GOSFORD HIGH SCHOOL

MATHEMATICS

Higher School Certificate

2009

Half Yearly Examination

Time Allowed: 2 Hours + 5 minutes reading time

Instructions:

- Start each **SECTION** in a new answer booklet, with each question on a new page.
- Write on one side of the page only.
- Write your name and number on each booklet.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used in all sections.
- A table of standard integrals is provided.

SECTION 1 (Start a new booklet)

Question 1: (8 Marks)

Marks

a. Evaluate, correct to 3 significant figures:

i. $4e^{-1.5}$ 1

ii. $\log_e 30$ 1

b. Find a primitive of $5e^{4x}$ 1

c. Factorise fully: 1

$xe^{2x} - 5xe^{x}$

d. If $y = \ln(3x + 2)$, find $\frac{dy}{dx}$ 1

e. If $x^{4.7} = 4$, evaluate $\log_x 16$ 1

f. Sketch, on a quarter page diagram, the graph of $y = 3^{-x}$ 2
(Show essential features with **TWO** points marked on the curve.)

SECTION 1 *(Continued)*

Question 2: (8 Marks) (Start a new page)

Marks

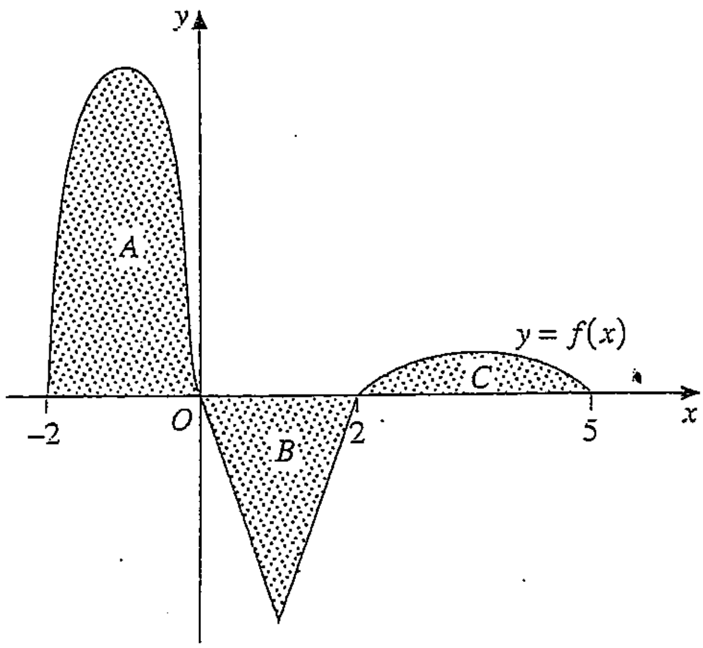
a. Find:

i. $\int x^5 - 2x^3 - 4\ dx$ 1

ii. $\int \frac{4}{x^3}\ dx$ 1

iii. $\int_0^1 (3x+2)^4 dx$ 2

b.



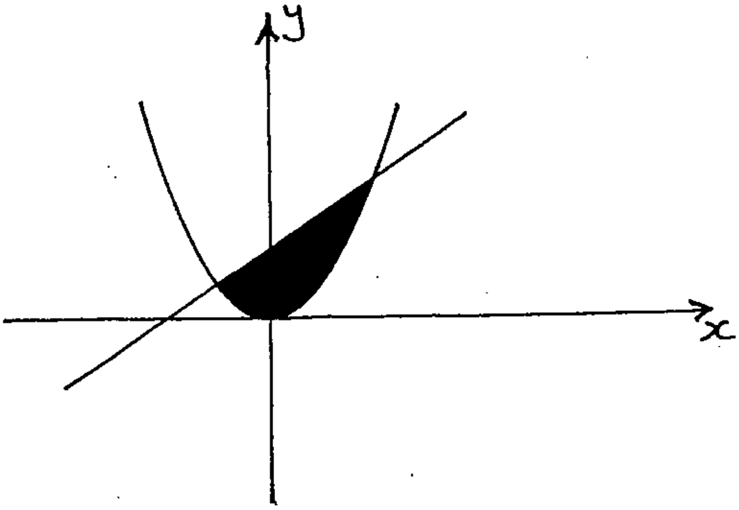
The graph of the function f is shown in the diagram. The shaded areas are bounded by $y = f(x)$ and the x axis. Shaded area A is 8 square units, shaded area B is 5 square units and shaded area C is 2 square units.

