YEAR 12 Mathematics
HSC Course
Assessment Task 2
March 2010

1. There are 3 questions.
2. Marks allocated to each question are indicated in brackets
3. Answer each question on your own paper showing all necessary working
4. Start each question on a new page
5. Calculators may be used
6. Time allowed - **90 minutes**

Topic	Mark
1. Question 1 (Geometrical Applications of Calculus)	/25
2. Question 2 (Integration)	/35
3. Question 3 (Trigonometric Functions)	/34

TOTAL **/94**

Question 1 (25 marks)

a) Consider the curve $y = \frac{4x^3 - x^4}{9}$

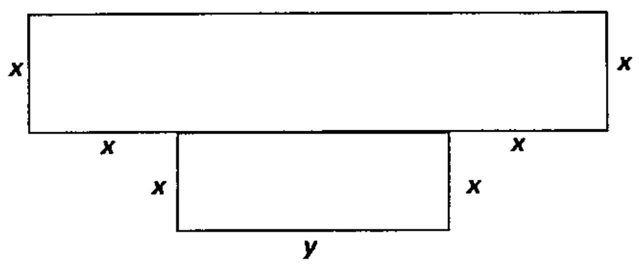
I. Find any stationary points, and determine their nature (4)

II. Find any points of inflexion (2)

III. For what values of x is the curve concave up? (2)

IV. Neatly sketch the curve in the domain $-3 \le x \le 4$ showing all important features (3)

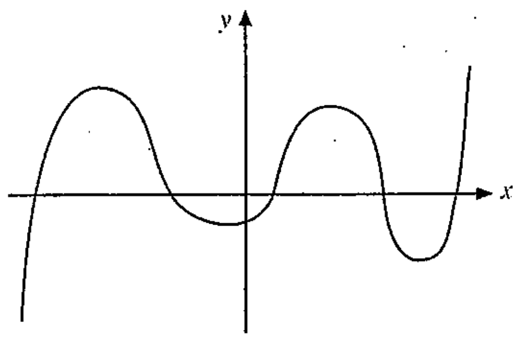
b) The enclosure below. In which all angles are right angles, is to be fenced with 120m of fencing. (all drawn lines are fences and measurements are all in metres)



I. If the total area enclosed is $A\ m^2$, show that $A = 2x^2 + 2xy$, and express this as a function of x alone. (3)

II. Find for what value of $\boldsymbol{x}$ the enclosure has a maximum area (Justify your answer using calculus) (3)

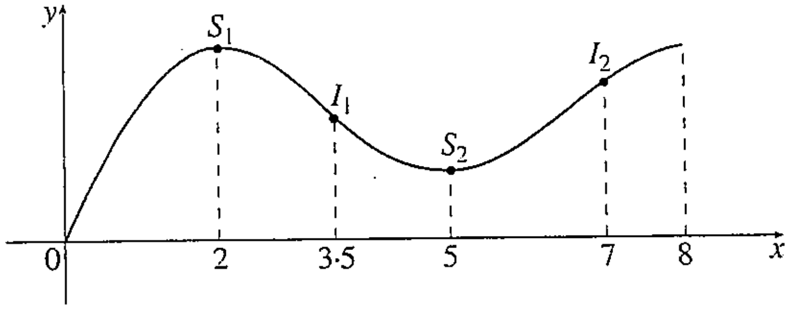
d)



The diagram above shows the graph of a function, $f(x)$.

I. Copy the diagram and on the same diagram sketch and clearly label $y = f'(x)$ (3)

e)



The diagram above shows the graph of function g(x) over the domain $0 \leq x \leq 8$. S_1 and S_2 are its two stationary points.
I_1 and I_2 are its two points of inflexion.

