GOSFORD HIGH SCHOOL

2008
YEAR 12 HALF YEARLY HIGHER SCHOOL CERTIFICATE

MATHEMATICS

General Instructions:

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Total marks: - 100

- Attempt Questions 1 - 8
- Questions are worth 12 or 13 marks each

Question 1 (13 Marks) **Marks**

a) Simplify $\dfrac{6a^2 - 9a}{3 - 2a}$ **2**

b) Find the value of $\dfrac{mn}{m - n}$ when $m = 6.92 \times 10^5$ and $n = 7.9 \times 10^4$ **2**

correct to 3 significant figures.

c) Solve $|3y - 2| < 7$ **2**

d) Solve $(x + 3)(x - 1) = 32$ **2**

e) If $2\cos^2\theta = 1$, find θ for $-180° \leq \theta \leq 180°$ **2**

f) An arc AB of a sector of a circle is of length $\dfrac{\pi}{4}$ metres, and subtends an angle of $30°$ at the centre O of the circle.

(i) Show that the exact length of the radius of the circle is 1.5 metres **1**

(ii) Find the area of the sector AOB **2**

Question 2 (12 Marks) Begin a New Booklet

a) Solve $2x^2 - x - 3 \geq 0$ **2**

b) For the quadratic equation $2x^2 - 9x + k = 0$, find:

(i) $\alpha + \beta$ **1**

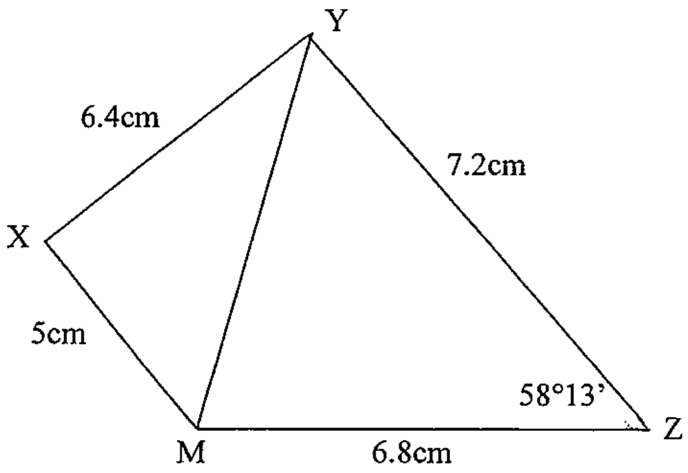
(ii) $\alpha\beta$

(iii) The value of k if the roots differ by 1 **1**

Question 2 continues on the next page . . . .

Question 2 continued **Marks**

c)



NOT TO SCALE

XYZM is a quadrilateral in which

XY=6.4cm, YZ=7.2cm, ZM=6.8cm

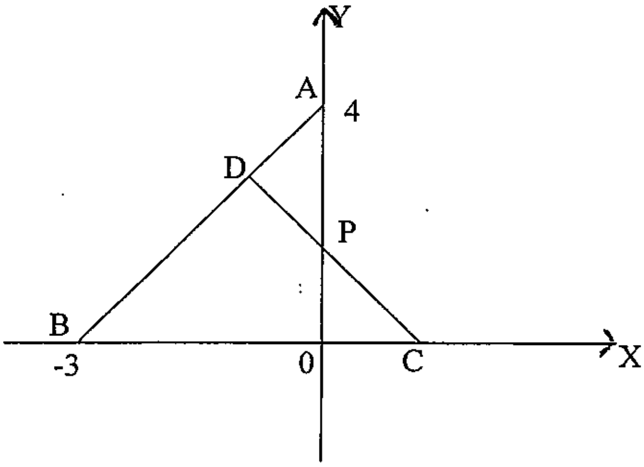
and XM=5cm. $\angle YZM = 58°13'$

i) Show that YM is 6.8 cm correct to 1 decimal place. **2**

ii) Show that $\angle MXY = 72°3'$ correct to the nearest minute. **2**

iii) Find the area of the quadrilateral XYZM correct to the nearest cm^2 **2**

Question 3 (12 Marks) Begin a new booklet.



