GOSFORD HIGH SCHOOL

SPECTEMUR AGENDO

Year 12

2011

HSC

MATHEMATICS

Assessment Task #2

Time Allowed: 90 minutes+5 minutes reading time

Instructions:

- Start each question on a new sheet of paper.
- Attempt questions 1-5.
- Board approved calculators may be used.
- Write using black or blue pen.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

QUESTION 1 (12 Marks) Start a new sheet of paper. **MARKS**

a) The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 12$

(i) find $\frac{d^2y}{dx^2}$ **1**

(ii) if the curve passes through (1,-2), find the equation of the curve **2**

(iii) find the values of x for which the curve both increases and is concave downwards **3**

(iv) find any point(s) of inflexion **2**

(v) sketch the curve labeling all critical points (do NOT include x – axis intercepts) **2**

b) Find exact values of x for which the gradient of the curve $y = 2x(x+3)^2$ is 14. **2**

QUESTION 2 (12 Marks) Start a new sheet of paper.

a) Find the primitive function of $x^{-2} - 2$ **2**

b) Find the indefinite integral of $\int (6x+7)^3\,dx$ **2**

c) Evaluate (i) $\int_{-1}^{2} (x^3 + x - 5)dx$ **3**

Find (ii) $\int \frac{dx}{\sqrt{9-2x}}$ **2**

d) Find the area bounded by the curve $y = \sqrt{x}$, the y-axis and the lines $y = 2$ and $y = 3$. **3**

QUESTION 3 (12 Marks) **Start a new sheet of paper.** **MARKS**

a) Evaluate $log_2 32$ 1

b) Solve $2^x = 1000$ to 3 significant figures 1

c) Sketch neatly, and stating Domain and Range, the graph of $y = lnx$ 2

d) Find $\frac{dy}{dx}$ if $y = ln\frac{2x+5}{3-x}$ 2

e) Evaluate $\int_1^e x^2 + \frac{2}{x}dx$ 2

f) (i) Sketch the curve $y = \frac{1}{3-x}$,showing any critical information 2

(ii) Find the area enclosed by: $y = \frac{1}{3-x}$, the x – axis and the lines $x = 1$ and $x = 2$. 2

QUESTION 4 (12 Marks) **Start a new sheet of paper.**

a) Sketch neatly, showing any critical points / values, $y = e^x - 1$ 2

b) Differentiate $y = e^{\sqrt{x}}$ 2

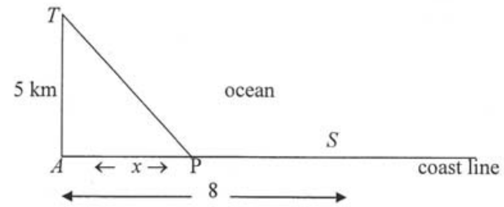
c) Find the stationary point of $y = xe^{-x}$ and determine its nature. 3

d) (i) Find the area bounded by the curve $y = e^{2x}$, the x ~~and y axes~~ and the lines $x = 1$ and $x = 3$ 2

(ii) Rotate this area around the x- axis and evaluate the *exact* volume of the solid formed 3

QUESTION5 (12 Marks) **Start a new sheet of paper.** **MARKS**

a) A natural gas pipe line is to be built connecting a coastal city S to an offshore island T which is 5 km from the closest coastline point A. The distance between A and the city S is 8 km. The pipeline is to be run from S to a point P then underwater to T. The cost of laying the pipeline is $75 000 per km on land and $100 000 per km under water.



Let $AP = x$

(i) show that the length of the pipeline (l) is $\sqrt{x^2 + 25} + (8 - x)$ 2

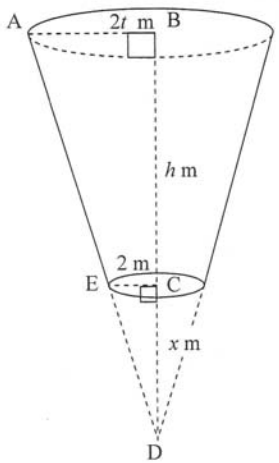
(ii) find an expression for the cost C of building the pipe line. 2

(iii) find where P should be located to minimise the cost of the pipeline 2

Question 5 continued over the page

QUESTION 5 (b) Continued **MARKS**

b) A truncated cone is to be used as a part of a hopper for a grain harvester. It has a total height of h metres. The top radius is to be t times greater than the bottom radius which is 2 metres.