Evaluate $\int_{-2}^{5} f(x)dx$ 1

SECTION 1 *(Continued)*

Question 2: *(Continued)*

c. The graphs $y = x^2$ and $y = x + 2$ are shown below:



i. Find the co-ordinates of the points of intersection of the two graphs. 1

ii. Hence find the area of the shaded region. 2

SECTION 2 (Start a new booklet)

Question 3: (8 Marks)

Marks

a. If $y = e^{\frac{x}{3}}$ and $\frac{dy}{dx} = ky,$ 1

find k.

b. Differentiate with respect to x:

i. xe^x 1

ii. $\frac{x}{\ln x}$ 2

iii. $(\log_e x)^3$ 1

c. Find:

i. $\int \frac{2x}{x^2+1}\,dx$ 1

ii. $\int_1^e \frac{1}{2x}\,dx$, answer in simplest exact form 2

SECTION 2 *(Continued)*

Question 4: (8 Marks) (Start a new page)

Marks

a. Express $105°$ in radians, giving your answer in simplest exact form. 1

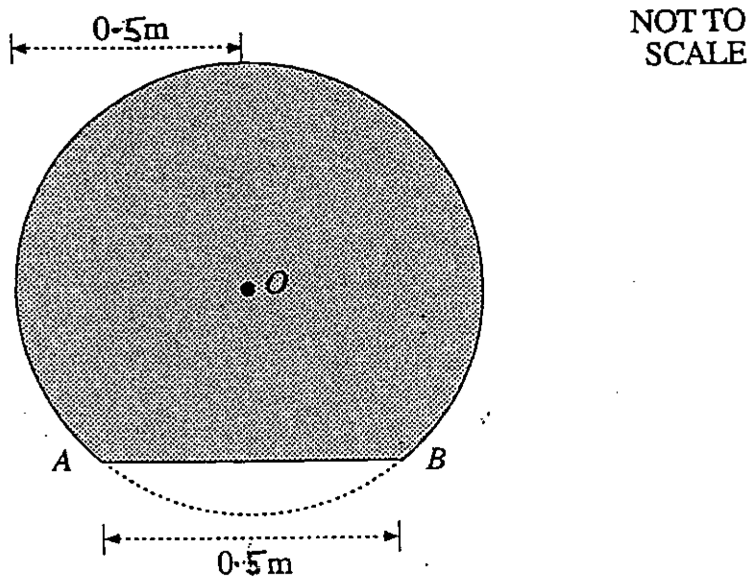
b. Change 1.25 radians into degrees and minutes, to the nearest minute. 1

c. A sector is formed from a circle of radius 5 cm by subtending an angle of $\frac{\pi}{6}$ radians at the centre. Find, in exact form, the:

i. Arc length of the sector 1

ii. Area of the sector 1

d.



A table-top is in the shape of a circle with a small segment removed as shown. The circle has centre 0 and radius 0.5 metres. The length of the straight edge AB is also 0.5 metres.

i. Explain why $\angle AOB = \frac{\pi}{3}$ radians. 1

ii. Find the area of the table top, giving your answer to 3 decimal places 3

SECTION 3 (Start a new booklet)

Question 5: (8 Marks)

Marks

a. If $f(x) = x^4 - 2x^3 + 12$, find the value of:

i. $f(-1)$ 1

ii. $f^1(-1)$ 1

iii. $f^{11}(-1)$ 1

b. The point (3,-11) is a turning point on the curve $y = ax^2 + bx + 7$. 3
Find the values of a and b.

