGC$ is cyclic (AO subtends equal $\angle$s at G and C - Angles in same segment)

1 mark correct reasoning

(ii) $\angle OGF = \angle OAC$ (exterior $\angle$ of cyclic quadrilateral AOGC is equal to opposite interior $\angle$)

1 mark correct reason

(iii) $BA = CA$ (tangents from external point)
$OB = OC$ (equal radii)
AO is common
$\therefore \triangle OBA \equiv \triangle OCA$ (SSS)
$\therefore \angle BAO = \angle CAO$ (corresponding sides in congruent $\triangle$s)

Also proved using RHS

1 mark correct reasoning

(iv) Let $\angle OGF = \theta$
$\therefore \angle OAC = \theta$ (from (ii))
and $\angle OAB = \theta$ (from (iii))
$\angle BOC = 180 - 2\theta$ ($\angle$ opposite $\angle BAC$ in cyclic quadrilateral)
$\angle BFC = 90 - \theta$ ($\angle$ at circumference is $\frac{1}{2}\angle$ at centre)
$\angle FGE + \theta + 90 = 180$ ($\angle AGE$ is a straight $\angle$)
$\angle FGE = 90 - \theta$
$= \angle BFC$
$\therefore BF \parallel AE$ (alternate $\angle$s equal)

1 mark (relates $\angle BFC$ to $\angle BOC$)
1 mark (establishes $\angle BFC = \angle FGE$ or $= \angle AGC$ or supplementary to $\angle FGA$)

1 mark reason

(b) (i) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = U\cos\beta$ $\dot{y} = -10(t-2) + U\sin\beta$

$x = U\cos\beta\,(t-2)$ $y = U\sin\beta(t-2) - 5(t-2)^2$

1 mark for finding correct integration from $\ddot{x}$ and $\ddot{y}$

1 mark for showing understanding that $T = t-2$ for 2nd particle

note: just substituting $t-2$ does not get mark

(ii) At time t seconds, 1st particle is at

$(80\cos\alpha t,\ 80\sin\alpha t - 5t^2)$

1 mark

particles collide when $t = 3$

i.e. 1st particle is at $(240\cos\alpha,\ 240\sin\alpha - 45)$

2nd particle is at $(U\cos\beta,\ U\sin\beta - 5)$

$U\cos\beta = 240\cos\alpha$ --(1)

$U\sin\beta = 240\sin\alpha - 40$ --(2)

1 mark

$(2) \div (1) \Rightarrow \quad \tan\beta = \tan\alpha - \frac{1}{6\cos\alpha}$

$\tan\alpha = \frac{5}{12} \Rightarrow \cos\alpha = \frac{12}{13}$

$\therefore \tan\beta = \frac{5}{12} - \frac{13}{72} = \frac{17}{72}$

1 mark

(iii) $U\cos\beta = 240 \times \frac{12}{13}$

$\cos\beta = \frac{72}{\sqrt{5473}}$

$\Rightarrow \quad U = \dfrac{\frac{2880}{13}}{\frac{72}{\sqrt{5473}}} = \dfrac{40\sqrt{5473}}{13}$

1 mark

2011 Ext. 2 Task 4

Q3 (a) Forces acting on ball : $\downarrow mg$ $\downarrow \frac{1}{20}v^2$

$\therefore$ Resultant force is

$$F = -mg - \frac{1}{20}v^2 \quad \text{(taking } \uparrow \text{ as positive)}$$

By Newton, $F = m\ddot{x}$

$$\therefore m\ddot{x} = -mg - \frac{1}{20}v^2$$ ✓ (must mention forces)

Subbing $m = 0.5$ kg, $g = 10ms^{-2}$, we get

$$0.5\ddot{x} = -0.5 \times 10 - \frac{1}{20}v^2, \text{ as required.}$$

(b) Method 1 – Constants of Integration

$$\ddot{x} = -10 - \frac{v^2}{10}$$

$$\therefore v\frac{dv}{dx} = -10 - \frac{v^2}{10} = -\frac{100+v^2}{10}$$ ✓

$$\therefore \frac{dv}{dx} = -\frac{100+v^2}{10v}$$

$$\therefore \frac{dx}{dv} = -\frac{10v}{100+v^2}$$

$$\therefore x = -5\ln(100+v^2) + C$$ ✓

At $t=0$, $x=0$, $v=10$:

$$\therefore C = 5\ln 200$$

$$\therefore x = 5\ln\left(\frac{200}{100+v^2}\right)$$ ✓

At max. height, $v=0$

$$\therefore x = 5\ln\left(\frac{200}{100}\right) = 5\ln 2$$

(as required to be shown) ✓

Q3 – b continued

b Method 2 – Definite Integral technique

$$\ddot{x} = -10 - \frac{v^2}{10}$$

$$\therefore v\frac{dv}{dx} = -\frac{100+v^2}{10v}$$ ✓

$$\frac{dx}{dv} = -\frac{10v}{100+v^2}$$

Initially $(t=0)$, $x=0$, $v=10$.

Let maximum height $= H$.

At $x = H$, $v = 0$.

