Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2015 Year 12 HSC
ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- **Working time – 55 minutes**
  Reading time – 5 minutes
- Write on both sides of the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Marks may be deducted for badly arranged or illegible work.

Class Teacher:
(Please tick or highlight)

○ Mr Lin

○ Mr Weiss

○ Mr Ireland

Student No./Name:____________________

(To be used by the exam markers only.)

Question	1	2	3	4	Total	Percent
Mark	/13	/13	/9	/5	/40	/100

Question 1 (13 marks) Commence a NEW page **Mark**

(a) A particle of mass 3kg moves horizontally in a straight line with velocity v metres per second under a constant force of magnitude 5 newtons and experiences a resistance of $2+3v$ newtons. The initial velocity was v_0 metres per second.

(i) Show that the acceleration, $\ddot{x}$, is given by $\ddot{x} = 1 - v$. 1

(ii) Show that $v = 1 - e^{-t} + v_0 e^{-t}$ 3

(iii) Find the terminal velocity. 1

(b)

(i) A particle of mass m falls from rest, from a point O, in a medium whose resistance to motion is mkv, where k is a positive constant, v is the velocity in metres per second after time t and g is the acceleration due to gravity.

Prove that the speed at time t is given by $v = \frac{g}{k}(1 - e^{-kt})$ 3

(ii) An identical particle is projected vertically upwards from O with initial velocity u metres per second in the same medium.

If this particle is released simultaneously with the first, prove that the speed of the first particle when the second is momentarily at rest is given by

$\frac{wu}{w+u}$, where w is the terminal velocity of the first particle. 5

Question 2 (13 marks) Commence a NEW page **Mark**

(a)

(i) If $a > 0$ and $b > 0$ prove that $\dfrac{a+b}{2} \geq \sqrt{ab}$ 2

(ii) If $a, b, c, d > 0$ prove that $\dfrac{a+b+c+d}{4} \geq (abcd)^{\frac{1}{4}}$ 2

(b) Show that $2ab \leq a^2 + b^2$ for all real numbers a and b. Hence deduce that $3(ab+bc+ca) \leq (a+b+c)^2$ for all real numbers a, b, c. 3

(c)

(i) Show that for $x > 0$, $x > \log_e(1+x)$. 2

(ii) Hence show that $e^{{}^nC_2} > n!$ for all positive integers $n = 1, 2, 3, \ldots$ 4

Question 3 (9 marks) Commence a NEW page **Mark**

(a)

(i) Show that $(1+x)^m (1-\frac{1}{x})^m = (x-\frac{1}{x})^m$ 1

(ii) By considering the term(s) independent of x in the expansion of the result from part (i), justify the result:

$$\binom{100}{0}^2 - \binom{100}{1}^2 + \binom{100}{2}^2 - \ldots + \binom{100}{100}^2 = \binom{100}{50}$$ 3

(iii) Hence or otherwise show that

$$\sum_{k=0}^{50} (-1)^k \binom{100}{k}^2 = \frac{1}{2}\binom{100}{50}\left[1 + \binom{100}{50}\right]$$ 3

(b) By considering the binomial expansion of $(1+i)^n$ show that

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = 2^{\frac{n}{2}} \cos\frac{n\pi}{4}$$ 2

Turn page for last question →

Question 4 (5 marks) Commence a NEW page **Mark**

A particle is projected from a point O with initial velocity V at an angle of elevation α with the horizontal (α is acute).

Ignoring air resistance, and taking the acceleration due to gravity as g, the equations of displacement are given as:

$$x = Vt\cos\alpha$$

$$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$$

DO NOT PROVE THESE

Show that there exists a point P on the projectile's trajectory such that OP is perpendicular to the direction of the motion at P if $\tan\alpha \geq 2\sqrt{2}$. 5

