Girraween High School

2017 Year 12 Half Yearly Examination

Mathematics (2 unit)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Reading time – 5 minutes
- Working time – 2 hours
- Write using a black pen.
- Board-approved calculators may be used.
- The BOSTES *Reference Sheet* may be used (see separate laminated sheet).
- All diagrams are **NOT TO SCALE**.
- **Total Marks: 100**

Use your separate ANSWER BOOKLET to complete this examination.

Write your NAME and your Mathematics TEACHER'S NAME on the cover of your ANSWER BOOKLET(S).

- **Section 1 (10 Marks):**
  - Attempt Questions 1 – 10.
  - Select A, B, C or D that best answers the question.
  - Answer on the *Multiple Choice Answer Sheet* inside your ANSWER BOOKLET.
- **Section 2 (90 Marks):**
  - Attempt Questions 11 – 16.
  - Show relevant mathematical reasoning and / or calculations.
  - Write on both sides of each sheet of paper in your separate ANSWER BOOKLET.
  - Start each new question on a NEW PAGE in your ANSWER BOOKLET.
  - If you need more paper, ask for another ANSWER BOOKLET.

SECTION I

10 Marks

- **Attempt all of Questions 1 – 10.**
- **Allow about 15 minutes for this section.**
- **Use the *Multiple Choice Answer Sheet* inside your ANSWER BOOKLET for Questions 1 – 10.**

1. Which of the following is e^{-3} written correct to three significant figures?

(A) 0.049

(B) 0.050

(C) 0.0497

(D) 0.0498

2. For what values of x is the curve $y = 4x^3 - 3x^2$ concave down?

(A) $x > \frac{1}{4}$

(B) $x < \frac{1}{4}$

(C) $x > \frac{3}{4}$

(D) $x < 0$

3. The primitive function of $x^{-2} - 2$ is:

(A) $-\frac{1}{x} - 2x + C$

(B) $\frac{1}{x} - 2x + C$

(C) $-\frac{1}{3x^3} - 2x + C$

(D) $\frac{1}{3x^3} - 2x + C$

4. For what values of m will the geometric series $1 + 2m + 4m^2 + 8m^3 + \dots$ have a limiting sum?

(A) $-1 < m < -1$

(B) $-\frac{1}{2} \leq m \leq \frac{1}{2}$

(C) $-\frac{1}{2} < m < \frac{1}{2}$

(D) $m < \frac{1}{2}$

5. What are the coordinates of the focus for the parabola $(x-2)^2 = -2\,(y-3)$?

(A) $\left(-2,\ 2\frac{1}{2}\right)$

(B) $\left(2,\ 2\frac{1}{2}\right)$

(C) $(2,\ 3)$

(D) $(-2, -3)$

6. A bag contains 4 blue marbles and 6 yellow marbles.

Three marbles are selected at random without replacement.

What is the probability that at least one of the marbles selected is blue?

(A) $\frac{1}{6}$

(B) $\frac{1}{2}$

(C) $\frac{5}{6}$

(D) $\frac{29}{30}$

7. Using the trapezoidal rule with 4 subintervals, which expression gives the approximate area under the curve $y = xe^x$ between $x = 1$ and $x = 3$?

(A) $\frac{1}{4}(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3)$

(B) $\frac{1}{4}(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3)$

(C) $\frac{1}{2}(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3)$

(D) $\frac{1}{2}(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3)$

8. Let $a = e^x$.

Which expression is equal to $\log_e(a^2)$?

(A) e^{x^2}

(B) e^{2x}

(C) x^2

(D) $2x$

9. What is the solution of $5^x = 4$?

(A) $x = \frac{4}{\log_e(5)}$

(B) $x = \frac{\log_e(4)}{5}$

(C) $x = \log_e\left(\frac{4}{5}\right)$

(D) $x = \frac{\log_e(4)}{\log_e(5)}$

10. What is the value of $\int_{-1}^{6} | x - 2 | \, dx$?

(A) $\frac{7}{2}$

(B) $\frac{25}{2}$

(C) $\frac{37}{2}$

(D) $\frac{63}{2}$

The examination continues on the next page.

SECTION II

90 Marks

Attempt all of Questions 11 – 16

Allow about 1 hour and 45 minutes for this section.

Use your separate ANSWER BOOKLET to complete this Section.

- Show relevant mathematical reasoning and / or calculations.
- Write on both sides of each sheet of paper.
- Start each new question on a NEW PAGE.
- If you need more paper, ask for another ANSWER BOOKLET.

Question 11 (16 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

(a) Differentiate:

(i) $y = 3x^5 - 2\sqrt{x}$ **2**

(ii) $y = \sqrt{e^{3x}}$ **2**

(iii) $y = \dfrac{e^x}{e^x + 1}$ **3**

(b) Find the equation of the tangent to the curve $y = 4\,e^{3x+1}$ at the y-intercept. **3**

Give your answer in gradient intercept form.

