Name: ______________________ Teacher: ______________

GOSFORD HIGH SCHOOL

2013

Year 12 HSC Mathematics

Assessment Task #2

INSTRUCTIONS:

TIME: *90 minutes + 5 minutes reading time*

- Write using black or blue pen.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In questions 8 – 11, show relevant mathematical reasoning and/or calculations

Questions 1 – 7	Multiple Choice	/7
Question 8	Trigonometric Functions	/12
Question 9	Integration	/12
Question 10	Integration	/12
Question 11	Trigonometric Functions	/12
TOTAL		/55

SECTION 1

7 Marks
Attempt Questions 1 – 7 on the multiple choice sheet provided.
Allow about 10 minutes for this section.

Question 1:

In evaluating a definite integral, a working step is shown by a student to be:

$$[x + tanx]_0^{\frac{\pi}{8}}$$

The answer to 2 decimal places is:

A. 0.39

B. 0.40

C. 0.80

D. 0.81

Questions 2:

What is the greatest value of the function

$$y = 3 - 2cosx$$

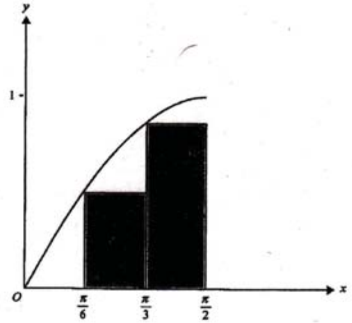
A. 1

B. 3

C. 5

D. 6

Question 3:



The area under the curve $y = sinx$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$ is approximated by 2 rectangles as shown above.

This approximation to the area is:

A. $\frac{\sqrt{3}\,\pi}{6}$

B. $\frac{\pi}{2}$

C. $\frac{(\sqrt{3}+1)\pi}{12}$

D. $\frac{(\sqrt{3}+1)\pi}{6}$

Question 4:

The function $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has a period of:

A. 5

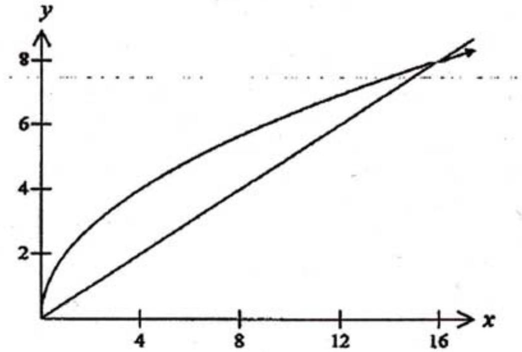
B. 10

C. $\frac{\pi}{5}$

D. $\frac{\pi}{10}$

Question 5:

The diagram below shows the graph of $y = 2\sqrt{x}$ and $y = \frac{x}{4}$



Which of the following is the correct expression for the volume of the solid of revolution when the area between the curve $y = 2\sqrt{x}$ and $y = \frac{x}{4}$ is rotated around the *x-axis*?

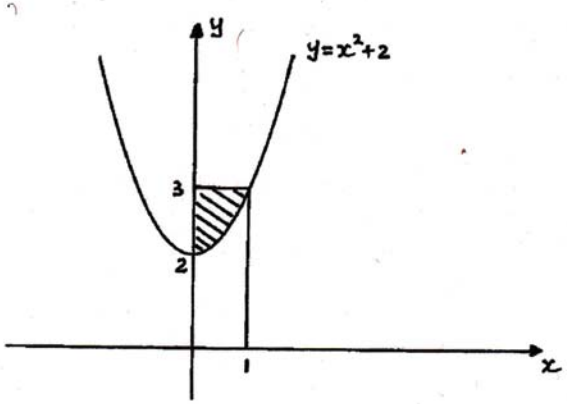
(A) $V = \int_0^8 (4y - \frac{y^2}{2})dy$

(B) $V = \int_0^{16} (2\sqrt{x} - \frac{x}{4})dx$

(C) $V = \pi\int_0^8 (16y^2 - \frac{y^4}{4})dy$

(D) $V = \pi\int_0^{16} (4x - \frac{x^2}{16})dx$

Question 6:



To find the area of the shaded region above, four different students proposed the following working:

i. $\int_0^1 x^2 + 2\ dx$

ii. $3 - \int_0^1 x^2 + 2\ dx$

iii. $\int_2^3 \sqrt{y-2}\ dy$

iv. $3 - \int_2^3 \sqrt{y-2}\ dy$

Which of the following is correct?