END OF EXAMINATION

2015 Y12 Extension 2 - Task 4 Suggested Solutions

1 (a) $R \longleftarrow \bullet \longrightarrow F$

(i) $m\ddot{x} = F - R$

$\therefore 3\ddot{x} = 5 - (2 + 3v)$

$= 3 - 3v$

$\therefore \ddot{x} = 1 - v$ ✓

(ii) $\dfrac{dv}{dt} = 1 - v$

$\therefore \dfrac{dt}{dv} = \dfrac{1}{1-v}$ ✓

$\therefore t = -\ln(1-v) + c$

at $t = 0$, $v = v_0$

$\therefore 0 = -\ln(1 - v_0) + c$

$\therefore c = \ln(1 - v_0)$

$\therefore t = \ln(1 - v_0) - \ln(1 - v)$

$t = \ln\left(\dfrac{1 - v_0}{1 - v}\right)$ ✓

$\therefore e^{t} = \dfrac{1 - v_0}{1 - v}$

$\therefore e^{-t} = \dfrac{1 - v}{1 - v_0}$

$\therefore 1 - v = e^{-t} - v_0 e^{-t}$

$\therefore v = 1 - e^{-t} + v_0 e^{-t}$. ✓

(iii) as $t \to \infty$, $e^{-t} \to 0$ $\therefore v \to 1$

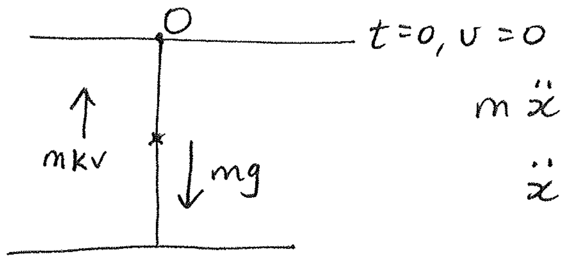
i.e. terminal velocity is 1 m/s. ✓

[alt. for terminal velocity, $\ddot{x} = 0$

$\therefore 1 - v = 0$ $\therefore v = 1$ m/s.].

$\boxed{1}$ - ctd. (b)

(i)



$$m\ddot{x} = mg - mkv$$
$$\ddot{x} = g - kv$$

$$\therefore \frac{dv}{dt} = g - kv$$
$$\frac{dt}{dv} = \frac{1}{g-kv}$$
$$\therefore t = -\frac{1}{k}\ln(g-kv) + c$$

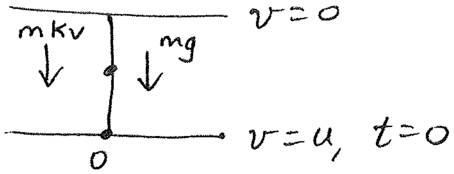
At $t=0, v=0$

$$\therefore 0 = -\frac{1}{k}\ln g + c \quad \therefore \quad c = \frac{1}{k}\ln g$$
$$\therefore t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$$
$$\therefore e^{kt} = \frac{g}{g-kv}$$
$$\frac{g-kv}{g} = e^{-kt}$$
$$g - kv = ge^{-kt}$$
$$kv = g - ge^{-kt}$$
$$\therefore v = \frac{g}{k}\left(1 - e^{-kt}\right), \text{ as required.}$$

(ii) as $t \to \infty$, $e^{-kt} \to 0$ $\therefore$ $v = \frac{g}{k}$ is the terminal velocity.

i.e. $w = \frac{g}{k}$.

1 (b)(ii)—ctd.



$$m\ddot{x} = -mg - mkv$$

$$\therefore \ddot{x} = -(g + kv)$$

$$\frac{dv}{dt} = -(g + kv) \qquad \therefore \frac{dt}{dv} = -\frac{1}{g+kv}$$

$$\therefore t = -\frac{1}{k}\ln(g + kv) + C$$

At $t = 0$, $v = u$ $\quad \therefore 0 = -\frac{1}{k}\ln(g + ku) + C$

$$\therefore C = \frac{1}{k}\ln(g + ku)$$

$$\therefore t = \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right)$$ ✓

When $v = 0$ (i.e. at maximum height),

$$t = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$$ ✓

Subbing this value in the expression for v in part (i) :–

$$v = \frac{g}{k}\left[1 - e^{-k\left(\frac{1}{k}\ln\left(\frac{g+ku}{g}\right)\right)}\right]$$ ✓

$$= \frac{g}{k}\left[1 - e^{-\ln\left(1 + \frac{k}{g}u\right)}\right]$$

$$v = w\left[1 - e^{-\ln\left(1 + \frac{u}{w}\right)}\right]$$

$$= w\left[1 - e^{-\ln\left(\frac{w+u}{w}\right)}\right]$$

$$= w\left[1 - e^{\ln\left(\frac{w}{w+u}\right)}\right]$$

$$= w\left[1 - \frac{w}{w+u}\right]$$

$$v = w\left[\frac{w+u-w}{w+u}\right] \qquad \therefore v = \frac{wu}{w+u} \quad \#$$ ✓