AB = $2t$ metres
BC = h metres
EC = 2 metres
CD = x metres

(i) If x is the height of the removed section of the original cone, show using similar triangles that $x = \dfrac{h}{t-1}$ **2**

(ii) Show that the volume of the truncated cone is given by $V = \left(\dfrac{4\pi h}{3}\right)\left(t^2 + t + 1\right)$ **2**

(iii) If the upper radius plus the lower radius plus the height of the truncated cone must total 12 metres, calculate the maximum volume of the hopper. **3**

END OF TEST ☺

Solutions 2011 Mathematics Task 2.

Q1 a) $\frac{dy}{dx} = 3x^2 - 12$

i) $\frac{d^2y}{dx^2} = 6x$ 1

ii) $y = x^3 - 12x + c$

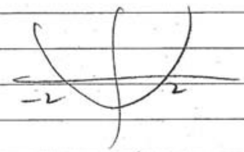
$(1,-2)$ lies on this

$-2 = 1 - 12 + c$

$9 = c$

$\therefore y = x^3 - 12x + 9$ 1

iii) Increasing $y' > 0$



$3x^2 - 12 > 0$

$x^2 - 4 > 0$

$(x-2)(x+2) > 0$

$x > 2$ or $x < -2$

Concave down

$y'' < 0$

$6x < 0$

$x < 0$

$\therefore$ $x < -2$ 3

iv) $6x = 0$ $y = 7$

$x = 0$

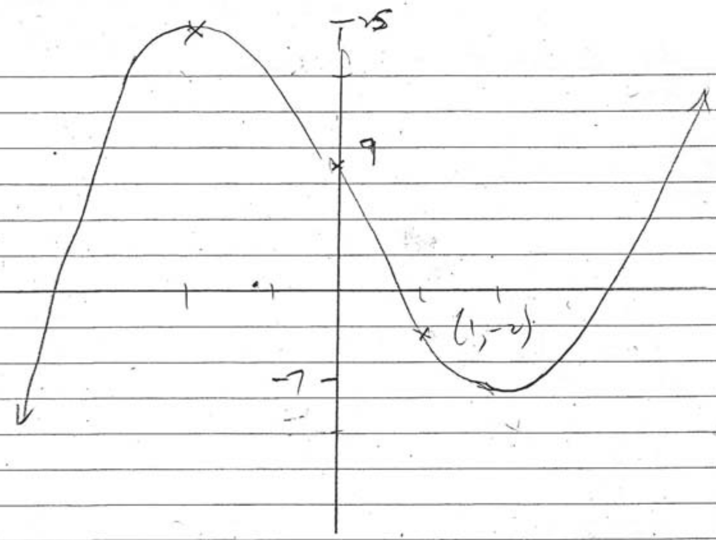
$(0, 9)$

check concavity change

x	0^-	0	0^+
y''	$-$	0	$+$

Concavity changes down to up.

$(0, 7)$ is a pt of inflexion.