(c) Express $3\log_e 2 + \log_e 3 + \log_e 5 - \log_e 30$ as a single logarithm. **3**

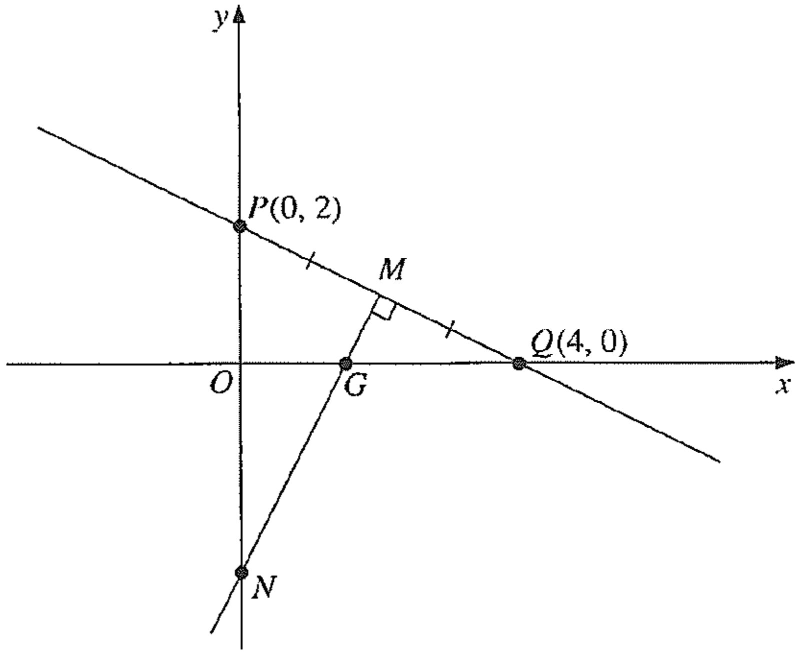
(d) Use Simpson's Rule with five function values to evaluate $\int_1^5 \log_{10} x \, dx$ correct to 4 significant figures. **3**

The examination continues on the next page.

Question 12 (15 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

(a)



The diagram shows the points P (0, 2) and Q (4, 0).

The point M is the midpoint of PQ.

The line MN is perpendicular to PQ and meets the x-axis at G and the y-axis at N.

(i) Show that the gradient of PQ is $-\frac{1}{2}$. **1**

(ii) Find the coordinates of M. **2**

(iii) Show that the equation of line MN is $2x - y - 3 = 0$. **2**

(iv) Show that N has coordinates $(0, -3)$. **1**

(v) Find the distance NQ. **2**

(vi) Find the equation of the circle with centre N and radius NQ. **2**

(vii) Hence show that the circle in part (vi) passes through the point P. **1**

(b) The third term of an arithmetic series is 32 and the sixth term is 17.

(i) Find the common difference. **2**

(ii) Find the sum of the first ten terms. **2**

The examination continues on the next page.

Question 13 (15 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

(a) Find the primitive of $\dfrac{e^{2x} - 1}{e^x}$. **3**

(b) Find the exact area under the curve $y = \dfrac{1}{2}(e^x + e^{-x})$ from $x = -2$ to $x = 2$. **3**

(c) The first term of a geometric series is 16 and the fourth term is $\dfrac{1}{4}$.

(i) Find the common ratio. **2**

(ii) Find the limiting sum of the series. **2**

(d) A layer of plastic cuts out 15% of the light and lets through the remaining 85%.

(i) Show that two layers of the plastic let through 72.25% of the light. **2**

(ii) How many layers of the plastic are required to cut out at least 90% of the light? **3**

The examination continues on the next page.

Question 14 (15 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

(a) Consider the function $y = 4x^3 - x^4$.

(i) Find the stationary points and determine their nature. **4**

(ii) Sketch the graph of the function, clearly showing the stationary points and the x- and y- intercepts **2**

(b) (i) Differentiate $y = x\,e^x - e^x$. **2**

(ii) Hence find the exact value of $\int_0^2 xe^x \, dx$. **2**

Question 14 continues on the next page.

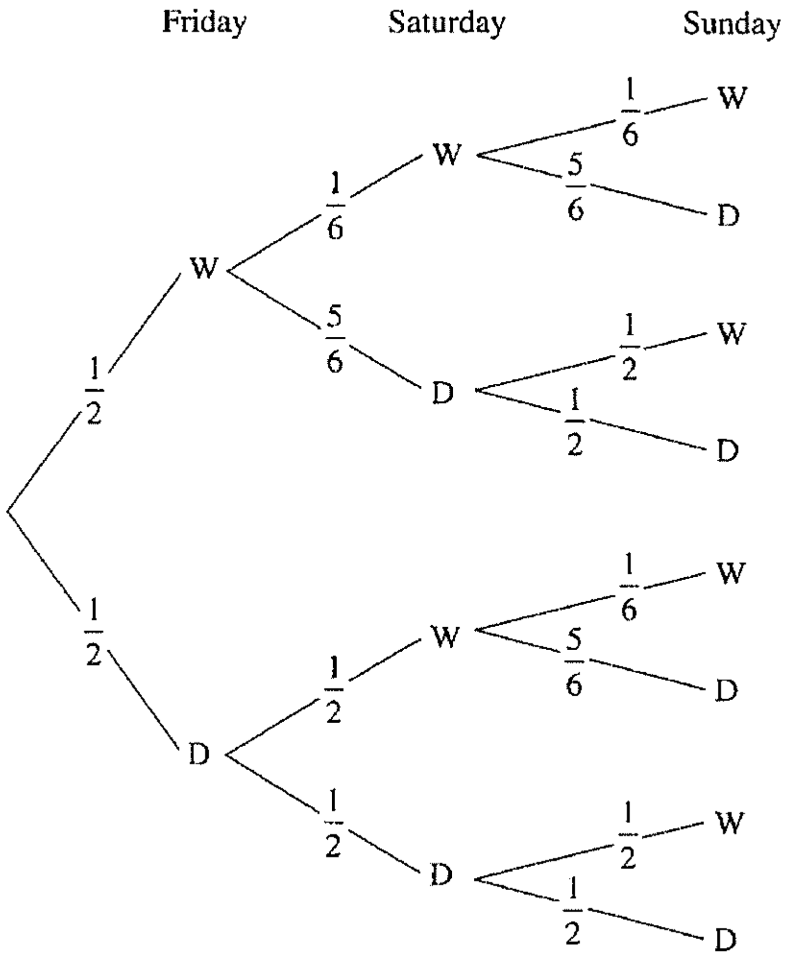
Question 14 (continued)

(c) Weather records for the town of Girraween suggest that:

- if a particular day is wet (W), the probability of the next day being dry is $\frac{5}{6}$
- if a particular day is dry (D), the probability of the next day being dry is $\frac{1}{2}$.