13

$\boxed{2}$ (a)

(i) $a>0,\ b>0 \quad \therefore \sqrt{a},\ \sqrt{b}$ exist

Then $(\sqrt{a}-\sqrt{b})^2 \geq 0$ ✓

$$a - 2\sqrt{a}\cdot\sqrt{b} + b \geq 0$$
$$a+b \geq 2\sqrt{a}\cdot\sqrt{b}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$ ✓

[other approaches also valid]

(ii) From (i), $\left(\frac{a+b}{2}\right)^2 \geq ab$

Using this pattern,

$$\left(\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\right)^2 \geq \frac{a+b}{2}\cdot\frac{c+d}{2}$$ ✓

$$\therefore \left(\frac{a+b+c+d}{4}\right)^2 \geq \frac{a+b}{2}\cdot\frac{c+d}{2} \qquad (*)$$

But also, $ab \leq \left(\frac{a+b}{2}\right)^2$ and $cd \leq \left(\frac{c+d}{2}\right)^2$

$$\therefore abcd \leq \left(\frac{a+b}{2}\right)^2\cdot\left(\frac{c+d}{2}\right)^2$$ ✓
$$= \left(\frac{a+b}{2}\cdot\frac{c+d}{2}\right)^2$$
$$\leq \left(\frac{a+b+c+d}{4}\right)^4 \quad \text{, from } (*)$$

$$\therefore (abcd)^{\frac{1}{4}} \leq \frac{a+b+c+d}{4}$$

as required.

$\boxed{2}$ (b) $(a-b)^2 \geq 0$

$\therefore\ a^2 - 2ab + b^2 \geq 0$

$\therefore\ a^2 + b^2 \geq 2ab$

Likewise,

$c^2 + a^2 \geq 2ca$

$b^2 + c^2 \geq 2bc$ ✓

Adding, $2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore\ a^2+b^2+c^2 \geq ab+bc+ca$

Thus $(a+b+c)^2 = a^2+b^2+c^2 + 2(ab+bc+ca)$ ✓

$\geq ab+bc+ca + 2(ab+bc+ca)$ ✓

$= 3(ab+bc+ca)$

(c) (i) Let $f(x) = x - \ln(1+x)$, $x > 0$

$f'(x) = 1 - \frac{1}{1+x}$

$= \frac{x}{1+x}$ ✓

> 0 for $x > 0$.

Also, $f'(x) = 0$ when $x = 0$

Since $f(0) = 0$ and $f(x)$ is an increasing function

for $x > 0$, $\therefore f(x) > 0$ for $x > 0$ ✓

i.e. $x > \ln(1+x)$ for $x > 0$.

$\boxed{2}$ (c) (ii) From (i),

$$1+2+3+\dots+(n-1) > \ln(1+1)+\ln(1+2)+\dots$$
$$\dots+\ln(1+n-1)$$

LHS = an A.P. with $a=1$, $l=n-1$, & $n-1$ terms

$$\therefore \text{LHS} = \frac{(n-1)[1+(n-1)]}{2}$$
$$= \frac{n(n-1)}{2}$$
$$= {}^nC_2$$

$$\text{RHS} = \ln(2\cdot 3\cdot 4 \cdots\cdots n)$$
$$= \ln n!$$

$$\therefore {}^nC_2 > \ln n!$$

$$\therefore e^{{}^nC_2} > n!$$, as e^x is an increasing function.

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$\boxed{3}$ (a)

(i) $(1+x)^m\left(1-\frac{1}{x}\right)^m = \left[(1+x)\left(1-\frac{1}{x}\right)\right]^m$

$= \left[1-\frac{1}{x}+x-1\right]^m$

$= \left[x-\frac{1}{x}\right]^m$

multiplies, then raises to power m. ✓

(ii) Let $m=100$.