NOT TO SCALE

In the diagram, AB=BC

and CD is perpendicular to AB

CD intersects the Y axis at P

a) Find the length of AB **1**

b) Show that the coordinates of C are (2 , 0) **1**

c) Show that the equation of CD is $3x + 4y = 6$ **2**

d) Find the coordinates of P **1**

e) Find the length of CP **2**

f) Prove that ΔADP is congruent to ΔCOP **3**

g) Hence, calculate the area of quadrilateral DPOB **2**

Question4 (13 Marks) Begin a new booklet

a) Differentiate with respect to x:

(i) $\dfrac{1}{x\sqrt{x}}$ **2**

(ii) $(4 - 3x)^6$ **2**

(iii) $\dfrac{\log x^2}{x}$ **2**

b) Evaluate $\lim\limits_{x \to -1} \dfrac{x^3 + 1}{x + 1}$ **2**

c) Write down the quadratic equation whose roots are $1 + \sqrt{3}$ and $1 - \sqrt{3}$ **2**

d) (i) Find the coordinates of the centre of the circle

$$x^2 + y^2 + 4x - 2y = 11$$ **2**

(ii) State the domain of this circle **1**

Question 5 (13 Marks) Begin a new booklet

a) Find $\displaystyle\int \frac{4x}{x^2 + 1}\, dx$ **1**

b) Evaluate $\displaystyle\int_0^1 e^{-3x}\, dx$ correct to 3 decimal places **2**

c) The equation of a parabola is given by $12y = x^2 - 6x - 15$

(i) Write this equation in standard form **1**

(ii) Sketch the parabola **1**

(iii) Find the coordinates of the vertex **1**

(iv) Find the coordinates of the focus **1**

d) For a particular curve, $\dfrac{d^2y}{dx^2} = 2$ for all points on the curve.

Find y in terms of x if $\dfrac{dy}{dx} = 10$ and $y = 3$ when $x = 0$ **3**

e) A and B are the points (1, 0) and (9, 0) respectively. P (x, y) is the point such that

$PA^2 + PO^2 = PB^2$, where O is the origin. Show that the locus of P is

$$x^2 + y^2 + 16x - 80 = 0$$ **3**

Question 6 (12 Marks) Start a new booklet

a) For the curve $y = x^3 - 3x^2 - 9x + 10$

(i) Find the stationary points **2**

(ii) Determine their nature **2**

(iii) Find the point of inflexion **2**

(iv) Sketch the curve in the domain $-2 \leq x \leq 4$ **3**

b) Find the equation of the normal to the curve $y = x^3 - 3x^2 - 9x + 10$

at the point where $x = 0$. **3**

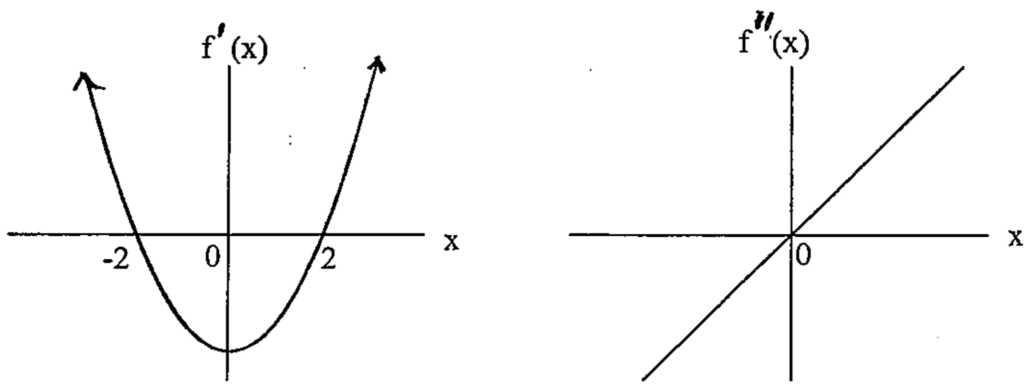
Question 7 (13 Marks) Begin a new booklet

a) (i) Differentiate $x.e^{2x}$ **2**

(ii) Hence, evaluate $\int_0^2 2x.e^{2x}\, dx$ as an exact value **3**

b) The area under the curve $y = e^x - 1$ from $x = 0$ to $x = 2$, is rotated about the X-axis. Find the exact volume generated. **3**

c)



These graphs show the first and second derivatives of a curve $y = f(x)$.
For which values of x is the function:

(i) increasing **1**

(ii) concave down? **1**

d) Use Simpson's Rule with 5 function values to estimate the value of

$$\int_1^5 \frac{x^2}{x^2 + 1}\, dx$$ correct to 2 decimal places **3**

Question 8 (12 Marks) Begin a new booklet

a) If $\log_e 2 = x$ and $\log_e 3 = y$, express in terms of x and y:

(i) $\log_e 0.25$ **1**

(ii) $\log_e\left(\frac{8}{9}\right)$ **1**

b) Solve for x: $2x\log_a 4 = \log_a 8$ (a is a positive constant) **2**

c) Find the area between the curve $y = \log_e x$, the Y axis and the lines

$y = 1$, and $y = 3$. (Answer to 3 significant figures) **2**

d) A cylinder of radius r and height h is **open at one end.**

(i) If the **volume** of the cylinder is to be $1000\ cm^2$, find an expression for h in terms of r. **1**

(ii) **Show** that the **surface area** of this open cylinder is given by

$$A = \pi r^2 + \frac{2000}{r}$$ **2**

(iii) Find the radius of the cylinder with least surface area. **3**

Give an exact answer, then an approximation to 2 significant figures

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Year 12 ½ Yrly 2008

Question 1 (13 marks).

a) $\frac{3a(2a-3)}{(3-2a)} = -3a$ 2

b) $\frac{(6\cdot92\times10^5)\times(7\cdot9\times10^4)}{6\cdot92\times10^5-7\cdot9\times10^4} = 89181\cdot077$

$= 89{,}200$ (3 s.f.) 2

c) $3y-2<7$ OR $3y-2>-7$

$3y<9$ $\quad 3y>-5$

$y<3$ $\quad y>-\frac{5}{3}$

$-\frac{5}{3}<y<3$ 2

d) $x^2+2x-3=32$

$x^2+2x-35=0$

$(x+7)(x-5)=0$

$\therefore x=-7$ OR $x=5$ 2

e) $2\cos^2\theta=1$

$\cos^2\theta=\frac{1}{2}$

$\cos\theta=\pm\frac{1}{\sqrt{2}}$

$\therefore \theta=-135°, -45°, 45°, 135°$ 2

f) i) $l=r\theta$ $\quad (30°=\frac{\pi}{6})$

$\frac{\pi}{4}=r\times\frac{\pi}{6}$

$r=1\cdot5$

$\therefore$ radius is 1·5 m. 1

ii) $A=\frac{1}{2}r^2\theta$

$A=\frac{1}{2}\times\left(\frac{3}{2}\right)^2\times\frac{\pi}{6}$

$A=\frac{3\pi}{16}$ 2

$\therefore$ Area is $\frac{3\pi}{16}$ m^2

Question 2 (12 marks)

a) For $(2x-3)(x+1)=0$

$x=\frac{3}{2}$ OR $x=-1$

Test $x=0$ in $2x^2-x-3\geq0$ 2

$-3\geq0$ F.

$\therefore x\leq-1$ and $x\geq\frac{3}{2}$

Q2 CONT'D

b) $2x^2-9x+k=0$

i) $\alpha+\beta=-\frac{b}{a}$

$=\frac{9}{2}$ 1

ii) $\alpha\beta=\frac{c}{a}$

$=\frac{k}{2}$ 1

iii) Roots are α and $(\alpha+1)$

$\alpha+(\alpha+1)=\frac{9}{2}$

$2\alpha=\frac{7}{2}$

$\alpha=\frac{7}{4}$

Roots are $\frac{7}{4}$ and $\frac{11}{4}$

Now $\frac{7}{4}\times\frac{11}{4}=\frac{k}{2}$

$k=\frac{77}{8}$ 2

c) i) $YM^2=6\cdot8^2+7\cdot2^2-2\times6\cdot8\times7\cdot2\times\cos58°13'$

$YM=6\cdot8$ cm (1 dp) as required 2

ii) $\cos\angle MXY=\frac{5^2+6\cdot4^2-6\cdot8^2}{2\times5\times6\cdot4}$

$\angle MXY=72°3'$ (as required) 2

iii) Area $\triangle XYM$: $A=\frac{1}{2}\times5\times6\cdot4\times\sin72°3'$

Area $\triangle YZM$: $A=\frac{1}{2}\times7\cdot2\times6\cdot8\times\sin58°13'$

$\therefore$ Area quad XYZM = 36 cm^2 (nearest cm^2) 2

Question 3 (12 marks)

a) $AB^2=3^2+4^2$

$AB=5$ units 1

b) $AB=BC$ (given)