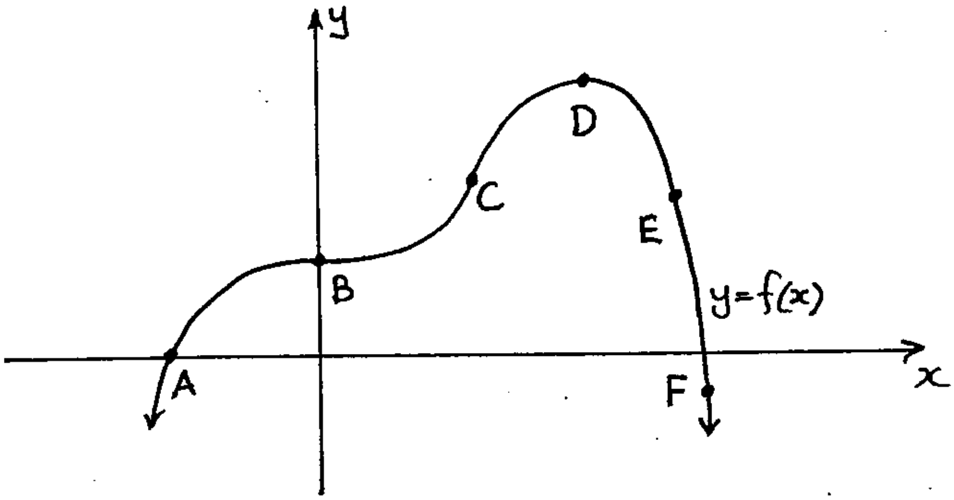
c. Show that $y = \dfrac{1}{(2x-1)^3}$ is monotonic decreasing for all x, except $x = \dfrac{1}{2}$ 2

SECTION 3 *(Continued)*

Question 6: (8 Marks) (Start a new page)

Marks

a. The diagram below shows the graph of $y = f(x)$, with 6 points on it A, B, C, D, E and F marked.



Write down the letter(s) which correspond to the point(s) where:

i. $f(x) = 0$ 1

ii. $f^{1}(x) = 0$ 1

iii. $f^{11}(x) = 0$ 1

iv. $f^{1}(x) < 0$ 1

b. Copy the graph in part (a) into your answer booklet and draw the graph of $y = f^{1}(x)$ on the same axes. (Clearly label your graphs.) 2

c. The gradient function of a curve is given by $\frac{dy}{dx} = 5x - 6$. 2

If the curve passes through (1,-3) find the equation of the curve.

SECTION 4 (Start a new booklet)

Question 7: (8 Marks)

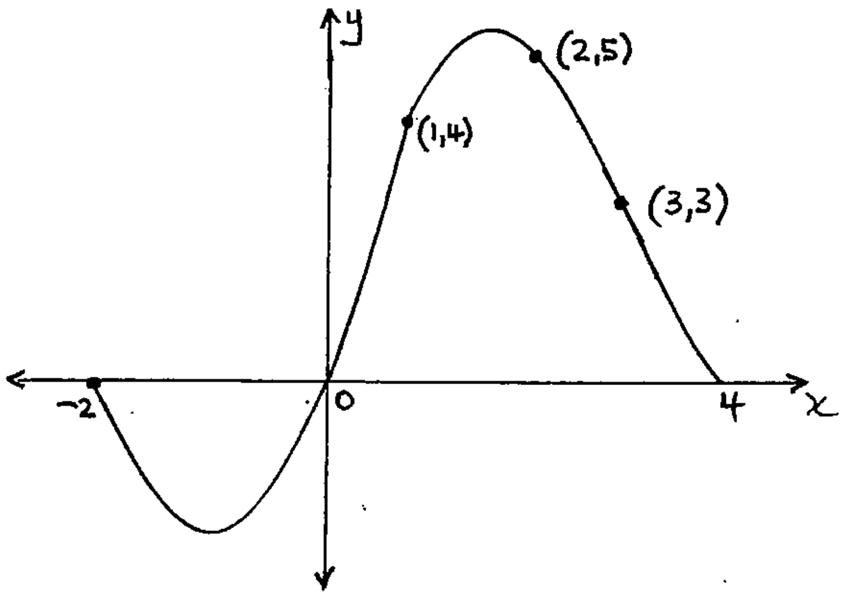
Marks

a. Find the exact volume generated when the curve $y = \sqrt{x+3}$, between $x = -2$ and $x = 2$ is rotated about the x axis. 2