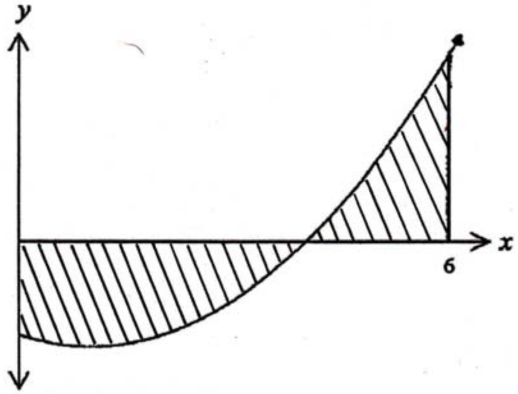
A. (ii) only

B. (ii) and (iii) only

C. (i) and (ii) only

D. (ii), (iii) and (iv) only

Question 7:



What is the correct expression for the area bounded by the *x-axis* and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$? *(As shown in the diagram above.)*

A. $A = \int_0^5 x^2 - 2x - 8dx + \left|\int_5^6 x^2 - 2x - 8dx\right|$

B. $A = \int_0^4 x^2 - 2x - 8dx + \left|\int_4^6 x^2 - 2x - 8dx\right|$

C. $A = \left|\int_0^5 x^2 - 2x - 8dx\right| + \int_5^6 x^2 - 2x - 8dx$

D. $A = \left|\int_0^4 x^2 - 2x - 8dx\right| + \int_4^6 x^2 - 2x - 8dx$

SECTION 2

48 Marks

- Attempt Questions 8 – 11 on your own writing paper, starting **each** question on a new sheet of paper.
- Allow about 1 hour and 20 minutes for this section.
- In Questions 8 – 11, your responses should include relevant mathematical reasoning and/or calculations.

Question 8: (12 marks) use a new sheet of writing paper.

a. Convert 1.25 radians into degrees and minutes. (*Answer to the nearest minute)* 1

b. Find the exact value of $cosec\ \frac{4\pi}{3}$ 1

c. Solve: $\sqrt{3}\tan x = 1$, 2

in the domain $-\pi \le x \le \pi$

d. Differentiate:

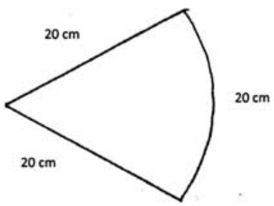
i. $3\sin(2-7x)$ 1

ii. $x^2 tanx$ 1

iii. $\frac{sinx}{1+cosx}$ 2

e. Find $\int_0^{\frac{\pi}{2}} 4\,sec^2\left(\frac{x}{2}\right) dx$ 2

f. The following sector: 2



Is cut from a circular piece of plastic and is **discarded.** Find the exact area of the **remaining** plastic.

Question 9: (12 Marks) Use a new sheet of writing paper.

a. Find a primitive of:

i. $6x^3 - 5x + 1$ 1

ii. $(3x+4)^8$ 1

b. Find: $\int \frac{x^{-1/2}}{4}\, dx$ 1

c. Evaluate:

i. $\int_1^2 \frac{x^4 - x^2 + 3}{3x^2}\, dx$ 2

ii. $\int_{-1}^0 \sqrt{1-2x}\, dx$ *(Leave in simplest exact form.)* 2

d. The table below gives values for $y = f(x)$

x	0	1	2	3
$f(x)$	2	1.5	1.41	1.37

Using the four function values in the table and the trapezoidal role approximate 2

$\int_0^3 f(x)\, dx$, correct to 1 decimal place.

e. Using Simpson's Rule with four strips find an approximate value for: 3

$$\int_0^2 \frac{4}{x^2+1}\, dx$$

(Answer correct to 3 significant figures.)

