Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2014
ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- **Working time – 55 minutes**
  Reading time – 5 minutes
- Write on both sides of the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators only
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions.

Class Teacher:
(Please tick or highlight)

- O Mr Lam
- O Ms Ζιάζιαρη
- O Mr Ireland

Student Number:

(To be used by the exam markers only.)

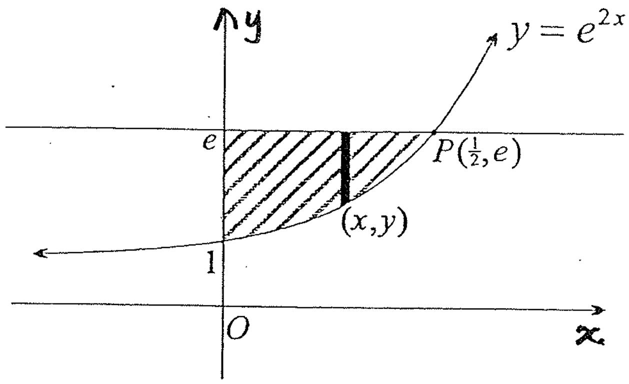
Section	A	B	C	D	E	Total	Percent
Mark	—/9	—/13	—/4	—/9	—/5	—/40	—/100

Section A (9 marks) **Marks**

(1) A solid has as its base the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$.

If each section perpendicular to the major axis is an equilateral triangle, then show that the volume of the solid is $128\sqrt{3}$ cubic units. **4**

(2) In the diagram shown below, $P(\frac{1}{2},e)$ is the point of intersection of the curve $y=e^{2x}$ and the line $y=e$.



Using the method of cylindrical shells, find the volume of the solid generated when the shaded region enclosed by the y-axis, the curve $y=e^{2x}$ and the line $y=e$ is rotated about the y-axis. **5**

Section B (13 marks)

(3) (a) Show that $a^2+b^2\geq 2ab$ for any two real numbers *a* and *b*. **1**

(b) Deduce that $a^2+b^2+c^2\geq ab+bc+ca$ for real numbers *a, b,c.* **1**

(c) If $a+b+c=6$ then show that $ab+bc+ca\leq 12$ **2**

(d) If *a, b* and c are positive, show that $(a+b)(b+c)(c+a)\geq 8abc$ **3**

(4) It can be shown that the arithmetic mean of *n* positive real numbers is always greater than or equal to their geometric mean. That is, if $a_1,a_2,...a_n$ are positive real numbers then $\frac{a_1+a_2+...+a_n}{n}\geq \sqrt[n]{a_1a_2...a_n}$. Using this result, show that $n!\leq\left(\frac{n+1}{2}\right)^n$ for any positive integer $n\geq 1$. *(induction is not needed)* **2**

(5) (a) Given that $f(x)=\sqrt{1+x}$, show that $f'(x)<\frac{1}{6}$ for $x>8$. **2**

(b) Hence or otherwise show that $\sqrt{1+x}\leq 3+\frac{x-8}{6}$ when $x\geq 8$. **2**

Section C (4 marks)

(6) Chris believes that if $x+\frac{1}{x}$ equals an integer, say N, then $x^n+\frac{1}{x^n}$ must also be an integer, for all positive integers n.

(a) Prove that this is true for $n=1$ and $n=2$. **1**

(b) Hence use strong induction to prove that if $x+\frac{1}{x}$ is an integer then

$x^n+\frac{1}{x^n}$ is an integer, for all integers $n \geq 1$. **3**

Section D (9 marks)

(7) (a) Show that $\binom{n}{r-1}=\binom{n+1}{r}-\binom{n}{r}$, where $r=1, 2, 3, \ldots, n$ **2**

(b) Hence evaluate $\sum_{n=3}^{100}\binom{n}{2}$ **2**

(8) This question uses the identity $(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots..\binom{n}{n}x^n$