Stationary pts $y' = 0$

$3x^2 - 12 = 0$

$x^2 = 4$

$x = \pm 2$

$(2, -7)$ $(-2, 25)$

Nature $x = 2$ $x = -2$

y'' $+$ y'' $-$

b) $y = 2x(x+3)^2$

$y = 2x(x^2 + 6x + 9)$

$= 2x^3 + 12x^2 + 18x$

$y' = 6x^2 + 24x + 18$

$14 = 6x^2 + 24x + 18$

$0 = 6x^2 + 24x + 4$

$0 = 3x^2 + 12x + 2$

$x = \frac{-12 \pm \sqrt{144 - 4 \times 3 \times 2}}{6}$

$x = \frac{-12 \pm \sqrt{120}}{6}$

$x = \frac{-6 \pm \sqrt{30}}{3}$

Question 2

a) $\frac{x^{-1}}{-1} - 2x + c$

$= -\frac{1}{x} - 2x + c$

b) $\int (6x+7)^3 \, dx = \frac{(6x+7)^4}{4 \times 6} + c$

$= \frac{(6x+7)^4}{24} + c$

c) i) $\int_{-1}^{2} x^3 + x - 5 \, dx$

$\left[\frac{x^4}{4} + \frac{x^2}{2} - 5x \right]_{-1}^{2}$

$= \left(\frac{16}{4} + \frac{4}{2} - 10 \right) - \left(\frac{1}{4} + \frac{1}{2} + 5 \right)$

$= 4 + 2 - 10 - \frac{1}{4} - \frac{1}{2} - 5$

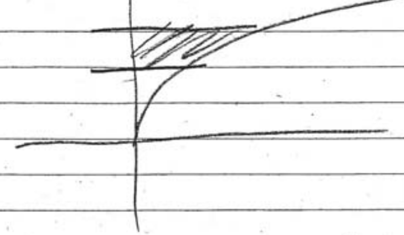
$= -9\frac{3}{4}$

ii) $\int (9-2x)^{-\frac{1}{2}} \, dx$

$= \frac{2(9-2x)^{\frac{1}{2}}}{-2} + c$

$= -\sqrt{9-2x} + c$

d) $y = \sqrt{x}$



$A = \int_{c}^{d} x \, dy$

$A = \int_{2}^{3} y^2 \, dy$

$= \left[\frac{y^3}{3} \right]_{2}^{3}$

$= \frac{27}{3} - \frac{8}{3}$

$= 6\frac{1}{3} \; u^2$

Question 3

a) $\log_2 32 = x$

$2^x = 32$

$x = 5$

$\log_2 32 = 5$ (1)

b) $2^x = 1000$

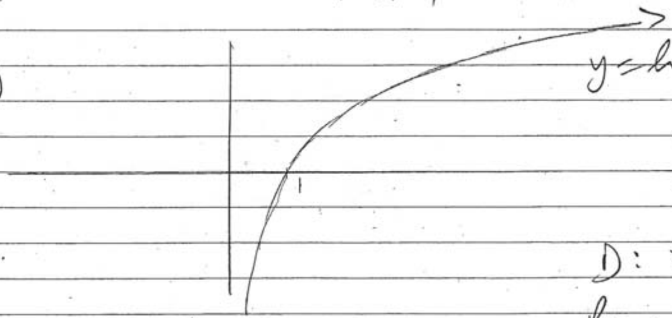
$x \ln 2 = \ln 1000$

$x = \dfrac{\ln 1000}{\ln 2}$

$= 9.96578$ (1)

$x \doteqdot 9.97$

c)



$y = \ln x$ 1

$D: x > 0$

Range: all real y 1

d) $y = \ln \dfrac{2x+5}{3-x}$

$y = \ln(2x+5) - \ln(3-x)$ 1

$\dfrac{dy}{dx} = \dfrac{2}{2x+5} - \dfrac{-1}{3-x}$ 1

$= \dfrac{2}{2x+5} + \dfrac{1}{3-x}$

OK $\dfrac{f'(x)}{f(x)}$

$\dfrac{2(3-x) + 2x+5}{(2x+5)(3-x)}$

$\dfrac{6 - 2x + 2x + 5}{(2x+5)(3-x)}$

$= \dfrac{11}{(2x+5)(3-x)}$

e) $$\int_1^e x^2 + \frac{2}{x}\,dx$$

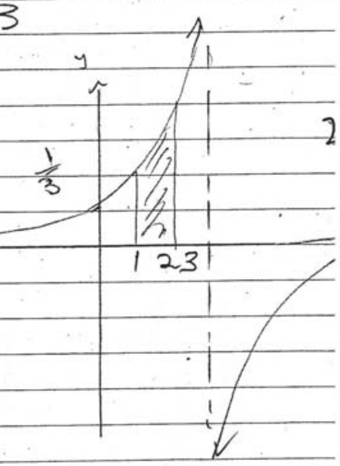
$$= \left[\frac{x^3}{3} + 2\ln x\right]_1^e \qquad 1$$