$AB=5$ units (shown in (a))

$\therefore BC=5$ units

Hence $C=(2,0)$ 1

c) Grad $AB=\frac{4}{3}$

Grad $CD=-\frac{3}{4}$

Eqn of CD: $y-y_1=m(x-x_1)$

$y-0=-\frac{3}{4}(x-2)$

$4y=-3x+6$

$\therefore 3x+4y=6$ as req'd. 2

d) When $x=0$, $0+4y=6$

$y=1\frac{1}{2}$

Q3 (CONT'D)

e) $CP^2 = 2^2 + \left(\frac{3}{2}\right)^2$

$= 4 + \frac{9}{4}$

$CP = \sqrt{\frac{25}{4}}$

$\therefore CP = 2\frac{1}{2}$ units — 2

f) Aim: To prove $\triangle ADP \equiv \triangle COP$

Proof: In $\triangle$'s ADP, COP

$\angle ADP = \angle COP$ (90°)

$\angle DPA = \angle OPC$ (vertically opp.)

Now $PA = (4 - 1\frac{1}{2})$ units

$= 2\frac{1}{2}$ units

and $CP = 2\frac{1}{2}$ units (shown in (e))

$\therefore PA = PC$

Hence $\triangle ADP \equiv \triangle COP$ (AAS) — 3

g) Area $\triangle ABO = 6$ units^2

Area $\triangle COP = \frac{1}{2} \times \frac{3}{2} \times 2$ units^2

$= 1\frac{1}{2}$ units^2

Now $\triangle COP \equiv \triangle ADP$

$\therefore$ Area $\triangle ADF = 1\frac{1}{2}$ units

Hence quad. DPOB $= 4\frac{1}{2}$ units^2 — 2

Question 4 (13 marks)

a) i) $\frac{d}{dx}\left(x^{-\frac{3}{2}}\right) = -\frac{3}{2}x^{-\frac{5}{2}}$

$= \frac{-3}{2\sqrt{x^5}}$ OR $-\frac{3}{2x^2\sqrt{x}}$ — 2

ii) $\frac{d}{dx}(4-3x)^6 = 6(4-3x)^5 \times -3$

$= -18(4-3x)^5$ — 2

iii) $\frac{d}{dx}\left(\frac{\log x^2}{x}\right) = \frac{d}{dx}\left[\frac{2\log x}{x}\right]$

$= \frac{x \cdot \frac{2}{x} - 2\log x \cdot 1}{x^2}$ — 2

$= \frac{2 - 2\log x}{x^2}$

b) $\lim_{x \to -1} \frac{x^3+1}{x+1} = \lim_{x \to -1} \frac{(x+1)(x^2-x+1)}{(x+1)}$

$= (-1)^2 + 1 + 1$

$= 3$ — 2

Q4 CONT'D

c) $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 - (1+\sqrt{3} + 1-\sqrt{3})x + (1+\sqrt{3})(1-\sqrt{3}) = 0$

$x^2 - 2x + 1 - 3 = 0$

ie. $x^2 - 2x - 2 = 0$ — 2

d) i) $x^2 + y^2 + 4x - 2y = 11$

$(x^2+4x) + (y^2-2y) = 11$

$(x^2+4x+4) + (y^2-2y+1) = 11+4+1$

$(x+2)^2 + (y-1)^2 = 16$

$\therefore$ Centre $= (-2, 1)$ — 2

ii) Radius $= 4$

$\therefore$ Domain is $\{x: -6 \le x \le 2\}$ — 1

Question 5 (13 marks)

a) $\int \frac{4x}{x^2+1}\,dx = 2\int \frac{2x}{x^2+1}\,dx$

$= 2\log(x^2+1) + C$ — 1

b) $\int_0^1 e^{-3x}\,dx = \left[-\frac{1}{3}e^{-3x}\right]_0^1$

$= -\frac{1}{3}(e^{-3} - 1)$

$= 0.317$ (3 dp) — 2

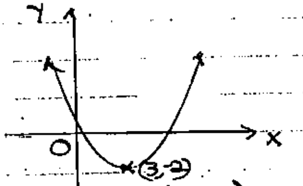
c) $12y = x^2 - 6x - 15$

i) $x^2 - 6x = 12y + 15$

$x^2 - 6x + 9 = 12y + 24$

$(x-3)^2 = 12(y+2)$ — 1

ii)