In a specific week Thursday is dry.

The tree diagram shows the possible outcomes for the next three days: Friday, Saturday and Sunday.



(i) Show that the probability of Saturday being dry is $\frac{2}{3}$. **1**

(ii) What is the probability of both Saturday and Sunday being wet? **2**

(iii) What is the probability of at least one of Saturday and Sunday being dry? **2**

End of Question 14.

The examination continues on the next page.

Question 15 (15 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

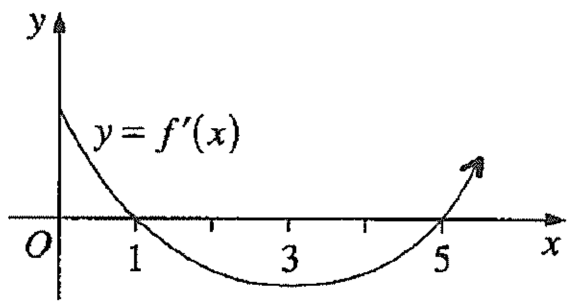
(a) Consider the equation $x^2 + (k+2)x + 4 = 0$.

For what values of k does the equation have:

(i) equal roots? 3

(ii) distinct real roots? 2

(b)

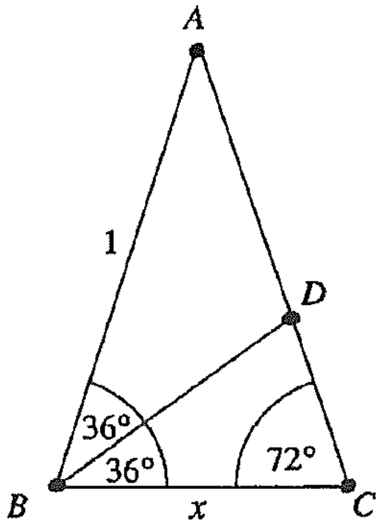


The diagram above shows the graph of the gradient function of the curve $y = f(x)$. 3

For what values of x does $f(x)$ have a local minimum?

Justify your answer.

(c)



NOT TO SCALE

In the diagram, ΔABC is an isosceles triangle where $\angle ABC = \angle BCA = 72°$ and $AB = AC = 1$ unit.

$\angle ABC$ is bisected by BD, and $BC = x$ units.

(i) Copy the diagram into your ANSWER BOOKLET. 1

(ii) Show that triangles ΔABC and ΔBCD are similar. 3

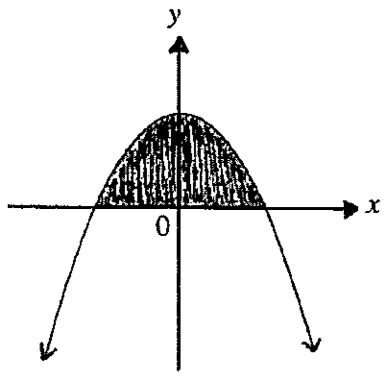
(iii) By using (ii), find the exact value of x. 3

The examination continues on the next page.

Question 16 (14 Marks) **Marks**

Start on a NEW PAGE in your ANSWER BOOKLET.

(a)



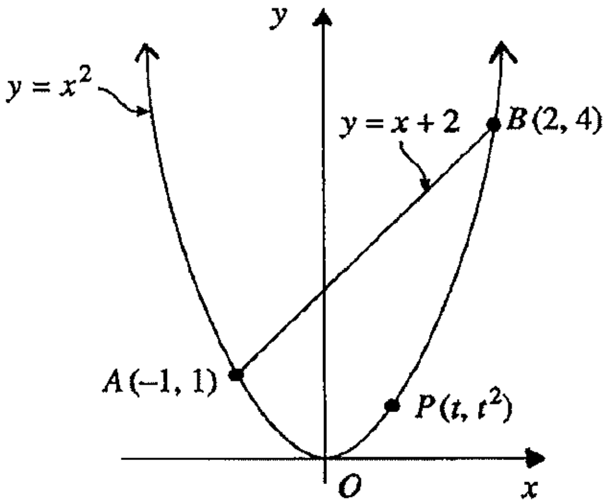
NOT TO SCALE

The shaded region lying between the curve $y = 1 - x^2$ and the x-axis is rotated about the x-axis. **4**

Find the exact volume of the solid formed.

(b) Solve $2\log_e x = \log_e(x + 12)$ **3**

(c)



NOT TO SCALE

In the diagram, $A(-1, 1)$ and $B(2, 4)$ are the points of intersection of the parabola $y = x^2$ with the line $y = x + 2$.

The point $P(t, t^2)$ is a variable point on the parabola below the line.

(i) Find the area of the parabolic segment APB, that is, the area below the line and above the parabola. **3**

(ii) Show that the maximum area of the triangle ΔAPB is three-quarters of the area of the parabolic segment APB. **4**

End of examination.