State all the intervals of *x* for which

I. $g'(x) > 0$ (3)

II. $g''(x) > 0$ (2)

Question 2 (35 marks)

a. Find

i. $\int (1-4x)\,dx$ (1)

ii. $\int (1-4x)^5\,dx$ (2)

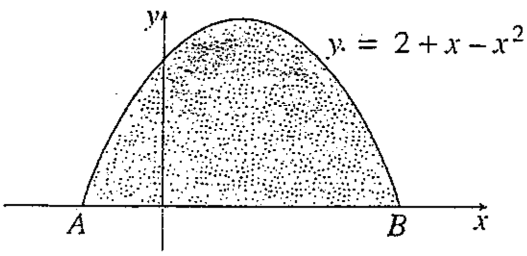
iii. $\int \frac{x^5+x^2+1}{x^2}\,dx$ (2)

iv. $\int (1-x^2)^2\,dx$ (2)

b. Use the table of standard integrals to evaluate

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$$ (2)

c.



i. Find the coordinates of A and B (2)

ii. Find the area bounded by the curve and the x axis (3)

d. i. Copy this table and complete the missing values in decimal form.

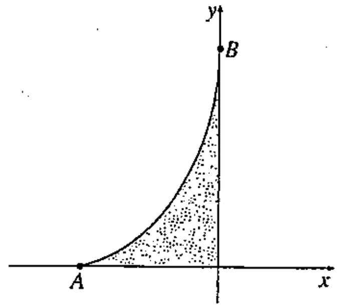
x	1	2	3	4	5
$\frac{3}{x(x+1)}$					

(2)

ii. Using all the above information, and Simpson's Rule, find an approximation to 2 decimal places for

$$\int_1^5 \frac{3}{x(x+1)}\,dx$$ (2)

e. The shaded region in the diagram below is bounded by the coordinate axes and part of the curve $y = (x+2)^3$.



i. What are the coordinates of A and B? (2)

ii. Find the volume of the solid formed when the region makes a revolution about the **x** axis. (3)

iii. If the region is revolved about the **y** axis, show that the volume, V, of the resulting solid is given by (2)

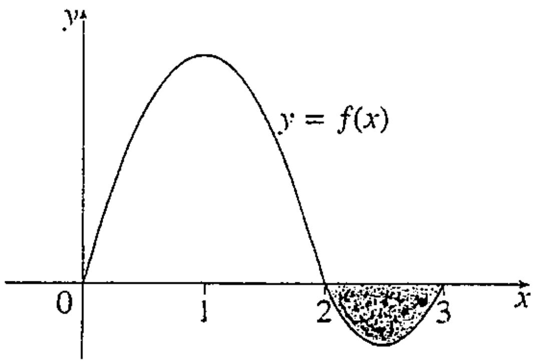
$$V = \pi\int_0^8 (y^{\frac{1}{3}} - 2)^2 dy$$

f. Consider the curves $y = (x-1)^2$ and $y = 10x - x^2 - 9$

i. Sketch the curves carefully on the same diagram, showing their point(s) of intersection (4)

ii. Find the area bounded by these two curves (4)

g.



Given that $\int_0^2 f(x)\,dx = 11$ and that $\int_0^3 f(x)\,dx = 7$

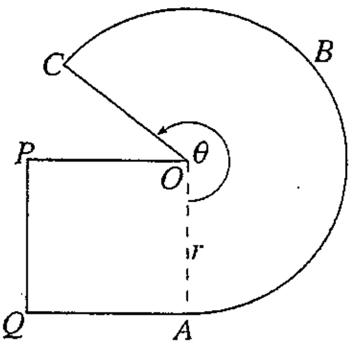
i. What is the area of the shaded region? (1)

ii. What is the value of $\int_2^3 f(x)\,dx$ (1)

Question 3 (34 marks)

a. Convert 18° to radians, giving your answer in terms of π. (2)

b.