(a) Use the sum of a geometric progression formula to simplify

$$1+(1+x)+(1+x)^2+(1+x)^3+\ldots+(1+x)^{n-1}$$ **1**

(b) Hence show that

$$1+(1+x)+(1+x)^2+\ldots+(1+x)^{n-1}=\binom{n}{1}+\binom{n}{2}x+\binom{n}{3}x^2+\ldots..\binom{n}{n}x^{n-1}$$ **1**

(c) Find $\int_{-1}^{0}\binom{n}{1}+\binom{n}{2}x+\binom{n}{3}x^2+\ldots..\binom{n}{n}x^{n-1}\,dx$ **1**

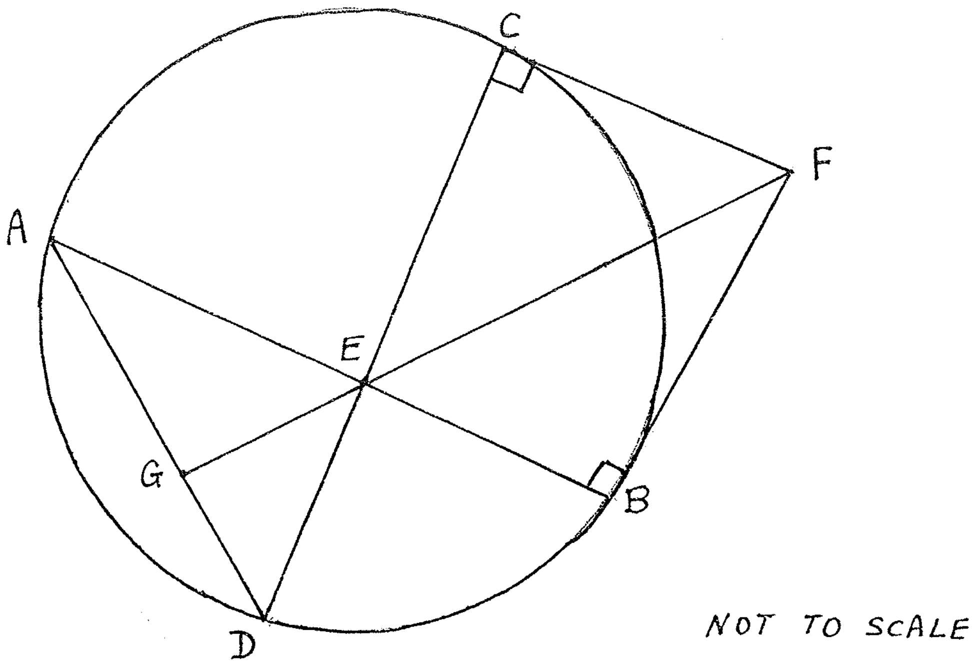
(d) Hence show that:

$$\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}\binom{n}{r}=\sum_{r=1}^{n}\frac{1}{r}$$ **2**

Please turn over for final question →

Section E (5 marks)

(9)



In the diagram, AB and CD are two chords in a circle. AB and CD intersect at E.
F is a point such that $\angle ABF$ and $\angle DCF$ are right angles.
FE produced meets AD at G.

Copy the diagram into your answer booklet. (Start this question at the top of a left-hand page, so that you – and we – don't have to keep turning the page!).

(a) State why $CFBE$ is a cyclic quadrilateral. **1**

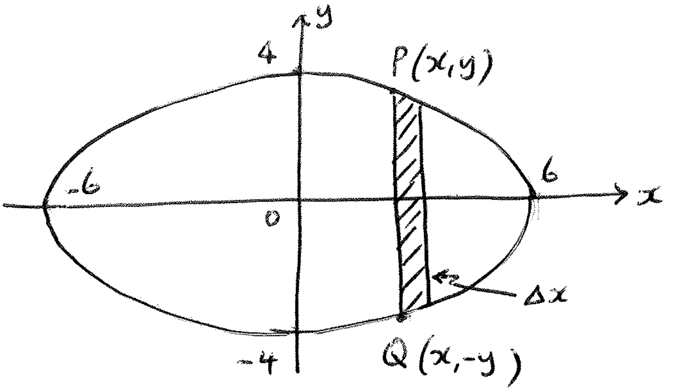
(b) Hence prove that $\angle BAD = \angle GFB$ **2**