Multiple choice:

① D ② B ③ A ④ C ⑤ B
⑥ C ⑦ A ⑧ D ⑨ D ⑩ B

Ques 1: $e^{-3} = 0.0497870683\,7$

$= 0.0498$ (3 sig fig's) **(D)**

Ques 2: $y = 4x^3 - 3x^2$

$\frac{dy}{dx} = 12x^2 - 6x$

$\frac{d^2y}{dx^2} = 24x - 6$

for concave down, $\frac{d^2y}{dx^2} < 0$

i.e. $24x - 6 < 0$

$24x < 6$

$x < \frac{1}{4}$ **(B)**

Ques 3: $\int x^{-2} - 2\,dx = \frac{x^{-1}}{-1} - 2x + C$

$= -\frac{1}{x} - 2x + C$ **(A)**

Ques 4: $1 + 2m + 4m^2 + 8m^3 + \ldots$

Geometric series with $a = 1$, $r = 2m$

for limiting sum, $|r| < 1$

i.e. $-1 < 2m < 1$

$-\frac{1}{2} < m < \frac{1}{2}$ **(C)**

Ques 5: $(x-2)^2 = -2(y-3)$

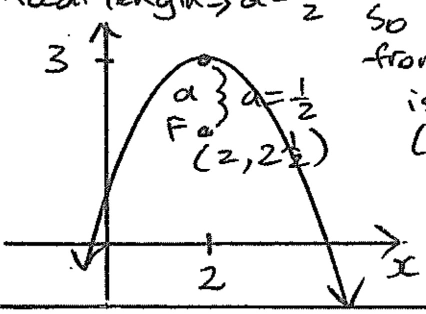
this is in the form $(x-x_1)^2 = -4a(y-y_1)$

which is concave down parabola, vertex (x_1, y_1)

so $4a = 2$ so concave down,

$a = \frac{1}{2}$ vertex $(2, 3)$

So focal length $\Rightarrow a = \frac{1}{2}$



So focus from sketch is $(2, 2\frac{1}{2})$

Ques 6: P(at least 1 blue)

$= 1 - P$(No blue, i.e. all 3 yellow)

$= 1 - \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right)$

$= 1 - \frac{1}{6}$

$= \frac{5}{6}$ **(C)**

Ques 7: $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2} = 0.5$

$y = xe^x$ (4 subintervals = 2 applications)

x	1	1.5	2	2.5	3
$f(x)$	$1e^1$	$1.5e^{1.5}$	$2e^2$	$2.5e^{2.5}$	$3e^3$

Area $\approx \frac{0.5}{2}\left[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right]$

$\approx \frac{1}{4}\left[1e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3\right]$ **(A)**

Ques 8: given $a = e^x$

so $\log_e(a^2) = 2\log_e(a)$

$= 2\log_e e^x$

$= 2x \cdot \log_e e$

$= 2x \times 1$

$= 2x$ **(D)**

Ques 9: $5^x = 4$

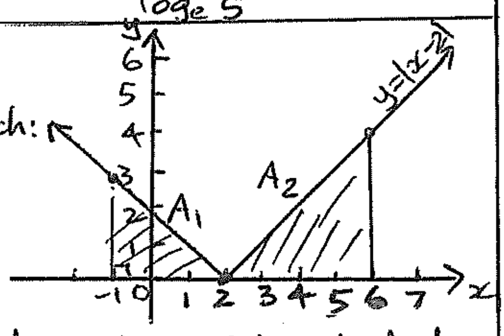
Taking logarithms to base e both sides:

$\log_e 5^x = \log_e 4$

$x\log_e 5 = \log_e 4$

$\therefore x = \frac{\log_e 4}{\log_e 5}$

Ques 10: $y = |x-2|$ is shown in sketch:



$\therefore \int_{-1}^{6} |x-2|\,dx$ = Area of two shaded triangles

$= A_1 + A_2$

$= \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 4 \times 4$

$= \frac{9}{2} + \frac{16}{2}$ **(B)**

$= \frac{25}{2}$

QUESTION 11 (16 Marks)

(a)(i) $y = 3x^5 - 2\sqrt{x}$

ie. $y = 3x^5 - 2x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = 15x^4 - 2.\frac{1}{2}x^{-\frac{1}{2}}$

$= 15x^4 - x^{-1/2}$

$= 15x^4 - \frac{1}{\sqrt{x}}$ (2m)

(ii) $y = \sqrt{e^{3x}}$

$y = (e^{3x})^{\frac{1}{2}}$

ie. $y = e^{\frac{3x}{2}}$

$\therefore \frac{dy}{dx} = \frac{3}{2}e^{\frac{3x}{2}}$

$= \frac{3}{2}\sqrt{e^{3x}}$ (2m)

(iii) $y = \frac{e^x}{e^x + 1}$

Let $u = e^x$ $\quad v = e^x + 1$

$\frac{du}{dx} = e^x$ $\quad \frac{dv}{dx} = e^x$

$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(e^x+1).(e^x) - (e^x).(e^x)}{(e^x+1)^2}$

$= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2}$

$\therefore \frac{dy}{dx} = \frac{e^x}{(e^x+1)^2}$ (3m)

(b) $y = 4e^{3x+1}$

$\frac{dy}{dx} = 4(3)(e^{3x+1})$

$\frac{dy}{dx} = 12(e^{3x+1})$

At y-intercept, $x=0$, $y = 4e^{0+1}$

$y = 4e$

when $x=0$, gradient tangent $m = 12e^{0+1}$

$m = 12e$

$(0, 4e)$, $m = 12e$:

Equation tangent: $y - y_1 = m(x - x_1)$

$y - 4e = 12e(x - 0)$

$\therefore y = 12ex + 4e$ (3m)

Ques11 continued:

(c) $3\log_e 2 + \log_e 3 + \log_e 5 - \log_e 30$

$= \log_e 2^3 + \log_e 3 + \log_e 5 - \log_e 30$

$= \log_e\left(\frac{8 \times 3 \times 5}{30}\right)$

$= \log_e 4$ (3m)

(d) $f(x) = \log_{10} x$

$h = \frac{b-a}{n}$ $\quad$ 5 function values

$= \frac{5-1}{4}$ $\quad$ = 4 subintervals

$\therefore h = 1$ $\quad$ (= 2 applications)

For $f(x) = \log_{10} x$, construct a table of values/weightings:

Weighting	(1)	4	2	4	(1)
x	1	2	3	4	5
$f(x)$	$\log_{10}1$	$\log_{10}2$	$\log_{10}3$	$\log_{10}4$	$\log_{10}5$

The formula for Simpson's Rule using the weighting method is:

$\int_a^b f(x)dx \approx \frac{h}{3}\{f(a) + 4\times(\text{function values}) + 2\times(\text{function values}) + f(b)\}$

Using values from the table:

$\int_1^5 \log_{10} x \, dx$

$\approx \frac{1}{3}\{\log_{10}1 + 4\times(\log_{10}2 + \log_{10}4) + 2\times(\log_{10}3) + \log_{10}5\}$

$= \frac{1}{3} \times 5.265572462$

$= 1.75519082 1\ldots$

$= 1.755$ (4 sig figs) (3m)

QUESTION 12 (15 Marks)

(a) (i) gradient PQ P (0,2) Q (4,0)

$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0-2}{4-0}$

$m_{PQ} = \frac{-2}{4}$

$m_{PQ} = -\frac{1}{2}$ as required (1m)

(ii) Midpoint PQ

$= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{0+4}{2}, \frac{2+0}{2}\right)$

∴ M has coordinates (2, 1) (2m)

(iii) MN ⊥ PQ

So if $m_{PQ} = -\frac{1}{2}$ from (i),

$m_{PQ} = 2$ (since $m_1 \times m_2 = -1$)

So $m_{PQ} = 2$, passing through (2,1):

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x-2)$

$y - 1 = 2x - 4$

∴ $2x - y - 3 = 0$ is equation of MN (2m)

(iv) N is on the y-axis, so when $x=0$: $0 - y - 3 = 0$

$y = -3$

∴ N has coordinates (0, −3) (1m)

(v) $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ N(0,−3) Q(4,0)

$= \sqrt{(4-0)^2 + (0-(-3))^2}$

$= \sqrt{4^2 + 3^2}$

$= \sqrt{25}$

∴ distance NQ = 5 units (2m)

(vi) centre N(0,−3), radius 5 units

$(x-0)^2 + (y-(-3))^2 = 5^2$

∴ $x^2 + (y+3)^2 = 25$

is equation of circle (2m)

Ques 12 continued:

(12)(a)(vii)

Point P(0,2) lies on circle

$x^2 + (y+3)^2 = 25$

METHOD 1

Distance PN

$= 2 + 3$

$= 5$

= radius of circle

∴ circle passes through P (1m)

METHOD 2

P(0,2)

substitute $x=0, y=2$:

$(0)^2 + (2+3)^2 = 25$

$25 = 25$

12)(b)(i) $T_3 = a + 2d = 32$ (1)

$T_6 = a + 5d = 17$ (2)

(2) − (1): $3d = -15$

∴ Common difference, $d = -5$ (2m)

(ii) $S_n = \frac{n}{2}(2a + (n-1)d)$

Need value of a, so from (1),

$a + 2\times(-5) = 32$

$a = 42$

∴ $S_{10} = \frac{10}{2}(2\times 42 + 9\times(-5))$

$= 5 \times (39)$

∴ $S_{10} = 195$ (2m)

QUESTION 13 (15 Marks)

(a) $\int \frac{e^{2x} - 1}{e^x}\,dx$

$= \int \frac{e^{2x}}{e^x} - \frac{1}{e^x}\,dx$

$= \int e^x - e^{-x}\,dx$

$= e^x + e^{-x} + C$ (3m)

Ques 13 continued:

13)(b) $y = \frac{1}{2}(e^x + e^{-x})$

y-intercept: $x = 0$, $y = \frac{1}{2}(e^0 + e^0)$

$= \frac{1}{2} \times 2$

$y = 1$

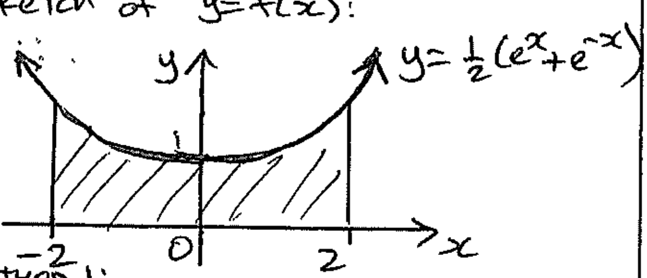
Even function: $f(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)})$

$= \frac{1}{2}(e^{-x} + e^x)$

$= f(x)$

$\therefore$ symmetrical about y-axis

Sketch of $y = f(x)$:



METHOD 1:

As $y = \frac{1}{2}(e^x + e^{-x})$ is an even function,

Area $= 2 \times \int_0^2 \frac{1}{2}(e^x + e^{-x})\,dx$

$= \left[e^x - e^{-x}\right]_0^2$

$= (e^2 - e^{-2}) - (e^0 - e^0)$

$= e^2 - \frac{1}{e^2} - (1 - 1)$

$\therefore$ Area $= e^2 - \frac{1}{e^2}$ square units

METHOD 2:

Area $= \int_{-2}^{2} \frac{1}{2}(e^x + e^{-x})\,dx$

$= \frac{1}{2}\left[e^x - e^{-x}\right]_{-2}^{2}$

$= \frac{1}{2}\left[(e^2 - e^{-2}) - (e^{-2} - e^2)\right]$

$= \frac{1}{2}\left[e^2 - \frac{1}{e^2} - \frac{1}{e^2} + e^2\right]$

$= \frac{1}{2}\left[2e^2 - \frac{2}{e^2}\right]$

$= \frac{1}{2} \times 2\left(e^2 - \frac{1}{e^2}\right)$ (3m)

Area $= e^2 - \frac{1}{e^2}$ square units

Ques 13 continued:

13)(c)(i) $T_1 = a = 16$ (1)