b. Find the area enclosed between the curve $y = x^3$, the y axis and the lines $y = 1$ and $y = 8$. 2

c. The diagram below is the graph of $y = f(x)$ for $-2 \le x \le 4$

(Not to scale)



i. Write down an expression for the exact area bounded by the curve $y = f(x)$ and the x axis. (You are **not** required to find the equation of the curve.) 1

ii. Use Simpson's Rule with 5 function values to approximate the area enclosed by the curve, the x axis and the lines $x = 0$ and $x = 4$ 3

SECTION 4 *(Continued)*

Question 8: (8 Marks) (Start a new page)

Marks

a. i. Complete the table of values for $y = 2^x$ 1

x	-1	0	1	2
y				

ii. Using the Trapezoidal rule and the table in Part (i), approximate the area enclosed by the curve $y = 2^x$, the x axis and the lines $x = -1$ and $x = 2$ 2

b. Show that a primitive of $x\sqrt{x}$ is $\dfrac{2x^2\sqrt{x}}{5}$ 2

c. Find the area between the curve $y = x^2 - 4$, the x axis and the ordinates $x = 1$ and $x = 3$ 3

SECTION 5 (Start a new booklet)

Question 9: (8 Marks)

Marks

a. Prove that the equation of the normal to the curve $y = \ln\left(\frac{1}{2x}\right)$ at the point where $x = e$ is given by: 3

$$y = ex - e^2 - \ln 2 - 1$$

b. Given $\log_8 81 - \log_2(\sqrt[3]{3}) = \log_x y$ 2

find the values of x and y.

c. Find the stationary points on the curve $y = x^3 e^x$ 3

and determine their nature.

SECTION 5 *(Continued)*

Question 10: (8 Marks) (Start a new page)

Marks

a. A farmer is building a wheat silo in the shape of a closed cylinder of radius r metres and height h metres. The silo is to be made from galvanised iron sheeting and is to have a capacity of $300m^3$.

Using the formulae $V = \pi r^2 h$ and
$S = 2\pi r^2 + 2\pi rh$

i. Find an expression for h in terms of r. 1

ii. show that the surface area S, is given by $S = \dfrac{2\pi r^3 + 600}{r}$ 1

iii. Hence, find the value of r, in exact form, that gives a minimum area of galvanised iron sheeting to be used. 2

b. For a continuous curve $y = f(x)$ it is known that:

- The curve has only one x intercept
- $\dfrac{d^2y}{dx^2} = 36x^2 - 96x + 48$
- (0,0) and (2,16) are stationary points

i. Find all x values that have points of inflexion. 3

ii. Sketch the curve $y = f(x)$, showing all intercepts and stationary points. 1

END OF TEST

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12 MATHEMATICS 2009 ½ YEARLY SOLNS

Q1 a.(i) $4e^{-1.5}$

$= 0.89252\ldots$ (calc)

$= 0.893$ (3.s.f)

(ii) $\log_e 30$

$= 3.401197\ldots$ (calc)

$= 3.40$ (3s.f)

b. Primitive $= \frac{5}{4}e^{4x}$

c. $xe^{2x} - 5xe^{x}$

$= xe^{x}(e^{x} - 5)$

d. $y = \ln(3x+2)$

$\frac{dy}{dx} = \frac{3}{3x+2}$

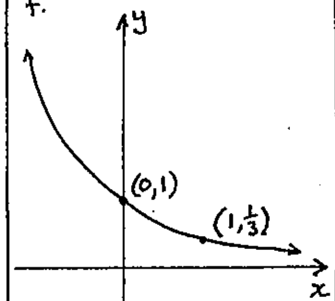
e. $x^{4.7} = 4$

$\therefore \log_x 4 = 4.7$

$\therefore \log_x 16 = 2\log_x 4$

$= 9.4$

f.