The diagram above shows a sector OABC of a circle, radius *r*, together with a square OPQA of side *r*.
Reflex $\angle AOC = \theta$ radians

i. If the perimeter of the sector is equal to the circumference of the circle radius *r*, prove that $\theta = 2\pi - 2$ (3)

ii. Show that the total area of the figure is equal to that of a circle radius *r*. (3)

c. Solve the following trigonometric equations for $0 \le \theta \le 2\pi$.

i. $\cos 2\theta = -\frac{\sqrt{3}}{2}$ (3)

ii. $\sec^2\theta + \tan\theta = 1$ (4)

d. i. Sketch a neat graph of $y = 2\cos x$ for $-\pi \le x \le \pi$ (3)

ii. The line $y = 1$ meets the curve in part i. at P and Q. Find the exact length of PQ. (3)

e. Differentiate with respect to x

i. $2\sin 3x$ (1)

ii. $\sin x \tan x$ (2)

iii. $\sqrt{\frac{x^2}{1+\tan^2 x}}$ (3)

f. For the function $y = 2\cos(2x - \frac{\pi}{4})$

i. State the amplitude (1)

ii. State the period (2)

iii. Solve $2\cos(2x - \frac{\pi}{4}) = \sqrt{3}$, for $0 \le x \le \pi$ (4)

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Q1/

(a) $y = \frac{4x^3 - x^4}{9}$

$y' = \frac{12x^2 - 4x^3}{9}$

St pts occur when $y' = 0$

$12x^2 - 4x^3 = 0$

$4x^2(3 - x) = 0$

$x = 0$ and $x = 3$

$y'' = \frac{24x - 12x^2}{9}$

when $x = 0$ $y'' = 0$

Test concavity either side

y''	$-$	0	$+$
x	$0-\epsilon$	0	$0+\epsilon$

∴ change in concavity

Horizontal POI at $(0,0)$

when $x = 3$ $y'' < 0$ ∴ Max st pt at $(3,3)$

(ii) Possible POI when $y'' = 0$

$24x - 12x^2 = 0$

$12x(2 - x) = 0$

∴ $x = 0$ and $x = 2$

x	$2-\epsilon$	2	$2+\epsilon$
y''	$+$	0	$-$

∴ change in concavity

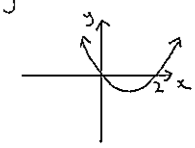
POI at $(2, \frac{16}{9})$ and $(0,0)$ proven in (i)

(iii) Concave up when $y'' > 0$

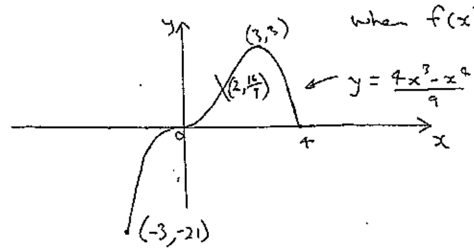
$\frac{24x - 12x^2}{9} > 0$

$0 > 12x^2 - 24x$

$0 > 12x(x-2)$



∴ $0 < x < 2$ when $f(x)$ is concave up



(b) Fencing length = 120

Fencing $= 4x + 2y + 2(2x + y)$

$= 4x + 2y + 4x + 2y$

$= 8x + 3y$ ∴ $8x + 3y = 120$

Area $= x(2x + y) + xy$

$= 2x^2 + xy + xy$

$= 2x^2 + 2xy$ but $8x + 3y = 120$

$y = \frac{120 - 8x}{3}$

∴ $A = 2x^2 + 2x\left(\frac{120 - 8x}{3}\right)$

$= 2x^2 + 80x - \frac{16x^2}{3}$

Stat. values occur when $\frac{dA}{dx} = 0$

(iii) $A = 2x^2 + 80x - \frac{16x^2}{3}$

$A = -\frac{10}{3}x^2 + 80x$

$\frac{dA}{dx} = -\frac{20}{3}x + 80$

$\frac{20x}{3} = 80$

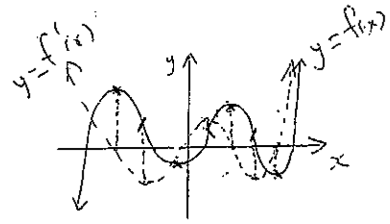
$20x = 240$

$x = 12$

$\frac{d^2A}{dx^2} = -\frac{20}{3}$ ∴ Concave up for all x

∴ Max area when $x = 12$

(d)



(Curve increasing)

(e)(i) $0 \leq x < 2$, $5 < x \leq 8$

(ii) $3.5 < x < 7$

(Curve concave up)