Question 10: (12 Marks) Use a new sheet of writing paper.

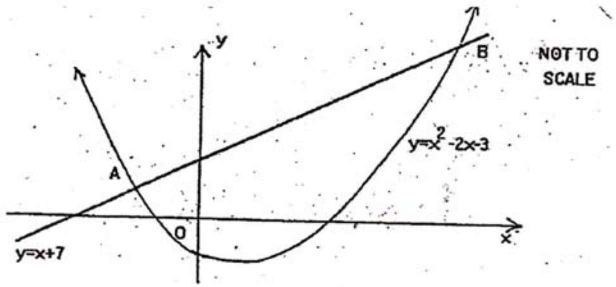
a. The gradient of a curve is given by 2

$$\frac{dy}{dx} = 2x - 3$$

Find the equation of the curve if it passes through the point (1,2)

b. The area under the curve $y = (x-1)^2$ and between the ordinates $x = 1$, $x = 3$ and the x axis is rotated about the x axis. Find the exact volume of the solid of revolution formed. 2

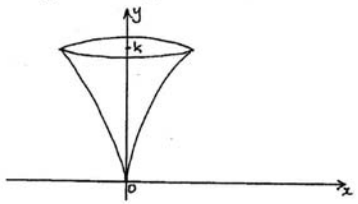
c. The diagram shows the graphs $y = x^2 - 2x - 3$ and $y = x + 7$. The graphs intersect at the points A and B.



i. Find the coordinates of A and B. 2

ii. Find the area enclosed by $y = x^2 - 2x - 3$ and $y = x + 7$ 2

d. The vase represented in the diagram is designed to hold $50\pi\ cm^3$ of water when full.



Its shape is determined by rotating the parabola $x = \frac{y^2}{30}$ about the y axis. If the depth of water in the vase is k centimetres:

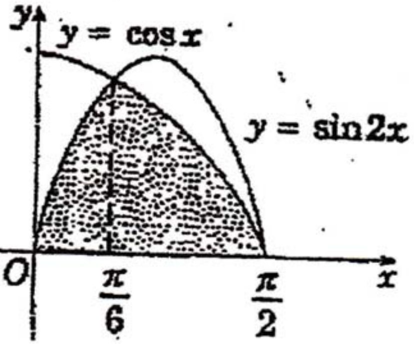
i. Write an expression for this volume and hence show that 2

$$\int_0^k y^4\, dy = 45\,000$$

ii. Find the value of k, correct to 1 decimal place. 2

Question 11: (12 Marks) Use a new sheet of writing paper.

a.



3

The diagram shows the graphs of the functions $y = cosx$ and $y = sin2x$ between $x = 0$ and $x = \frac{\pi}{2}$

Calculate the area of the shaded region.

b. On a $\frac{1}{3}$ page number plane, draw a neat sketch of $y = \sin(x + \frac{\pi}{4})$ in the domain $0 \le x \le 2\pi$. 2

(Clearly indicate and label all important features.)

c. i. Show that $sin\theta\ tan\theta = sec\theta - cos\theta$ 2

ii. Hence, solve $sin\theta\ tan\theta = 0$ for $0 \le x \le 2\pi$ 2

d. i. Differentiate $sin^3 4x$ 1

ii. Hence, evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} cos4x - cos^3 4x\ dx$ 2

(Leave answer in exact form.)