$T_4 = ar^3 = \frac{1}{4}$ (2)

Substitute $a = 16$ into (2),

$16r^3 = \frac{1}{4}$

$r^3 = \frac{1}{64}$

$\therefore r = \frac{1}{4}$ (2m)

(c)(ii) $a = 16$, $r = \frac{1}{4}$

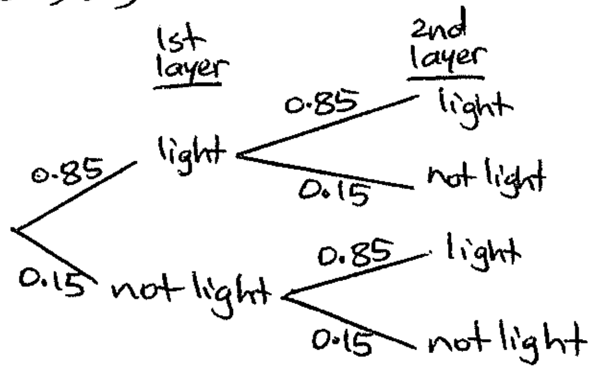
$S_\infty = \frac{a}{1-r}$

$= \frac{16}{1 - \frac{1}{4}}$

$= \frac{16}{\frac{3}{4}}$

$\therefore$ limiting sum $= 21\frac{1}{3}$ (2m)

(d)(i)



Amount of light for 2 layers $= 0.85 \times 0.85$

$= 0.7225$

$= 72.25\%$ (2m)

(ii) Need to find n layers of plastic that allow at least 10% of light:

$0.85^n \leq 0.1$

Taking logarithm to base e of both sides:

$\log_e 0.85^n \leq \log_e 0.1$

$n \log_e 0.85 \leq \log_e 0.1$

Note that $\log_e 0.85$ is negative, so

$n \geq \frac{\log_e 0.1}{\log_e 0.85}$

$n \geq 14.16810399$

$\therefore$ require at least 15 layers (3m)

QUESTION 14 (15 Marks)

(a)(i) $y = 4x^3 - x^4$

$\frac{dy}{dx} = 12x^2 - 4x^3$

$\frac{d^2y}{dx^2} = 24x - 12x^2$

Stationary points when $\frac{dy}{dx} = 0$:

$12x^2 - 4x^3 = 0$

$4x^2(3 - x) = 0$

$\therefore x = 0$ or $x = 3$

When $x=0$: $y = 0$

When $x=3$: $y = 4\times 3^3 - 3^4 = 27$

$\therefore$ two stationary points are $(0,0)$ and $(3,27)$

At $x=0$: $\frac{d^2y}{dx^2} = 0$

This is a possible point of inflexion. Check for change of concavity.

When $x=-1$: $\frac{d^2y}{dx^2} = -24 - 12 = -36$, $\frac{d^2y}{dx^2} < 0$ concave down

When $x=1$: $\frac{d^2y}{dx^2} = 24 - 12 = 12$, $\frac{d^2y}{dx^2} > 0$ concave up

$\therefore$ change of concavity and $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

$\therefore$ At $(0,0)$, horizontal point of inflexion.

At $x=3$: $\frac{d^2y}{dx^2} = 24\times 3 - 12\times 9 = -36$

$\therefore \frac{d^2y}{dx^2} < 0$

$\therefore$ concave DOWN

$\therefore$ At $(3,27)$, MAXIMUM turning point. (4m)

Ques 14(a)(ii) continued:

14)(a)(ii) x-intercepts when $y=0$:

$4x^3 - x^4 = 0$

$x^3(4 - x) = 0$

so x-intercepts at $x=0$, and $x=4$

y-intercepts when $x=0$

$y = 0 - 0$

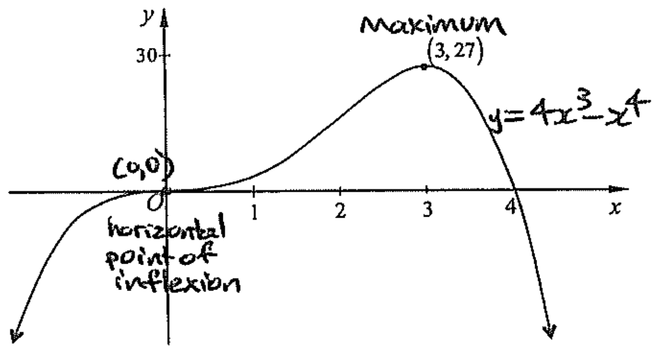
$y = 0$

So y-intercept at $(0,0)$

$(3,27)$ Maximum turning point

$(0,0)$ horizontal point of inflexion.

So sketch of $y = 4x^3 - x^4$ is:



(2m)

14)(b)(i) $y = xe^x - e^x$

so $\frac{dy}{dx} = \frac{d}{dx}(xe^x) - \frac{d}{dx}(e^x)$ ($u = x$, $v = e^x$)

$= [x.e^x + e^x.1] - e^x$

$= xe^x + e^x - e^x$

$\therefore \frac{dy}{dx} = xe^x$ (2m)

(ii) $\therefore \int xe^x\,dx = xe^x - e^x + C$

So $\int_0^2 xe^x\,dx = \left[xe^x - e^x\right]_0^2$

$= (2e^2 - e^2) - (0 - e^0)$

$= e^2 + 1$ (2m)

Ques 14 continued:

14) (c)(i) P(Sat dry)

$= P(WD)$ or $P(DD)$

$= \frac{1}{2} \times \frac{5}{6} + \frac{1}{2} \times \frac{1}{2}$

$= \frac{5}{12} + \frac{1}{4}$

$\therefore$ P(Sat dry) $= \frac{2}{3}$ as required (1m)

(c)(ii) P(Sat + Sun Wet)