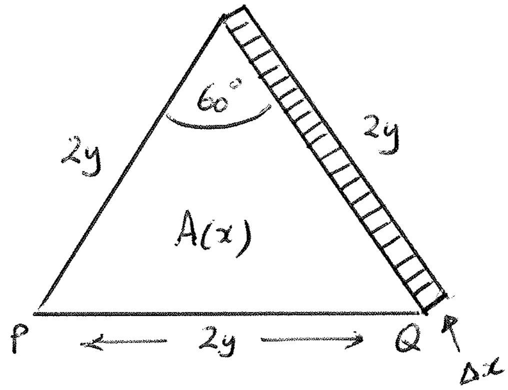
(c) Hence prove that FG is perpendicular to AD. **2**

END OF EXAM

2014 Extension 2 - Task 4 - {SUGGESTED SOLUTIONS}

Q1 $\quad \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$





$$A(x) = \tfrac{1}{2}\cdot 2y \cdot 2y \cdot \sin 60^\circ$$

$$\therefore \ A(x) = y^2\sqrt{3}$$

But $\quad y^2 = 16\left(1 - \dfrac{x^2}{36}\right)$

$$\therefore \ A(x) = 16\sqrt{3}\left(1 - \frac{x^2}{36}\right)$$

$$\& \therefore \ V(x) \doteqdot 16\sqrt{3}\left(1 - \frac{x^2}{36}\right)\cdot \Delta x$$

Thus $\quad V = \lim\limits_{\Delta x \to 0} \displaystyle\sum_{x=-6}^{x=6} 16\sqrt{3}\left(1 - \frac{x^2}{36}\right)\cdot \Delta x$

$$\therefore \ V = 16\sqrt{3}\int_{-6}^{6}\left(1 - \frac{x^2}{36}\right)dx$$

$$= 32\sqrt{3}\int_{0}^{6}\left(1 - \frac{x^2}{36}\right)dx$$

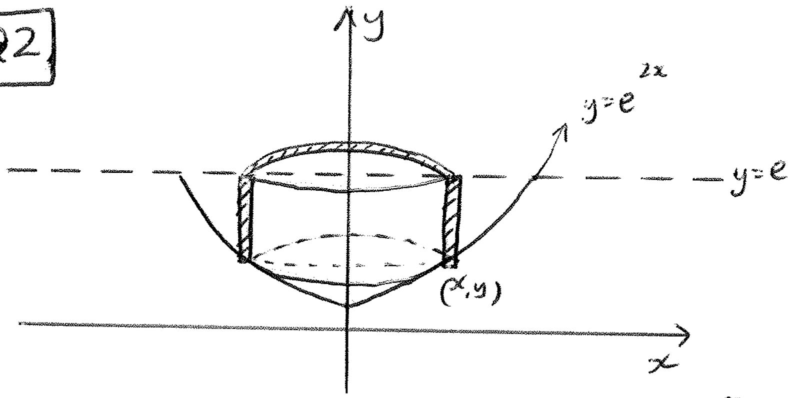
$$= 32\sqrt{3}\left[x - \frac{x^3}{108}\right]_0^6$$

$$= 32\sqrt{3}\left[6 - \frac{216}{108}\right]$$

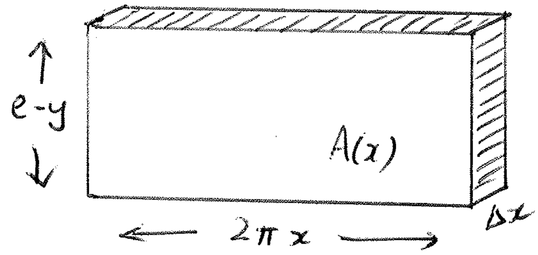
$$\therefore \ V = 128\sqrt{3} \text{ units}^3. \ \#$$