Thus $$\int_0^H dx = \int_{10}^{0} \frac{-10v}{100+v^2}\,dv$$ ✓ uses initial conditions

$$\therefore [x]_0^H = \left[-5\ln(100+v^2)\right]_{10}^{0}$$ ✓

$$H - 0 = -5\ln 100 + 5\ln 200$$

$$H = 5\ln\frac{200}{100}$$ ✓

$$\therefore H = 5\ln 2$$, as required.

Q3 continued.

(c) Method 1 – Constants of Integration:

$$\ddot{x} = -10 - \frac{v^2}{10}$$

$$\therefore \frac{dv}{dt} = -\frac{100+v^2}{10}$$

$$\frac{dt}{dv} = -\frac{10}{100+v^2}$$

$$\therefore t = -10 \cdot \frac{1}{10} \tan^{-1}\frac{v}{10} + C$$ ✓

At $t=0$, $x=0$, $v=10$:

$$\therefore C = \tan^{-1}\frac{10}{10} = \tan^{-1} 1 = \pi/4$$

Thus $t = \frac{\pi}{4} - \tan^{-1}\frac{v}{10}$ ✓

At max. height, $v=0$ $\therefore t = \frac{\pi}{4} - \tan^{-1} 0 = \frac{\pi}{4}$ secs. ✓

Method 2 – Definite Integrals:

Let T = time to maximum height.

$$\therefore \ddot{x} = \frac{dv}{dt} = -\frac{100+v^2}{10}$$

$$\therefore \frac{dt}{dv} = -\frac{10}{100+v^2}$$

$$\therefore \int_0^T dt = -10 \int_{10}^{0} \frac{dv}{100+v^2}$$ ✓

$$\therefore T - 0 = -10 \left[\frac{1}{10}\tan^{-1}\frac{v}{10}\right]_{10}^{0}$$ ✓

$$\therefore T = -10\left[\frac{1}{10}\tan^{-1} 0 - \frac{1}{10}\tan^{-1} 1\right]$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$ secs, as required. ✓

Q3 continued.

(d) <u>Method 1 – Constants of Integration:</u>

Falling, we measure $\downarrow$ as positive, and so at $t=0$, $x=0$ and $v=0$.

Thus $\ddot{x} = 10 - \frac{v^2}{10}$ ✓

$\therefore\ v\frac{dv}{dx} = \frac{100-v^2}{10}$ $\therefore\ \frac{dv}{dx} = \frac{100-v^2}{10v}$

$\therefore\ \frac{dx}{dv} = \frac{10v}{100-v^2}$

$\therefore\ x = -5\ln(100-v^2) + C$

At $t=0$, $x=0$, $v=0$:

$\therefore\ C = 5\ln 100$

$\therefore\ x = 5\ln\left(\frac{100}{100-v^2}\right)$ ✓

"Returns to start point" means $x = 5\ln 2$ (from (b))

$\therefore\ 5\ln 2 = 5\ln \frac{100}{100-v^2}$

$\therefore\ 2 = \frac{100}{100-v^2}$

$\therefore\ v^2 = 50$ $\therefore\ |v| = \text{speed} = \sqrt{50}$ m/s. ✓

Terminal velocity arises when $\ddot{x} = 0$

$\therefore\ 10 - \frac{v^2}{10} = 0$ $\therefore\ v^2 = 100,\ v = 10$ ✓

Since $\sqrt{50} < 10$, final velocity does not reach terminal velocity. ✓

Q3 continued

(d) <u>Method 2 – Definite Integrals:</u>

For downwards motion, take $\downarrow$ as positive.

At $t=0$, $x=0$, $v=0$.

When hitting the ground, we have $x = 5\ln 2$ (from (b)), and let $v = v_f$ (i.e. final velocity)

Equation of motion is $\ddot{x} = 10 - \frac{v^2}{10}$ ✓

(since $m\ddot{x} = mg - \frac{v^2}{20}$)

$\therefore\ v\frac{dv}{dx} = 10 - \frac{v^2}{10} = \frac{100-v^2}{10}$

$\therefore\ \frac{dv}{dx} = \frac{100-v^2}{10v}$

$\therefore\ \frac{dx}{dv} = \frac{10v}{100-v^2}$

Thus $\int_0^{5\ln 2} dx = \int_0^{v_f} \frac{10v}{100-v^2}\,dv$

$\therefore\ [x]_0^{5\ln 2} = \left[-5\ln(100-v^2)\right]_0^{v_f}$ ✓

$\therefore\ 5\ln 2 = 5\ln 100 - 5\ln(100-v_f^{\,2})$

$\therefore\ 2 = \frac{100}{100-v_f^{\,2}}$ $\therefore\ v_f^{\,2} = 50$ $\therefore\ v_f = \sqrt{50}$ ✓

For terminal velocity $\ddot{x} = 0$, $\therefore\ 10 - \frac{v^2}{10} = 0$ $\therefore\ v^2 = 100$, $v = 10$ ✓

Thus $v_f < 10$ ie $v_f <$ terminal velocity, as required. ✓