NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 2)

2013 HSC Course Assessment Task 4
Monday August 12, 2013

General instructions

- Working time – 60 min
  (plus 5 minutes reading time)
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M4A – Mr Choy
- ◯ 12M4B – Mr Weiss
- ◯ 12M4C – Ms Ziaziaris

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	**Total**	%
MARKS	$\overline{15}$	$\overline{15}$	$\overline{13}$	$\overline{43}$	

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 1 (15 Marks) Commence a NEW page. **Marks**

(a) Find the value of k in the expansion of $(1+kx)^{10}$ if the coefficient of x^8 is 42 times the coefficient of x^6 in the expansion. **3**

(b) Find the (numerically) greatest term in the expansion of $(2+5x)^{11}$ when $x = \dfrac{2}{3}$. **3**

(c) i. For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, show that **2**

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}(1-x) + \binom{n}{2}x^{n-2}(1-x)^2 + \cdots \binom{n}{n}(1-x)^n = 1$$

ii. Deduce that **1**

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

(d) Use the expansion of $(1+x)^n$ to show that **3**

$$\sum_{r=0}^{n} \frac{\binom{n}{r}}{r+1} = \frac{2^{n+1}-1}{n+1}$$

(e) By considering the coefficients of x^{n+1} from both sides of the identity **3**

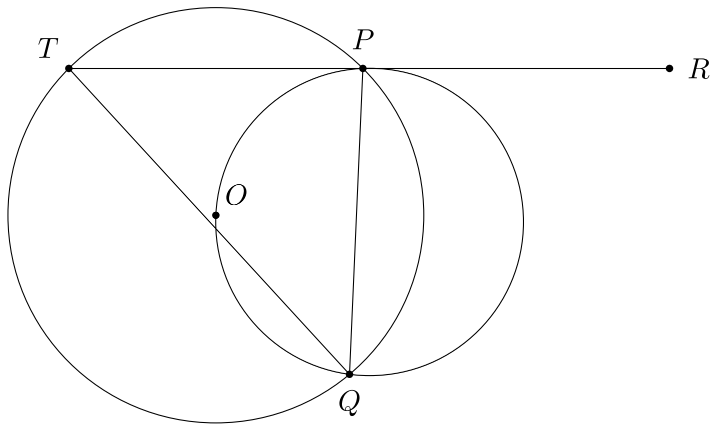
$$(1+x)^n(1+x)^n = (1+x)^{2n}$$

prove that

$$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \cdots + \binom{n}{n-1}\binom{n}{n} = \frac{2n!}{(n-1)!(n+1)!}$$

Question 2 (15 Marks) Commence a NEW page. **Marks**

(a) Two circles intersect at P and Q as shown. The smaller circle passes through the centre, O, of the larger circle.



The tangent to the smaller circle RPT cuts the larger circle at T. PQ bisects $\angle RQO$. Let $\angle PTQ = \alpha$.

i. Show that $\triangle PQT$ is isosceles. **3**

ii. Show that P is the midpoint of RT. **2**

(b) If a, b, c and $d > 0$, show that

i. $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ **2**

ii. $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$ **2**

iii. $(a+b+d)(b+c+d)(c+a+d)(a+b+c) \geq 81abcd$ **2**

(c) Consider the function $f(x) = (ax-b)^2 + (cx-d)^2$ where a, b, c and $d \in \mathbb{R}$.

i. Explain why $f(x) > 0$ for all x. **1**

ii. Hence or otherwise, prove that **3**

$$|ab + cd| \leq \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}$$

Examination continues overleaf. . .

Question 3 (13 Marks) Commence a NEW page. **Marks**

(a) The Fibonacci sequence $1, 1, 2, 3, 5, 8, \cdots$ can be defined as $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_n + u_{n+1}$ for integers $n > 0$.

i. Use induction to prove that $u_n < a^n$, for any $a > \dfrac{1+\sqrt{5}}{2}$. **3**

ii. Assuming that $\dfrac{u_{n+1}}{u_n}$ approaches a limit as $n \to \infty$. Show that **4**

$$\frac{u_{n+1}}{u_n} \to \frac{1+\sqrt{5}}{2}$$

(b) i. Write out the binomial expansion of $\left(1 + \dfrac{1}{n}\right)^n$, where $n \in \mathbb{Z}^+$. **1**

ii. Show that the $(k+1)$-th term, T_{k+1}, is given by **2**

$$T_{k+1} = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)$$

iii. Let the $(k+1)$-th term of the expansion $\left(1 + \dfrac{1}{n}\right)^{n+1}$ be U_{k+1}. **3**
Show that $U_{k+1} > T_{k+1}$.

End of paper.