— 1

iii) Vertex $= (3, -2)$ — 1

iv) Focus $= (3, 1)$ — 1

d) $\frac{d^2y}{dx^2} = 2$

$\frac{dy}{dx} = 2x + C$

when $x=0$, $\frac{dy}{dx} = 10$

$\therefore C = 10$

Hence $\frac{dy}{dx} = 2x + 10$

Now $y = x^2 + 10x + C$

when $x=0$, $y = 3$

$3 = C$

$\therefore y = x^2 + 10x + 3$ — 3

Q5 CONT'D

e) Dist $PA = \sqrt{(x-1)^2 + (y-0)^2}$

" $PO = \sqrt{(x-0)^2 + (y-0)^2}$

" $PB = \sqrt{(x-9)^2 + (y-0)^2}$

Now $PA^2 + PO^2 = PB^2$

$\therefore (x-1)^2 + y^2 + x^2 + y^2 = (x-9)^2 + y^2$

$x^2 - 2x + 1 + y^2 + x^2 + y^2 = x^2 - 18x + 81 + y^2$

$x^2 + 16x + y^2 - 80 = 0$

$\therefore$ The locus of P is $x^2 + y^2 + 16x - 80 = 0$ 3

Question 6 (12 marks)

a) $y = x^3 - 3x^2 - 9x + 10$.

i) $\frac{dy}{dx} = 3x^2 - 6x - 9$

$\frac{d^2y}{dx^2} = 6x - 6$

s.p. occur when $\frac{dy}{dx} = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$\therefore x = 3$ OR $x = -1$

when $x = 3$, $y = -17$

when $x = -1$, $y = 15$.

$\therefore$ Stationary points are $(3,-17)$ + $(-1,15)$ 2

ii) when $x = 3$, $\frac{d^2y}{dx^2} > 0$

when $x = -1$, $\frac{d^2y}{dx^2} < 0$

$\therefore (3,-17)$ is a minimum t.p

and $(-1,15)$ is a maximum t.p 2

iii) Possible point of inflexion occurs when $\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$x = 1$

when $x = 1$, $y = -1$

For $x < 1$, $\frac{d^2y}{dx^2} < 0$

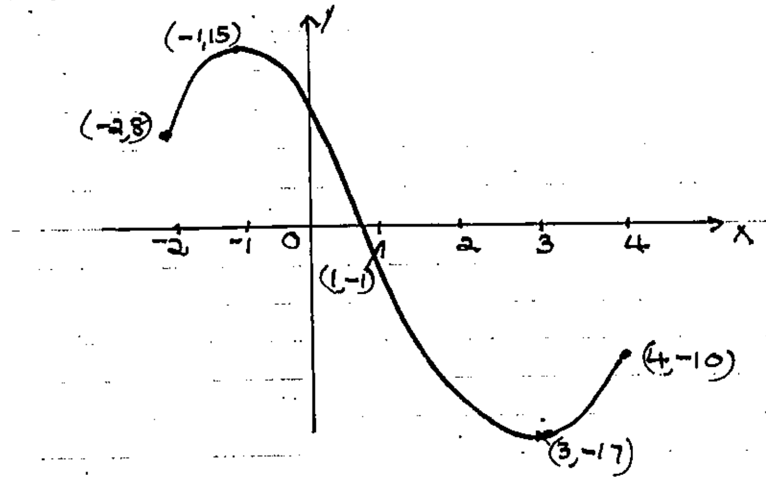
For $x > 1$ $\frac{d^2y}{dx^2} > 0$

A change in concavity occurs

$\therefore (1,-1)$ is an inflexion point 2.

iv) When $x = -2$, $y = 8$

Q6 CONT'D



3

b) $y = x^3 - 3x^2 - 9x + 10$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

when $x = 0$, $\frac{dy}{dx} = -9$.

$m_{tangent} = -9$

$m_{normal} = \frac{1}{9}$

when $x = 0$, $y = 10$.