Q2 (a) (i) $x - 2x^2 + c$

(ii) $\dfrac{(1-4x)^6}{-24} + c$

(iii) $\int x^3 + 1 + x^{-2}\,dx = \dfrac{x^4}{4} + x - x^{-1} + c$

$= \dfrac{x^4}{4} + x - \dfrac{1}{x} + c$

(iv) $\int 1 - 2x^2 + x^4\,dx$

$= x - \dfrac{2x^3}{3} + \dfrac{x^5}{5} + c$

(b) $\left[\dfrac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}}$

$= \dfrac{1}{2}\tan\dfrac{\pi}{4} - \dfrac{1}{2}\tan 0$

$= \dfrac{1}{2}(1) - \dfrac{1}{2}(0)$

$= \dfrac{1}{2}$

(c) $y = 2 + x - x^2$

(i) cut x axis when $y = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ and $x = -1$

$\therefore A(-1, 0)$

$B(2, 0)$

(ii) $A = \int_{-1}^{2} 2 + x - x^2\,dx$

$= \left[2x + \dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-1}^{2}$

$= 2(2) + \dfrac{(2)^2}{2} - \dfrac{(2)^3}{3} - \left[2(-1) + \dfrac{(-1)^2}{2} - \dfrac{(-1)^3}{3}\right]$

$= 4 + 2 - \dfrac{8}{3} - \left[-2 + \dfrac{1}{2} + \dfrac{1}{3}\right]$

$= 4\frac{1}{2}\ u^2$

(d) (i)

x	1	2	3	4	5
$\dfrac{3}{x(x+1)}$	$\dfrac{3}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{4}$	$\dfrac{3}{20}$	$\dfrac{1}{10}$
	1.5	0.5	0.25	0.15	0.1

(ii) $\dfrac{h}{3}\left(f_{1st} + f_{last} + 2f_{odd} + 4f_{even}\right)$

$= \dfrac{1}{3}\left(1.5 + 0.1 + 2(0.25) + 4(0.5 + 0.15)\right)$

$= 1.57$ (to 2 dec places)

(e) $y = (x+2)^3$

cuts y axis when $x = 0$

$y = 8$

cuts x axis when $y = 0$

$0 = (x+2)^3$

$x = -2$

(i) $\therefore A(-2, 0)$ and $B(0, 8)$

(ii) $\pi\int_{-2}^{0} y^2\,dx = \pi\int_{-2}^{0} (x+2)^6\,dx$

$= \dfrac{\pi}{7}\left[(x+2)^7\right]_{-2}^{0}$

$= \dfrac{\pi}{7}\left[2^7 - 0\right]$

$= \dfrac{128\pi}{7}$

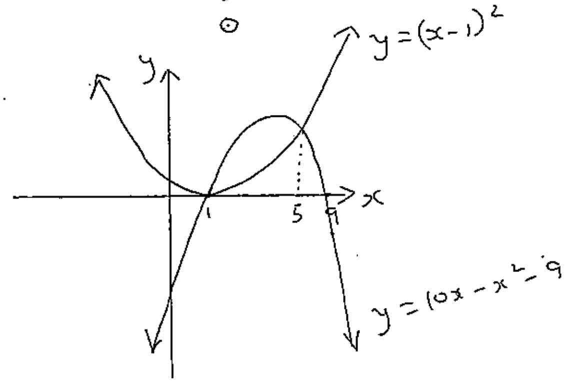
(iii) $V = \pi\int_0^8 x^2\,dy$

$y^{\frac{1}{3}} = x + 2$

$(y^{\frac{1}{3}} - 2) = x \qquad (y^{\frac{1}{3}} - 2)^2 = x^2$

$\therefore V = \pi\int_0^8 (y^{\frac{1}{3}} - 2)^2\,dy$

(f)



$y = -x^2 + 10x - 9$

$y = -(x^2 - 10x + 9)$

$= -(x-9)(x-1)$

$(x-1)^2 = 10x - x^2 - 9$

$x^2 - 2x + 1 = 10x - x^2 - 9$

$2x^2 - 12x + 10 = 0$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 1$ and $x = 5$