/4

Q2



$\Delta V = 2\pi x(e-y)\cdot\Delta x$



$\therefore \Delta V = 2\pi x(e-e^{2x})\cdot\Delta x$

$$V = \lim_{\Delta x\to 0}\sum_{x=0}^{\frac{1}{2}} 2\pi x(e-e^{2x})\Delta x$$

$$\therefore V = \int_0^{\frac{1}{2}} 2\pi x(e-e^{2x})\,dx$$

Now $\displaystyle\int_0^{\frac{1}{2}} 2\pi x e\,dx = 2\pi e\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{\pi e}{4}$

And $\displaystyle\int_0^{\frac{1}{2}} 2\pi x e^{2x}\,dx = 2\pi\int_0^{\frac{1}{2}} x\,\frac{d}{dx}\left(\tfrac{1}{2}e^{2x}\right)dx$

$$= 2\pi\left\{\left[\tfrac{1}{2}xe^{2x}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\tfrac{1}{2}e^{2x}\,dx\right\}$$

$$= 2\pi\left[\tfrac{1}{2}\cdot\tfrac{1}{2}e^{1} - \left[\tfrac{1}{4}e^{2x}\right]_0^{\frac{1}{2}}\right]$$

$$= 2\pi\left[\frac{e}{4} - \left(\frac{e}{4}-\frac{1}{4}\right)\right]$$

$$= \frac{\pi}{2}$$

Thus $V = \dfrac{\pi e}{4} - \dfrac{\pi}{2}$

$\left[\text{ie } V = \dfrac{\pi}{4}(e-2)\right]$ #

/5

$\boxed{Q3}$ (a) $(a-b)^2 \geq 0$

$\therefore a^2 - 2ab + b^2 \geq 0$ ✓

$\therefore a^2 + b^2 \geq 2ab$.

(b) Likewise,

$a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

Adding the 3 equations, ✓

$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2ac + 2bc$

$\therefore a^2 + b^2 + c^2 \geq ab + bc + ca$.

(c) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$ ✓

$\therefore \quad 6^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\therefore \quad 36 \geq 3(ab + bc + ca)$ from (b) ✓

$\therefore \quad ab + bc + ca \leq 12$

(d) Since $x^2 + y^2 \geq 2xy$, x, y real (from (a))

$\therefore \quad a + b \geq 2\sqrt{a} \cdot \sqrt{b}$ (letting $a = x^2$, $b = y^2$) ✓

Likewise

$b + c \geq 2\sqrt{b} \cdot \sqrt{c}$

$a + c \geq 2\sqrt{a} \cdot \sqrt{c}$ ✓

$\therefore$ Multiplying, $(a+b)(b+c)(a+c) \geq 2\sqrt{a} \cdot \sqrt{b} \cdot 2\sqrt{b} \cdot \sqrt{c} \cdot 2\sqrt{a} \cdot \sqrt{c}$ ✓

$= 8abc$.

/7

Q4

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Thus $$\frac{1+2+3+\cdots+n}{n} \geq \sqrt[n]{1\cdot 2\cdot 3\cdots n}$$ ✓