LHS of (i) $= (1+x)^{100}\left(1-\frac{1}{x}\right)^{100}$

$= \left[\binom{100}{0}+\binom{100}{1}x+\dots+\binom{100}{r}x^r+\dots+\binom{100}{100}x^{100}\right]$

$\times\left[\binom{100}{0}-\binom{100}{1}\cdot\frac{1}{x}+\dots+(-1)^r\binom{100}{r}\frac{1}{x^r}+\dots+\binom{100}{100}\frac{1}{x^{100}}\right]$

✓ multiplies expansions (*)

So, coeff. of x^0 in LHS is

$\binom{100}{0}\times\binom{100}{0}+\binom{100}{1}\times-\binom{100}{1}+\dots+(-1)^r\binom{100}{r}\times\binom{100}{r}+\dots+\binom{100}{100}\times\binom{100}{100}$

i.e. $\binom{100}{0}^2-\binom{100}{1}^2+\binom{100}{2}^2-\dots+(-1)^r\binom{100}{r}^2+\dots+\binom{100}{100}^2$

✓ identifies x^0 coeffs

RHS of (i) $= \left(x-\frac{1}{x}\right)^{100}$

General term is $\binom{100}{r}x^{100-r}\left(-\frac{1}{x}\right)^r$

$= (-1)^r\binom{100}{r}x^{100-2r}$

$\therefore$ coeff. of x^0 occurs when $100-2r=0$
i.e. $r=50$

✓ Shows how x^0 term is located

$\therefore$ Coeff. is $(-1)^{50}\binom{100}{50} = \binom{100}{50}$.

$\therefore$ LHS = RHS.

(*) ALT: LHS $= \sum_{k=0}^{100}\binom{100}{k}x^k\cdot\sum_{r=0}^{100}\binom{100}{r}(-1)^r x^{-r}$ $\Rightarrow$ const. term when $x^k\cdot x^{-r}=0$ x $k=r$; etc.

$\boxed{3}$ (a) (iii)

LHS in (ii) $= \binom{100}{0}^2 - \binom{100}{1}^2 + \binom{100}{2}^2 - \cdots - \binom{100}{49}^2 + \binom{100}{50}^2 - \binom{100}{51}^2 + \cdots$

$$\cdots + \binom{100}{98}^2 - \binom{100}{99}^2 + \binom{100}{100}^2$$

$$= 2\left[\binom{100}{0}^2 - \binom{100}{1}^2 + \binom{100}{2}^2 - \cdots - \binom{100}{49}^2\right] + \binom{100}{50}^2$$

as $\binom{100}{r} = \binom{100}{100-r}$

✓ explicit use of symmetry relation

$$= 2\left[\binom{100}{0}^2 - \binom{100}{1}^2 + \cdots - \binom{100}{49}^2 + \binom{100}{50}^2\right] - \binom{100}{50}^2$$

✓ incorporates $\binom{100}{50}^2$ inside $\sum$ expression

$$= 2\left[\sum_{K=0}^{50} (-1)^K \binom{100}{K}^2\right] - \binom{100}{50}^2$$

$$\therefore\ 2\left[\sum_{K=0}^{50} (-1)^K \binom{100}{K}^2\right] - \binom{100}{50}^2 = \binom{100}{50}, \text{ from (ii)}.$$

$$\therefore\ \sum_{K=0}^{50} (-1)^K \binom{100}{K}^2 = \frac{1}{2}\left[\binom{100}{50}^2 + \binom{100}{50}\right]$$

$$= \frac{1}{2}\binom{100}{50}\left[\binom{100}{50} + 1\right] \quad \#$$

✓ completes with no error.

$\boxed{3}$ (b) $(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]^n = \left(\sqrt{2}\ \text{cis}\ \pi/4\right)^n$

$$= (\sqrt{2})^n\ \text{cis}\ \frac{n\pi}{4} \quad \text{by de Moivre.}$$

$$= 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + i \cdot 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

✓ uses de Moivre

Also, $(1+i)^n = 1 + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \cdots$

$$= 1 + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i + \binom{n}{4} + \binom{n}{5}i - \cdots$$

[as $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ etc]

$$= 1 - \binom{n}{2} + \binom{n}{4} + \cdots + i\left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right]$$

✓ accurately expands + equates real parts.