Q1 a) $(1+Kx)^{10} = {}^{10}C_0 + {}^{10}C_1 Kx + {}^{10}C_2 (Kx)^2 + \dots + {}^{10}C_6 (Kx)^6 + \dots {}^{10}C_8 (Kx)^8$

$\therefore 42\ {}^{10}C_6\ K^6 = {}^{10}C_8\ K^8$ 1

$$42 \times \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = \frac{10\times 9}{2\times 1} K^2$$

$$\frac{42\times 10\times 8\times 7}{45\ 93} = K^2$$ 1

$196 = K^2$

$\therefore K = \pm 14$ 1

b) $(2+5x)^{11}$

$$\frac{T_{r+1}}{T_r} = \frac{{}^{11}C_r\ 2^{11-r}(5x)^r}{{}^{11}C_{r-1}\ 2^{12-r}(5x)^{r-1}}$$

$$= \frac{11!}{r!(11-r)!} \cdot \frac{5x}{2} \times \frac{(12-r)!(r-1)!}{11!}$$ 1

$$= \frac{12-r}{r} \times \frac{5}{2} \times \frac{2}{3} > 1$$

$60 - 5r > 3r$ 1

$60 > 8r$

$r < 7\frac{3}{4}$

$r = 7$

$\therefore$ greatest term $\frac{11!}{4!7!}\ 2^4\ 5^7 \times \frac{2}{3}^7$ 1

c) i) $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + \dots + {}^nC_r a^{n-r}b^r + \dots + {}^nC_n b^n$

Let $a = x$ and $b = 1-x$

$1 = {}^nC_0 x^n + {}^nC_1 x^{n-1}(1-x) + \dots + {}^nC_r x^{n-r}(1-x)^r + \dots + {}^nC_n (1-x)^n$

1

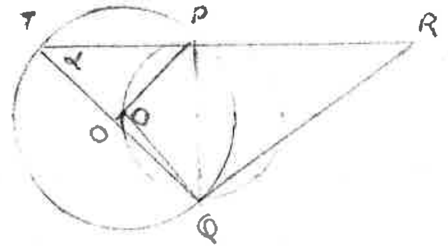
ii) Let $x = \frac{1}{2}$

$1 = {}^nC_0 (\frac{1}{2})^n + {}^nC_1 (\frac{1}{2})^{n-1}(\frac{1}{2}) + \dots {}^nC_r (\frac{1}{2})^{n-r}(\frac{1}{2})^r + \dots + {}^nC_n (\frac{1}{2})^n$

$1 = {}^nC_0 \frac{1}{2^n} + {}^nC_1 \frac{1}{2^n} + \dots + {}^nC_r \frac{1}{2^n} + \dots + {}^nC_n \frac{1}{2^n}$

$\therefore 2^n = {}^nC_0 + {}^nC_1 + \dots + {}^nC_r + \dots + {}^nC_n$ 1

Q2 a) i)



$P\hat{O}Q = 2\,P\hat{T}Q = 2\alpha$ (angle at centre 2× angle at circumference)

$Q\hat{P}R = P\hat{O}Q = 2\alpha$ (alt. segment thm)

In $\triangle PTQ$

$T\hat{Q}P + P\hat{T}Q = Q\hat{P}R$ (ext angle of $\triangle$ = interior opposites)

$T\hat{Q}P = P\hat{T}Q$

$\therefore \triangle PTQ$ is isos (2 angles equal)

ii) $\triangle PQO$ is isos (radii equal)

$\therefore O\hat{P}Q = O\hat{Q}P$

$\therefore P\hat{Q}O = \frac{1}{2}(180 - 2\alpha)$ (angle sum of $\triangle$)

$= 90 - \alpha$

In $\triangle PQR$

$P\hat{Q}R = P\hat{Q}O = 90 - \alpha$ (PQ bisects $R\hat{Q}O$)

$P\hat{R}Q = 180 - 2\alpha - (90 - \alpha)$ (angle sum of $\triangle$)

$= 90 - \alpha$

$\therefore \triangle PRQ$ is isos (2 angles equal)

$\therefore PR = PQ$

But $PQ = PT$ from (i) isos $\triangle$

$\therefore TP = PR$

and P is the midpoint of RT.

Q2) i) The sum of 2 squares is positive.

c) ii) $f(x) = a^2x^2 - 2abx + b^2 + c^2x^2 - 2cdx + d^2$

$= (a^2+c^2)x^2 - (2ab+2cd)x + b^2 + d^2$

Since $f(x) \geq 0$ for all x,

the function has no real roots, ie $\Delta \leq 0$

$$\Delta = (-(2ab+2cd))^2 - 4(a^2+c^2)(b^2+d^2) \leq 0$$

$$4(ab+cd)^2 - 4(a^2+c^2)(b^2+d^2) \leq 0$$

$$(ab+cd)^2 \leq (a^2+c^2)(b^2+d^2)$$

$$\sqrt{(ab+cd)^2} \leq \sqrt{a^2+c^2}\,\sqrt{b^2+d^2}$$

ie $|ab+cd| \leq \sqrt{a^2+c^2}\,\sqrt{b^2+d^2}$ #

Q3 a)

ii) If $\frac{U_{n+1}}{U_n}$ approaches a limit as $n \to \infty$

then $\frac{U_{n+2}}{U_{n+1}}$ approaches the same limit.

So as $n \to \infty$ $\frac{U_{n+2}}{U_{n+1}} = \frac{U_{n+1}}{U_n}$

ie $\frac{U_{n+1} + U_n}{U_{n+1}} = \frac{U_{n+1}}{U_n}$ since $U_{n+2} = U_n + U_{n+1}$

$\therefore$ $1 + \frac{U_n}{U_{n+1}} = \frac{U_{n+1}}{U_n}$

Let $\frac{U_{n+1}}{U_n} = x$ as $n \to \infty$

$\therefore$ $1 + \frac{1}{x} = x$

ie $x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$

but $x > 0$ $\therefore$ limit is $\frac{1+\sqrt{5}}{2}$

ie as $n \to \infty$, $\frac{U_{n+1}}{U_n} \to \frac{1+\sqrt{5}}{2}$