Q2 a(i) $\int x^5 - 2x^3 - 4\,dx$

$= \frac{x^6}{6} - \frac{x^4}{2} - 4x + c$

(ii) $\int 4x^{-3}\,dx$

$= -\frac{4}{2}x^{-2} + c$

$= -\frac{2}{x^2} + c$

(iii) $\int_0^1 (3x+2)^4\,dx$

$= \frac{1}{5}\left[\frac{(3x+2)^5}{3}\right]_0^1$

$= \frac{1}{15}(5^5 - 2^5)$

$= \frac{3093}{15} = \frac{1031}{5}$

b. $\int_{-2}^{5} f(x)\,dx$

$= 8 + (-5) + 2$

$= 5$

c.(i) $y = x^2$ —(1)

$y = x + 2$ —(2)

Solving simultaneously:

Sub (1) in (2): $x^2 = x + 2$

$\therefore x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2, -1$

$y = 4, 1$

$\therefore$ Pts of intersection are $(2,4)$ and $(-1,1)$

(ii) $A = \int_{-1}^{2} x + 2 - x^2\,dx$

$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= 4\frac{1}{2}$ units²

Q3 a. $y = e^{x/3}$

$\frac{dy}{dx} = \frac{1}{3}e^{x/3}$

$= \frac{1}{3}y$

$\therefore k = \frac{1}{3}$

b(i) let $y = xe^{x}$

$y' = e^{x}.1 + x.e^{x}$

$= e^{x}(1+x)$

(ii) $y = \frac{x}{\ln x}$

$y' = \frac{\ln x.1 - x.\frac{1}{x}}{(\ln x)^2}$

$= \frac{\ln x - 1}{(\ln x)^2}$

(iii) $y = (\log_e x)^3$

$y' = 3(\ln x)^2.\frac{1}{x}$

$= \frac{3(\ln x)^2}{x}$

c. (i) $\int \frac{2x}{x^2+1}\,dx$

$= \ln(x^2+1) + c$

(ii) $\int_1^e \frac{1}{2x}\,dx$

$= \frac{1}{2}\left[\ln x\right]_1^e$

$= \frac{1}{2}\left[\ln e - \ln 1\right]$

$= \frac{1}{2}(1 - 0)$

$= \frac{1}{2}$

Q4 a. $105° = \frac{105\pi}{180}$

$= \frac{7\pi}{12}$ radians

b. $1.25 \times \frac{180}{\pi}$

$= 71.6197\ldots°$ (calc)

$= 71°37'$

c. (i) $l = r\theta$

$= 5 \times \frac{\pi}{6}$

$= \frac{5\pi}{6}$ cm

(ii) $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 25 \times \frac{\pi}{6}$

$= \frac{25\pi}{12}$ cm²

d(i) ΔAOB is equilateral

$\therefore \angle AOB = 60°$

$= \frac{\pi}{3}$ radians as required

(ii) Area of table top

= Area of circle − Area of minor segment

$= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$= 0.25\pi - 0.125\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= \frac{\pi}{4} - \frac{1}{8}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{4} - \frac{\pi}{24} + \frac{\sqrt{3}}{16}$

$= 0.7627516\ldots$ (calc)

$= 0.763$ m² (3d.p)

Q5 a. $f(x) = x^4 - 2x^3 + 12$

(i) $f(-1) = (-1)^4 - 2(-1)^3 + 12$

$= 1 + 2 + 12$

$= 15$

(ii) $f'(x) = 4x^3 - 6x^2$

$f'(-1) = 4(-1)^3 - 6(-1)^2$

$= -4 - 6$

$= -10$

(iii) $f''(x) = 12x^2 - 12x$

$f''(-1) = 12(-1)^2 - 12(-1)$

$= 12 + 12$

$= 24$

b. $y = ax^2 + bx + 7$

T.P's occur when $y' = 0$

$y' = 2ax + b$

$\therefore 2ax + b = 0$

Since $(3, -11)$ is a T.P

$6a + b = 0$ —(1)