$$LHS = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

$$RHS = \sqrt[n]{n!}$$

Thus $$\sqrt[n]{n!} \leq \frac{n+1}{2}$$

$$\therefore n! \leq \left(\frac{n+1}{2}\right)^n \quad \#$$ ✓

/2

Q5 (a) $f(x) = \sqrt{1+x}$ $\therefore$ $f'(x) = \dfrac{1}{2\sqrt{1+x}}$ ✓

$$< \frac{1}{2\sqrt{1+8}} \quad \text{as } x > 8$$

$$= \frac{1}{6}$$ ✓

(b) Let $g(x) = 3 + \dfrac{x-8}{6}$ $\therefore$ $g'(x) = \dfrac{1}{6}$

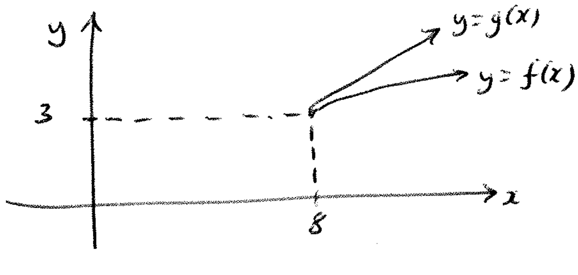
Now $f(8) = \sqrt{1+8} = 3$

$g(8) = 3 + 0 = 3$ $\therefore$ $f(8) = g(8)$. ✓

Since $f'(x) < g'(x)$ when $x > 8$,

and $f(8) = g(8)$

$\therefore$ $f(x) \leq g(x)$ when $x \geq 8$. ✓



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Q6

Let $x + \frac{1}{x} = N$.

(a) For $n=1$, $x^1 + \frac{1}{x^1} = x + \frac{1}{x}$

$= N$ $\therefore$ true for $n=1$

For $n=2$, $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 \cdot x \cdot \frac{1}{x}$

$= N^2 - 2$, an integer

$\therefore$ true for $n=2$.

✓

(b) Assume $x^n + \frac{1}{x^n}$ is an integer, for all integers $1, 2, 3, \ldots, k$, when $x + \frac{1}{x}$ is an integer.

Then $x^{k+1} + \frac{1}{x^{k+1}} = \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - x^k \cdot \frac{1}{x} - \frac{1}{x^k} \cdot x$

$= \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right)$.

But $x^k + \frac{1}{x^k}$ and $x^{k-1} + \frac{1}{x^{k-1}}$ are both integers, by our hypothesis;

and $x + \frac{1}{x} = N$, an integer.

Thus $x^{k+1} + \frac{1}{x^{k+1}}$ is an integer.

✓✓✓

We have proved that the statement is true for $n = k+1$ if it is true for $n = 1, 2, 3, \ldots k$.

Since it's true for $n=1$ and 2, it is true for all $n \geq 1$, by induction.

4

Q7 (a) $\binom{n+1}{r} - \binom{n}{r} = \dfrac{(n+1)!}{r!\,(n+1-r)!} - \dfrac{n!}{r!\,(n-r)!}$

$$= \frac{n!}{r!\,(n+1-r)!} \cdot \left[(n+1) - (n+1-r)\right]$$

$$= \frac{n!\,r}{r!\,(n+1-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \binom{n}{r-1} \text{, as required.}$$

(b) $\binom{n}{2} = \binom{n+1}{3} - \binom{n}{3}$ from (a)

$$\therefore \sum_{n=3}^{100} \binom{n}{2} = \sum_{n=3}^{100} \binom{n+1}{3} - \sum_{n=3}^{100} \binom{n}{3}$$

$$= \sum_{n=4}^{101} \binom{n}{3} - \sum_{n=3}^{100} \binom{n}{3}$$

$$= \binom{101}{3} - \binom{3}{3}$$

$$= 166\,650 - 1$$

$$\therefore \quad = 166\,649. \quad \#$$

4

$\boxed{Q8}$ (a) $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^{n-1} = \dfrac{1\left[(1+x)^n - 1\right]}{(1+x) - 1}$

(as $r = 1+x$,
$a = 1$
& n terms)

$= \dfrac{(1+x)^n - 1}{x}$ ✓

(b)

$1 + (1+x) + \cdots + (1+x)^{n-1} = \dfrac{(1+x)^n - 1}{x}$

$= \dfrac{\left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right] - 1}{x}$

$= \binom{n}{1} + \binom{n}{2}x + \cdots + \binom{n}{n}x^{n-1}$ ✓

(c) $\displaystyle\int_{-1}^{0} \binom{n}{1} + \binom{n}{2}x + \cdots + \binom{n}{n}x^{n-1}\,dx$

$= \left[\binom{n}{1}x + \frac{1}{2}\binom{n}{2}x^2 + \frac{1}{3}\binom{n}{3}x^3 + \cdots + \frac{1}{n}\binom{n}{n}x^n\right]_{-1}^{0}$

$= \binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + \frac{(-1)^{n+1}}{n}\binom{n}{n}$. ✓

P.T.O. →

$\boxed{Q8}$ – ctd.