$$= \frac{e^3}{3} + 2\ln e - \left(\frac{1}{3} + 2\ln 1\right)$$

$$= \frac{e^3}{3} + 2 - \frac{1}{3} - 0$$

$$= \frac{e^3}{3} + \frac{5}{3} = \frac{e^3+5}{3} \qquad 1$$

f) i) $$y = \frac{-1}{x-3} \qquad 2$$



$$A = -\int_1^2 \frac{-1}{3-x}\,dx$$

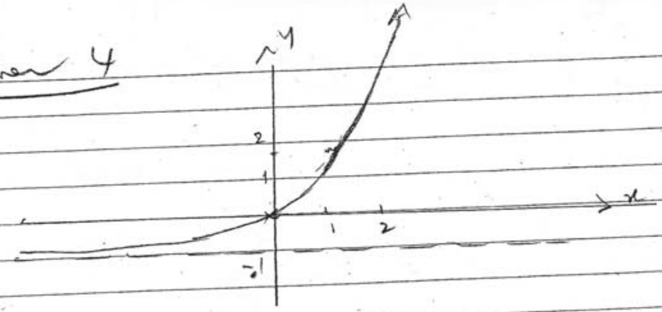
$$= \left[-\ln(3-x)\right]_1^2$$

$$= -\ln 1 + \ln 2$$

$$= \ln 2 \quad u^2 \qquad 2$$

Question 4

a)



b) $$y = e^{x^{\frac{1}{2}}}$$

$$y' = e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$y' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

c) $$y = xe^{-x}$$

$$y' = x \cdot -e^{-x} + e^{-x}$$

$$y' = e^{-x}(1-x)$$

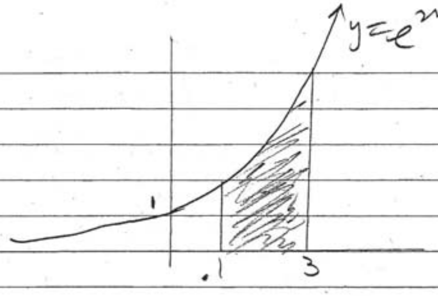
Stat at $x = 1$ $y = \frac{1}{e}$

Nature

x	1^-	1	1^+
y'	$+$	0	$-$

$\left(1, \frac{1}{e}\right)$ is a maximum turning point

d) i)



$$A = \int_1^3 e^{2x} \, dx$$

$$= \left[\frac{e^{2x}}{2} \right]_1^3 = \frac{e^6}{2} - \frac{e^2}{2} \quad u^2$$

ii) $$V = \pi \int_2^3 y^2 \, dx \qquad y^2 = \left(e^{2x}\right)^2 = e^{4x}$$

$$V = \pi \int_2^3 e^{4x} \, dx$$

$$= \pi \left[\frac{e^{4x}}{4} \right]_2^3$$

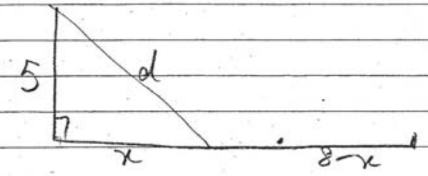
$$= \pi \, \frac{e^{12} - e^8}{4}$$

$$= \frac{\pi}{4} \cdot e^8 \left(e^4 - 1\right) \quad u^3$$

Question 5

a) \$ 75 000 on land   \$ 100 000 under H_2O

i)