Eqn of normal:

$y - y_1 = m(x - x_1)$

$y - 10 = \frac{1}{9}(x - 0)$

$9y - 90 = x$

$x - 9y + 90 = 0$ 3

Question 7 (13 marks)

a) i) $\frac{d(x.e^{2x})}{dx} = 1.e^{2x} + x.2e^{2x}$

$= e^{2x}(1 + 2x)$

OR $e^{2x} + 2xe^{2x}$ 2

ii) $\int_0^2 2x\,e^{2x}\,dx = \left[x.e^{2x} - \frac{1}{2}e^{2x}\right]_0^2$

$= \left(2e^4 - \frac{1}{2}e^4\right) - \left(0 - \frac{1}{2}e^0\right)$

$= 2e^4 - \frac{1}{2}e^4 + \frac{1}{2}$

$= \frac{3e^4 + 1}{2}$ 3

Q7 CONT'D

b) $y = e^x - 1$

$y^2 = (e^x - 1)^2$

$y^2 = e^{2x} - 2e^x + 1$

$\therefore V = \pi \int_0^2 (e^{2x} - 2e^x + 1)\,dx$

$V = \pi \left[\frac{1}{2}e^{2x} - 2e^x + x\right]_0^2$

$V = \pi \left[\left(\frac{1}{2}e^4 - 2e^2 + 2\right) - \left(\frac{1}{2} - 2 + 0\right)\right]$

$V = \pi \left[\frac{1}{2}e^4 - 2e^2 + 2 - \frac{1}{2} + 2\right]$

$V = \pi \left(\frac{1}{2}e^4 - 2e^2 + 3\frac{1}{2}\right)$

OR $V = \frac{\pi}{2}(e^4 - 4e^2 + 7)$ units^3 3

c) i) Increasing for $x < -2$, $x > 2$ 1

ii) Concave down when $x < 0$ 1

d)

x	1	2	3	4	5
$f(x)$	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{9}{10}$	$\frac{16}{17}$	$\frac{25}{26}$

$A \doteqdot \frac{1}{3}\left\{\frac{1}{2} + \frac{25}{26} + 2 \times \frac{9}{10} + 4\left(\frac{4}{5} + \frac{16}{17}\right)\right\}$

$A = 3.41$ (2 d.p.) 3

Question 8 (12 marks)

a) $\log_e 2 = x$ $\log_e 3 = y$

i) $\log_e 0.25 = \log_e 2^{-2}$

$= -2\log_e 2$

$= -2x$ 1

ii) $\log_e \frac{8}{9} = \log_e 8 - \log_e 9$

$= 3\log_e 2 - 2\log_e 3$

$= 3x - 2y$ 1

b) $2x \log_a 4 = \log_a 8$

$2x \cdot 2\log_a 2 = 3\log_a 2$

$4x \log_a 2 = 3\log_a 2$

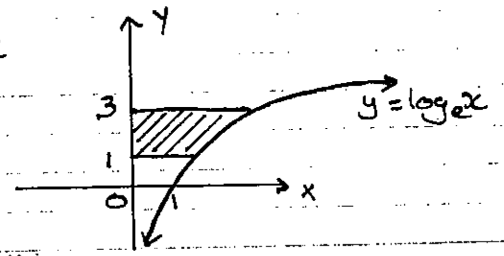
$4x = 3$

$x = \frac{3}{4}$ 2

Q8 CONT'D

c) $y = \log_e x$

$x = e^y$



$A = \int_1^3 e^y\,dy$

$A = [e^y]_1^3$

$A = e^3 - e$

$A = 17.4$ (3 s.f.) 2

d) i) $V = \pi r^2 h$

$1000 = \pi r^2 h$

$\therefore h = \frac{1000}{\pi r^2}$ 1

ii) $SA = \pi r^2 + 2\pi r h$

$SA = \pi r^2 + 2\pi r \times \frac{1000}{\pi r^2}$

ie $A = \pi r^2 + \frac{2000}{r}$ as req'd. 2

iii) $A = \pi r^2 + 2000 r^{-1}$

$\frac{dA}{dr} = 2\pi r - 2000 r^{-2}$

$\frac{d^2A}{dr^2} = 2\pi + 4000 r^{-3}$

A max. or min. occurs when $\frac{dA}{dr} = 0$

$2\pi r - \frac{2000}{r^2} = 0$

$2\pi r^3 - 2000 = 0$

$\pi r^3 = 1000$

$r^3 = \frac{1000}{\pi}$

$r = \frac{10}{\sqrt[3]{\pi}}$

when $r = \frac{10}{\sqrt[3]{\pi}}$, $\frac{d^2A}{dr^2} > 0$

Hence a minimum (least) surface area occurs when $r = \frac{10}{\ldots}$

$\therefore$ Radius is 6.8 cm (2 s.f.) 3