(d)

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}\binom{n}{r} = \binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + \frac{(-1)^{n+1}}{n}\binom{n}{n}$$

$$= \int_{-1}^{0} \binom{n}{1} + \binom{n}{2}x + \cdots + \binom{n}{n}x^{n-1}\,dx$$

$$= \int_{-1}^{0} 1 + (1+x) + (1+x)^2 + \cdots + (1+x)^{n-1}\,dx$$

$$= \left[x + \frac{(1+x)^2}{2} + \frac{(1+x)^3}{3} + \cdots + \frac{(1+x)^n}{n}\right]_{-1}^{0}$$

$$= \left(0 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(-1 + 0 + 0 \cdots + 0\right)$$

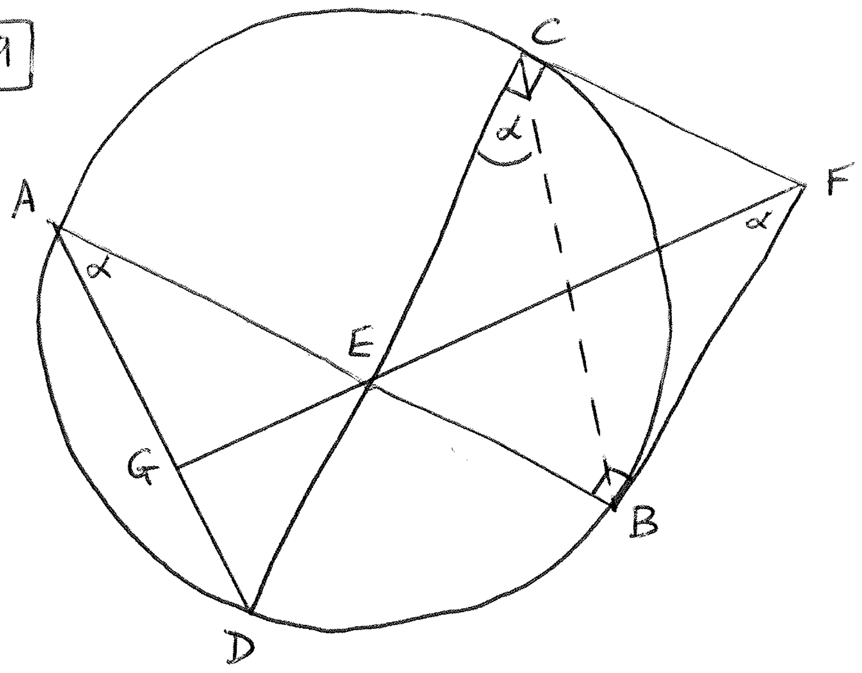
$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= \sum_{r=1}^{n} \frac{1}{r}. \qquad \#$$

✓ ✓

/5

Q9



(a) CFBE is cyclic as opposite angles $\angle ECF$ and $\angle EBF$ are supplementary. ✓

(b) Draw BC; Let $\angle ECB = \alpha$.

Then $\angle EFB = \alpha$ (angles in same segment of the circle thru CFBE). ✓

i.e. $\angle GFB = \alpha$.

Also, $\angle BAD = \alpha$ (angles in same segment of original circle ACBD)

$\therefore \angle BAD = \angle GFB$. ✓

(c) Since $\angle BAD = \angle GFB$,

$\therefore$ AFBG is a cyclic quadrilateral (BG subtends equal angles at two points, A and F, on the same side of BG). ✓

Hence $\angle AGF = \angle ABF$ (angles in same segment of circle AFBG) ✓

$= 90°$. #

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