Using Pythagoras $d^2 = 5^2 + x^2$

$d = \sqrt{25 + x^2}$

$\therefore$ Length $= \sqrt{25 + x^2} + (8 - x)$

ii) $C = 100\,000 \sqrt{25 + x^2} + 75\,000 (8 - x)$

$C = 25\,000 \left\{ 4 \sqrt{25 + x^2} + 3(8 - x) \right\}$

iii) For minimum cost $\frac{dC}{dx} = 0$

$$\frac{dC}{dx} = 25000 \left\{ 4 \times \tfrac{1}{2} (25 + x^2)^{-\frac{1}{2}} \cdot 2x + -3 \right\}$$

$$0 = \frac{4x}{\sqrt{25 + x^2}} - 3$$

$$3 \sqrt{25 + x^2} = 4x$$

$$9(25 + x^2) = 16x^2$$

$$225 + 9x^2 = 16x^2$$

$$225 = 7x^2$$

$$x^2 = \frac{225}{7}$$

$x = 5.669$ Km

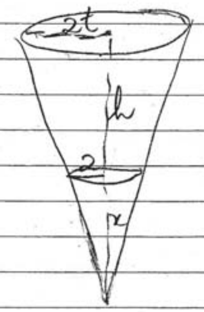
$x \doteqdot 5.67$ Km from A

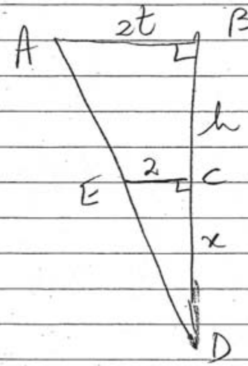
Show cost is in fact minimum

x	5	5.7	6
C'	$-.17$	0	0.07
	$-$	0	$+$

$\therefore$ cost is minimum

5b)





In $\triangle EDC$, $\triangle ADB$

1. $\angle D$ is common
2. $\angle ECD = \angle ABD$ (both $90°$)

$\therefore \triangle EDC \parallel\!\mid \triangle ABD$ (2 angles test)

$$\therefore \frac{x}{x+h} = \frac{2}{2t} = \frac{1}{t}$$

$$tx = x+h$$

$$tx - x = h$$

$$x(t-1) = h$$

$$x = \frac{h}{t-1} \quad \text{as required}$$

$V =$ Large $-$ small cone

$$= \frac{1}{3}\pi (2t)^2 (x+h) - \frac{1}{3}\pi (2)^2 \cdot x$$

$$= \frac{\pi}{3}\left\{4t^2x + 4t^2h - 4x\right\}$$

$$= \frac{4\pi}{3}\left\{x\left(t^2-1\right) + t^2h\right\}$$

$$= \frac{4\pi}{3}\left\{\frac{h}{t-1}(t-1)(t+1) + t^2h\right\}$$

$$= \frac{4\pi}{3}\left\{h(t+1) + t^2h\right\}$$

$$= \frac{4\pi h}{3}\left\{t+1+t^2\right\} \quad \text{as required}$$

iii)

$$2 + 2t + h = 12$$

$$h = 10 - 2t$$

$$V = \frac{8\pi}{3}(5-t)(t+1+t^2)$$

$$= \frac{8\pi}{3}(5t + 5 + 5t^2 - t^2 - t - t^3)$$

$$= \frac{8\pi}{3}(4t^2 + 4t - t^3 + 5)$$

For max volume $\frac{dV}{dt} = 0$

$$\frac{dV}{dt} = \frac{8\pi}{3}\left\{8t + 4 - 3t^2\right\}$$

$$3t^2 - 8t - 4 = 0$$

$$t = \frac{8 \pm \sqrt{64+48}}{6} = \frac{8 \pm \sqrt{112}}{6} = \frac{4 \pm 2\sqrt{7}}{3}$$

3.097 or -0.4305

$$\frac{d^2V}{dt^2} = \frac{8\pi}{3}\left\{8 - 6t\right\}$$

$t = 3.097$ $\frac{d^2V}{dt^2} = \frac{8\pi}{3}\left\{8 - 18.58\right\}$

negative

$\therefore$ Max volume when $t = 3.097$

$t = -0.4305$ $\frac{d^2V}{dt^2} = \frac{8\pi}{3}\left\{8 - -2.583\right\}$

pos $\therefore$ min volume

This Maximum volume is

$$V = \frac{8\pi}{3}\left\{4\times 3.097^2 + 4\times 3.097 - 3.097^3 + 5\right\}$$

$$= \frac{8\pi}{3}\left(26.049\right)$$

$V = 218.22\ m^3$