Also $(3, -11)$ is a point on the curve

$\therefore -11 = 9a + 3b + 7$

$\therefore 9a + 3b = -18$ —(2)

Solving simultaneously

(1)×3 : $18a + 3b = 0$ —(3)

(2)−(3) $-9a = -18$

$\therefore a = 2$

$b = -12$

c. If $y = \frac{1}{(2x-1)^3}$

$x \neq \frac{1}{2}$

$y = (2x-1)^{-3}$

$y' = -3(2x-1)^{-4} \times 2$

$= \frac{-6}{(2x-1)^4}$

Since $(2x-1)^4 > 0$ for all x except $x = \frac{1}{2}$

$\therefore y' < 0$ for all x except $x = \frac{1}{2}$

$\therefore$ always monotonic decreasing.

Q6 a.

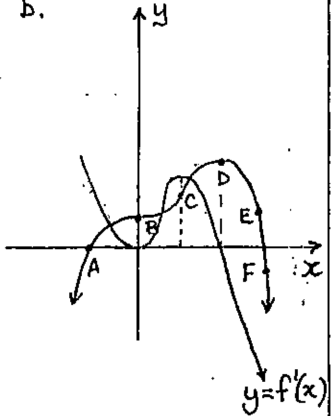
(i) A

(ii) B and D

(iii) C and B

(iv) E and F

b.



c. $\frac{dy}{dx} = 5x - 6$

$y = \frac{5x^2}{2} - 6x + c$

when $x = 1$, $y = -3$

$-3 = \frac{5}{2} - 6 + c$

$\frac{1}{2} = c$

$\therefore$ equation of curve is $y = \frac{5x^2}{2} - 6x + \frac{1}{2}$

Q7 a. $V = \pi \int y^2 \, dx$

$V = \pi \int_{-2}^{2} x + 3 \, dx$

$= \pi \left[\frac{x^2}{2} + 3x \right]_{-2}^{2}$

$= \pi \left[(2+6) - (2-6) \right]$

$= \pi (8 + 4)$

$= 12\pi$ units^3

b. $A = \int_1^8 y^{1/3} \, dy$

$= \frac{3}{4} \left[y^{4/3} \right]_1^8$

$= \frac{3}{4}(16 - 1)$

$= \frac{45}{4}$ units^2

c. (i) $A = \left| \int_{-2}^{0} f(x)\,dx \right| + \int_0^4 f(x)\,dx$.

(ii)

x	0	1	2	3	4
y	0	4	5	3	0
	y_0	y_1	y_2	y_3	y_4

$A \doteq \frac{h}{3} \left[y_0 + y_4 + 4(y_1 + y_3) + 2y_2 \right]$

$= \frac{1}{3} \left[0 + 0 + 4(4+3) + 10 \right]$

$= \frac{38}{3}$ units^2

Q8 a.

(i)

x	-1	0	1	2
y	$\frac{1}{2}$	1	2	4
	y_0	y_1	y_2	y_3

(ii) $A \doteq \frac{h}{2} \left[y_0 + y_3 + 2(y_1 + y_2) \right]$

$= \frac{1}{2} \left[\frac{1}{2} + 4 + 2(1+2) \right]$

$= \frac{21}{4}$ units^2

b. $\int x\sqrt{x} \, dx$

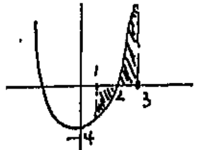
$= \int x^{3/2} \, dx$

$= \frac{x^{5/2}}{5/2} + c$

$= \frac{2}{5}\sqrt{x^5}$

$= \frac{2x^2\sqrt{x}}{5}$ as req.

c.