$A = \int_1^5 10x - x^2 - 9 - (x-1$

$A = \int_1^5 10x - x^2 - 9 - (x^2 - 2x + 1)\,dx$

$= \int_1^5 10x - x^2 - 9 - x^2 + 2x - 1\,dx$

$$\therefore \int_1^5 -2x^2+12x-10\,dx$$

$$= \left[-\frac{2x^3}{3}+6x^2-10x\right]_1^5$$

$$= \left[-\frac{2(5)^3}{3}+6(5)^2-10(5)-\left(-\frac{2}{3}+6-10\right)\right]$$

$$= 21\tfrac{1}{3}\ u^2$$

(g) (i) $4u^2$
(ii) -4

Question 3

(a) $180° = \pi^c$

$1° = \frac{\pi}{180}$

$18 = \frac{18\pi}{180}$

$= \frac{\pi}{10}^c$

(b) $C = 2\pi r$ Sector $OABC = r\theta + 2r$

$\therefore\ 2\pi r = r\theta + 2r$

~~$2\pi r = r(\theta+2)$~~

$2\pi r = r\theta + 2r$

$2\pi r - 2r = r\theta$

$r(2\pi - 2) = r\theta$

$\therefore\ \theta = 2\pi - 2$

(ii) Area figure $= r^2 + \frac{1}{2}r^2\theta$

Area circle $= \pi r^2$

But Area figure $= r^2 + \frac{1}{2}r^2(2\pi - 2)$ from (i)

$= r^2 + \pi r^2 - r^2$

$= \pi r^2$

C. (i) $\cos 2\theta = -\frac{\sqrt{3}}{2}$ $0 \le 2\theta \le 2\pi$

ref $\angle = \frac{\pi}{6}$

$2\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$

$\therefore\ \theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

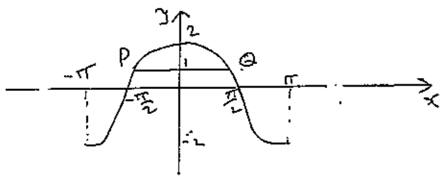
(ii) $1 + \tan^2\theta + \tan\theta = 1$

$\tan^2\theta + \tan\theta = 0$

$\tan\theta(1 + \tan\theta) = 0$

$\tan\theta = 0$ OR $\tan\theta = -1$

$\theta = 0, \pi, 2\pi$ $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

(d) 

(3)

(i) $2\cos x = 1$

$\cos x = \frac{1}{2}$ $-\pi \le x \le \pi$

$x = \pm\frac{\pi}{3}$

(3)

$\therefore$ distance $PQ = \frac{2\pi}{3}$

(e) (i) $\frac{d}{dx} = 6\cos 3x$ (1)

(ii) $\frac{d}{dx} = \sin x . \sec^2 x + \tan x . \cos x$ (2)

$\sin x . \frac{1}{\cos^2 x} + \tan x . \cos x$

$\tan x . \sec x + \tan x . \cos x$

$\tan x(\sec x + \cos x)$

(iii) $\frac{x}{\sec x} = x\cos x$

$\frac{d}{dx}\, x\cos x = x . -\sin x + \cos x . 1$

$= -x\sin x + \cos x$

$= \cos x - x\sin x$ (3)

(f) (i) $a = 2$

(ii) Period $= \frac{2\pi}{2}$

$= \pi$

(iii) $\cos\left(2x - \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$ $0 \le x \le \pi$

$0 \le 2x \le 2\pi$

$-\frac{\pi}{4} \le 2x \le 2\pi - \frac{\pi}{4}$

ref $\angle = \frac{\pi}{6}$

$2x - \frac{\pi}{4} = -\frac{\pi}{6}, \frac{\pi}{6}$

$2x = -\frac{\pi}{6} + \frac{\pi}{4}, \frac{\pi}{6} + \frac{\pi}{4}$

$2x = \frac{\pi}{12}, \frac{5\pi}{12}$

$x = \frac{\pi}{24}, \frac{5\pi}{24}$