$= P(WWW)$ or $P(DWW)$

$= \frac{1}{2} \times \frac{1}{6} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{6}$

$= \frac{1}{72} + \frac{1}{24}$

$= \frac{1}{18}$ (2m)

(c)(iii) P(at least one of Sat or Sun DRY)

$= 1 - P$(both wet)

$= 1 - P$(Sat + Sun wet)

$= 1 - \frac{1}{18}$

$= \frac{17}{18}$ (2m)

QUESTION 15 (14 Marks)

(a) $x^2 + (k+2)x + 4 = 0$

(i) $\Delta = b^2 - 4ac$

$= (k+2)^2 - 4 \times 1 \times 4$

$= k^2 + 4k + 4 - 16$

$\therefore \Delta = k^2 + 4k - 12$

For equal roots, $\Delta = 0$

i.e. $k^2 + 4k - 12 = 0$

$(k+6)(k-2) = 0$ (3m)

$\therefore$ for equal roots, $k = -6$ or $k = 2$

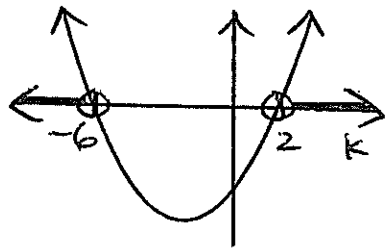
Ques 15 (a) continued:

15)(a)(ii) For distinct real roots,

$\Delta > 0$

i.e. $k^2 + 4k - 12 > 0$

$(k+6)(k-2) > 0$



This solution is

$k < -6$ or $k > 2$

(2m)

15)(b) from the diagram, stationary points are when

$f'(x) = 0$

which occurs at $x = 1$ and $x = 5$.

To find the MINIMUM turning point, use either First or Second derivative test.

First derivative Test (METHOD 1)

At $x = 1$:

from graph:

when $x = 0$, $f'(0) > 0$ when $x = 2$, $f'(2) < 0$

+ / 0 \ −

$\therefore$ At $x = 1$, MAXIMUM turning point.

At $x = 5$:

From graph:

when $x = 4$, $f'(4) < 0$ when $x > 5$, $f'(x) > 0$

− \ 0 / +

$\therefore$ At $x = 5$, MINIMUM turning point (as required) (3m)

Ques 15(b) continued:

Second Derivative Test (METHOD 2)

At $x = 1$: $f'(x)$ is decreasing from graph

i.e. gradient of $f'(x)$ is negative – i.e. $f''(x) < 0$

$\therefore f''(x)$ is concave DOWN

$\therefore$ At $x = 1$, MAXIMUM turning point.

At $x = 5$:

$f'(x)$ is increasing from graph

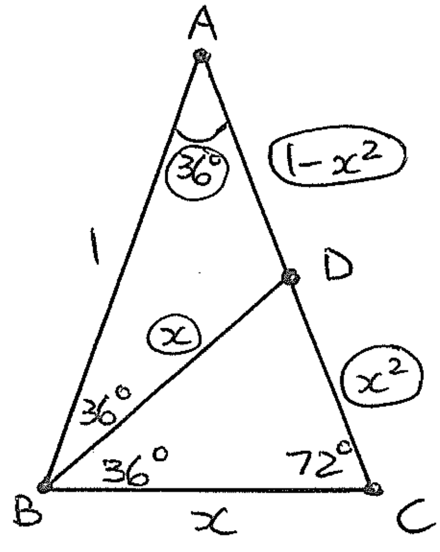
i.e. gradient of $f'(x)$ is positive – i.e. $f''(x) > 0$

$\therefore f''(x)$ is concave UP.

$\therefore$ at $x = 5$, MINIMUM turning point.

15) (c)

(i)



Copy diagram (1m)

Ques 15(c) continued:

15)(c)(ii)

In $\triangle ABC$: $\angle ABC = \angle ACB = 72^\circ$ (given)

In $\triangle BCD$:

$\angle BDC = 180^\circ - (36^\circ + 72^\circ)$
$= 180^\circ - 108^\circ$ (angle sum of triangle)

$\therefore \angle BDC = 72^\circ$

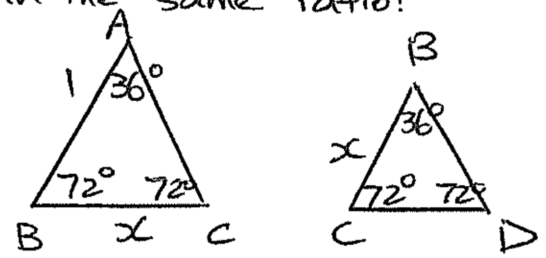
So in $\triangle BCD$: $\angle BDC = \angle BCD = 72^\circ$

$\therefore \triangle ABC \,|||\, \triangle BCD$ (equiangular)

(3m)

(c)(iii)

Since $\triangle ABC$ and $\triangle BCD$ are similar, the corresponding sides are in the same ratio:



So $\dfrac{BC}{AB} = \dfrac{x}{1} = \dfrac{CD}{BC}$

i.e. $\dfrac{CD}{BC} = \dfrac{x}{1}$

but $BC = x$, so $\dfrac{CD}{x} = \dfrac{x}{1}$

$\therefore CD = x^2$

Now since $BC = x$, we have $CD = x^2$

But $\triangle ABC$ is isosceles,

so if $AB = 1$, $AC = 1$ (sides opposite equal angles are equal)

But $AC = AD + CD$

$1 = AD + x^2$

$\therefore AD = 1 - x^2$

In $\triangle ABC$, $\angle BAC = 180^\circ - (72^\circ + 72^\circ)$ (angle sum of triangle)
$= 36^\circ$