So, equating real parts,

$$2^{\frac{n}{2}} \cdot \cos\frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots \quad \#$$

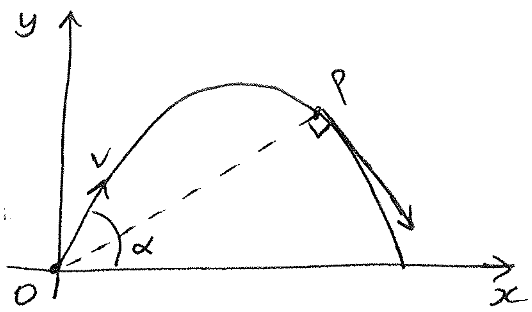
/9

2015 X.2 T.4.

4 If P is on the trajectory, then it has

x-coord. $= Vt\cos\alpha$

y-coord. $= -\frac{1}{2}gt^2 + Vt\sin\alpha$



Thus $$m_{OP} = \frac{-\frac{1}{2}gt^2 + Vt\sin\alpha}{Vt\cos\alpha}$$

At P, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt + V\sin\alpha}{V\cos\alpha}$$

✓ No marks without correct $\frac{dy}{dx}$!

Thus at P, $$\frac{-gt + V\sin\alpha}{V\cos\alpha} \cdot \frac{-\frac{1}{2}gt^2 + Vt\sin\alpha}{Vt\cos\alpha} = -1$$

✓ uses ⊥ condition

$$\therefore (-gt + V\sin\alpha)\cdot\left(-\tfrac{1}{2}gt^2 + Vt\sin\alpha\right) = -V^2t\cos^2\alpha$$

$$\therefore \tfrac{1}{2}g^2t^3 - Vgt^2\sin\alpha - \tfrac{1}{2}Vg\sin\alpha t^2 + V^2t\sin^2\alpha + V^2t\cos^2\alpha = 0$$

$$\tfrac{1}{2}g^2t^3 - \tfrac{3}{2}Vg\sin\alpha\, t^2 + V^2t(\sin^2\alpha + \cos^2\alpha) = 0$$

$$g^2t^2 - 3Vg\sin\alpha\, t + 2V^2 = 0 \quad (\text{as } t \neq 0)$$

✓ reduces to quadratic

For solutions to exist (i.e. for such a point P to exist),

$$\Delta \geq 0$$

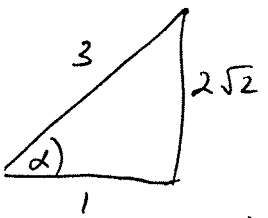
$$\therefore 9V^2g^2\sin^2\alpha - 8V^2g^2 \geq 0$$

$$9\sin^2\alpha - 8 \geq 0$$

$$\sin^2\alpha \geq \frac{8}{9}$$

$$\therefore \sin\alpha \geq \frac{2\sqrt{2}}{3} \quad (\text{as } \sin\alpha \geq 0 \text{ only})$$

✓ correct Δ used correctly



$$\therefore \tan\alpha \geq 2\sqrt{2}.$$ #

✓ final derivation

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4 ALT: One could derive the Cartesian equation,

$$y = -\frac{g}{2v^2}\sec^2\alpha \cdot x^2 + x\tan\alpha$$

then get $\frac{dy}{dx} = -\frac{g\sec^2\alpha\, x}{v^2} + \tan\alpha$

✓ correct derivative (terminating error)

And $m_{op} = \dfrac{-\frac{g\sec^2\alpha\, x^2}{2v^2} + x\tan\alpha}{x}$

$$= \frac{-g\sec^2\alpha\, x}{2v^2} + \tan\alpha$$

Then use the fact that

$\frac{dy}{dx} \cdot m_{op} = -1$ to get

$$\left(-\frac{g\sec^2\alpha}{v^2}x + \tan\alpha\right)\left(\frac{-g\sec^2\alpha\, x}{2v^2} + \tan\alpha\right) = -1$$

✓ uses $\perp$ condition

.....

& thus that

$$g^2\sec^2\alpha\, x^2 - 3gv^2\tan\alpha\, x + 2v^4 = 0$$

✓ derives correct quadratic in usable form

For a solution, $\Delta \geq 0$

$$\therefore 9g^2v^4\tan^2\alpha - 8g\sec^2\alpha\, v^4 \geq 0$$

$$\therefore 9\tan^2\alpha - 8(\tan^2\alpha + 1) \geq 0$$

$$\tan^2\alpha - 8 \geq 0$$

$$\tan\alpha \geq 2\sqrt{2} \quad \text{(as } \alpha \text{ acute).}$$

#

✓ correct Δ, used correctly

✓ final answer

/5