$A = \left| \int_1^2 x^2 - 4 \, dx \right| + \int_2^3 x^2 - 4 \, dx$

$= \left| \left[\frac{x^3}{3} - 4x \right]_1^2 \right| + \left[\frac{x^3}{3} - 4x \right]_2^3$

$= \left| \left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right) \right| + (9 - 12) - \left(\frac{8}{3} - 8\right)$

$= \left| -\frac{5}{3} \right| + \frac{7}{3}$

$= 4$ units^2

Q9 a. $y = \ln\left(\frac{1}{2x}\right)$

$x = e$
$y = -\ln 2e$
$= -\ln 2 - 1$

$= \ln(2x)^{-1}$

$= -\ln 2x$

$\therefore y' = -\frac{2}{2x}$

$= -\frac{1}{x}$

when $x = e$

$y' = -\frac{1}{e} = m_{Tangent}$

$\therefore m_{normal} = e$

equation of normal

$\Rightarrow y - (-\ln 2 - 1) = e(x - e)$

$y + \ln 2 + 1 = ex - e^2$

$y = ex - e^2 - \ln 2 - 1$ as required

b. $\log_8 81 - \log_2(\sqrt[3]{3})$

$= \frac{\log_2 81}{\log_2 8} - \log_2 3^{1/3}$

$= \frac{\log_2 3^4}{\log_2 2^3} - \frac{1}{3}\log_2 3$

$= \frac{4\log_2 3}{3\log_2 2} - \frac{1}{3}\log_2 3$

$= \frac{4\log_2 3}{3} - \frac{1}{3}\log_2 3$

$= \log_2 3 \quad \therefore x = 2, y = 3$

c. $y = x^3 e^x$

Stat. pts. occur when $y' = 0$

$y' = e^x \cdot 3x^2 + x^3 \cdot e^x$

$= x^2 e^x (3 + x)$

$y' = 0$ when $x = 0, -3$

$\therefore$ Stat. pts occur at $(0,0)$ and $\left(-3, -\frac{27}{e^3}\right)$

to determine nature

at $(0,0)$:

x	0^-	0	0^+
y'	+	0	+

$\therefore$ Horizontal Inflexion at $(0,0)$ since $y'' = 0$ at $x = 0$.

at $\left(-3, -\frac{27}{e^3}\right)$:

x	-3^-	-3	-3^+
y'	\	—	/

$\therefore$ Min T.P at $\left(-3, -\frac{27}{e^3}\right)$

Q10 a.

(i) $V = \pi r^2 h$

$300 = \pi r^2 h$

$\frac{300}{\pi r^2} = h$

(ii) $S = 2\pi r^2 + 2\pi r \left(\frac{300}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{600}{r}$

$= \frac{2\pi r^3 + 600}{r}$ as required

(iii) Min value when $S' = 0$

$S' = 4\pi r - \frac{600}{r^2}$

$= \frac{4\pi r^3 - 600}{r^2}$

$S' = 0$ when $4\pi r^3 - 600 = 0$

$4\pi r^3 = 600$

$r^3 = \frac{150}{\pi}$

$r = \sqrt[3]{\frac{150}{\pi}}$

$S'' = 4\pi + \frac{1200}{r^3}$

when $r = \sqrt[3]{\frac{150}{\pi}}$ $S'' > 0$

$\therefore$ a minimum area when $r = \sqrt[3]{\frac{150}{\pi}}$

b. Points of inflexion occur when $y'' = 0$

$36x^2 - 96x + 48 = 0$

$3x^2 - 8x + 4 = 0$

$(3x - 2)(x - 2) = 0$

$\therefore x = \frac{2}{3}, 2$.

Test for concavity change

$x = \frac{2}{3}$ ✓

x	$\frac{2}{3}^-$	$\frac{2}{3}$	$\frac{2}{3}^+$
y''	+	0	−

$x = 2$ ✓ (H.I)

x	2^-	2	2^+
y''	−	0	+

$\therefore \left(\frac{2}{3}, 0\right)$ is a pt. inflex. $(2, 16)$ is H.I.

(ii)