$\therefore$ in $\triangle ABD$, $\angle ABD = 36^\circ = \angle BAD$

$\therefore \triangle ABD$ is isosceles

$\therefore AD = BD$ (sides opposite equal angles are equal)

Ques 15(c)(iii) continued:

In $\triangle BCD$:

$$\angle BDC = 180^\circ - (36^\circ + 72^\circ) = 72^\circ$$

$\therefore \triangle BCD$ is isosceles

$\therefore BC = x = BD$ (sides opposite equal angles are equal)

Now since $AD = BD$,

we have $AD = 1 - x^2$

and $BD = x$

$\therefore 1 - x^2 = x$

or $x^2 - x - 1 = 0$

From quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

but x is a length, so $x > 0$

so only solution is

$$x = \frac{1 + \sqrt{5}}{2}$$

(3m)

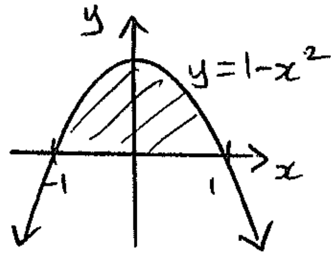
QUESTION 16 (14 Marks)

(a) $y = 1 - x^2$

when $y = 0$, $1 - x^2 = 0$

$(1-x)(1+x) = 0$

$x = 1$ or $x = -1$



$$\therefore \text{Volume} = \pi \int_{-1}^{1} y^2 \, dx$$

$$= \pi \int_{-1}^{1} (1 - x^2)^2 \, dx$$

$$= \pi \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1}$$

$$= \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) \right]$$

$$= \pi \left[\frac{8}{15} - \left(-\frac{8}{15}\right) \right]$$

$$= \frac{16\pi}{15} \text{ cubic units}$$

(4m)

(b) $2\log_e x = \log_e (x + 12)$

From logarithm laws,

$$\log_e x^2 = \log_e (x + 12)$$

Equating logarithms:

$$x^2 = x + 12$$

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

So $x = -3$ or $x = 4$

$x = -3$ is NOT a solution since for logarithms, $x > 0$.

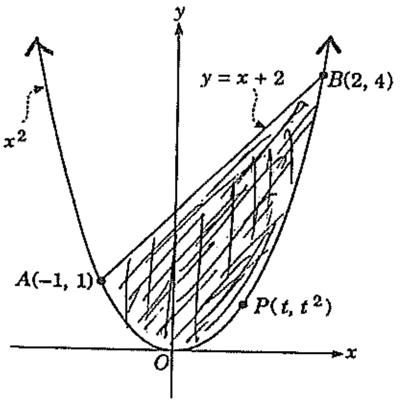
So only solution is $x = 4$

(3m)

Ques 16 continued:

16)(c)

(i)



Area of parabolic segment APB

$= \begin{pmatrix}\text{Area under} \\ \text{line } y=x+2\end{pmatrix} - \begin{pmatrix}\text{Area under} \\ \text{parabola} \\ y=x^2\end{pmatrix}$

$= \int_{-1}^{2} (x+2) - x^2 \, dx$

$= \int_{-1}^{2} x+2-x^2 \, dx$

$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$

$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= \frac{9}{2}$

∴ Area $= 4\frac{1}{2}$ square units (3m)

c)(ii) To find area ΔAPB:

① Need length of AB

② perpendicular distance from $P(t, t^2)$ to line AB ($y = x+2$)

① Length of AB — A(−1, 1), B(2, 4)

$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(2-(-1))^2 + (4-1)^2}$

$= \sqrt{3^2+3^2}$

$= \sqrt{18}$

$= \sqrt{9} \times \sqrt{2}$

∴ length AB $= 3\sqrt{2}$ units

Question 16(c)(ii) continued:

② Perpendicular distance from $P(t, t^2)$ to line AB (i.e. $x - y + 2 = 0$)

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$d = \frac{|1 \times t - 1 \times t^2 + 2|}{\sqrt{1^2 + (-1)^2}}$

$d = \frac{|t - t^2 + 2|}{\sqrt{2}}$

$\therefore d = \frac{t - t^2 + 2}{\sqrt{2}}$

$\therefore$ Area $\Delta APB = \frac{1}{2} \times AB \times d$

$= \frac{1}{2} \times 3\sqrt{2} \times \frac{(t - t^2 + 2)}{\sqrt{2}}$

$\therefore A = \frac{3}{2}(t - t^2 + 2)$

For maximum area, need to solve $\frac{dA}{dt} = 0$.

$\frac{dA}{dt} = \frac{3}{2}(1 - 2t)$

$\frac{dA}{dt} = 0$ when $\frac{3}{2}(1-2t) = 0$

$1 - 2t = 0$

$2t = 1$

$t = \frac{1}{2}$

Note that $\frac{d^2A}{dt^2} = \frac{3}{2} \times (-2)$

$= -3$

i.e. $\frac{d^2A}{dt^2} < 0$ concave DOWN

$\therefore$ Area is a MAXIMUM when $t = \frac{1}{2}$

↳ continued...

Ques 16 (c) (ii) continued:

when $t = \frac{1}{2}$:

$$A = \frac{3}{2}(t - t^2 + 2)$$
$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4} + 2\right)$$
$$= \frac{3}{2} \times \frac{9}{4}$$
$$= \frac{27}{8}$$

$\therefore$ Maximum Area of $\triangle APB$
$= 3\frac{3}{8}$ square units

But from question, required to find
$\frac{3}{4} \times$ (Area parabolic segment APB)

$= \frac{3}{4} \times 4\frac{1}{2}$ (from (i))

$= \frac{27}{8}$

$= 3\frac{3}{8}$ square units.

$\therefore$ Maximum area of $\triangle APB$
$= \frac{3}{4} \times$ (area of parabolic segment APB)

As required (4m)