END OF ASSESSMENT TASK

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12 MATHS ASSESS TASK #2 2013 SOLUTIONS

SECTION 1

1/ D

$\left[x + \tan x\right]_0^{\frac{\pi}{8}} = \left(\frac{\pi}{8} + \tan\frac{\pi}{8}\right) - 0$

Using calculator and RAD. MODE

answer $= 0.8069126..$ (calc)

$= 0.81$ (2d.p)

2/ C

Since $-1 \leq \cos x \leq 1$, the maximum value of $3 - 2\cos x$ will occur when $\cos x = -1$

i.e $3 - 2x - 1 = 5$

3/ C

The area of the 2 rectangles is given by

Area $= \frac{\pi}{6} \times \sin\frac{\pi}{6} + \frac{\pi}{6} \times \sin\frac{\pi}{3}$

$= \frac{\pi}{6} \times \frac{1}{2} + \frac{\pi}{6} \times \frac{\sqrt{3}}{2}$

$= \frac{\pi}{6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

$= \left(\frac{\sqrt{3}+1}{12}\right)\pi$

4/ B

The period of $y = a\sin nx$ is $\frac{2\pi}{n}$

$\therefore$ Period $= \frac{2\pi}{\pi/5}$

$= 10$

5/ D

Using $V = \pi\int_0^{16} y_1^2 - y_2^2\,dx$

$= \pi\int_0^{16} 4x - \frac{x^2}{16}\,dx$

6/ B

(ii) gives a correct answer

(i.e area of rectangle – area under the curve)

(iii) gives a correct answer

(i.e finding the area between the curve, the y axis and $y=2$, $y=3$.)

7/ D

x intercepts of $y = x^2 - 2x - 8$ are $x = 4, -2$

$\therefore$ missing intercept in diagram is 4 and area is to be calculated

$\left|\int_0^4\right| + \int_4^6 \quad \therefore D$

SECTION 2

Q8 a) $1.25 \times \frac{180}{\pi}$

$= 71.61972439\ldots$ (calc)

$= 71^\circ 37'$ (nearest min.)

b) $\operatorname{cosec} \frac{4\pi}{3} = -\frac{2}{\sqrt{3}}$

c) $\sqrt{3}\tan x = 1$

$\tan x = \frac{1}{\sqrt{3}}$

for domain $0 \le x \le 2\pi$:

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

for domain $-\pi \le x \le \pi$

$x = \frac{\pi}{6}, -\frac{5\pi}{6}$

d) (i) let $y = 3\sin(2-7x)$

$y' = -21\cos(2-7x)$

(ii) let $y = x^2 \tan x$

(Product Rule) $y' = \tan x \cdot 2x + x^2 \cdot \sec^2 x$

$= 2x\tan x + x^2\sec^2 x$

(iii) let $y = \frac{\sin x}{1+\cos x}$

(Using quotient rule)

$y' = \frac{(1+\cos x)\cdot\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$= \frac{\cos x + 1}{(1+\cos x)^2}$

$= \frac{1}{1+\cos x}$

e) $\int_0^{\frac{\pi}{2}} 4\sec^2\frac{x}{2}\,dx$

$= \left[8\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= 8\tan\frac{\pi}{4} - 0$

$= 8$

f) Area of remaining plastic

= Area of circle − Area of Sector

$= \pi r^2 - \frac{1}{2}r^2\theta$

$= \pi \times 20^2 - \frac{1}{2}\times 20^2 \times 1$

$= 400\pi - 200 \text{ cm}^2$

Q9

a) (i) $\frac{6x^4}{4} - \frac{5x^2}{2} + x$

$= \frac{3x^4}{2} - \frac{5x^2}{2} + x$

(ii) $\frac{(3x+4)^9}{3\times 9}$

$= \frac{(3x+4)^9}{27}$

b) $\int \frac{x^{-\frac{1}{2}}}{4}\,dx = \frac{x^{\frac{1}{2}}}{2} + c$

(this form is consistent with question)

does not need:

$\frac{\sqrt{x}}{2} + c$

c) $\int_1^2 \frac{x^4 - x^2 + 3}{3x^2}\,dx$

$= \int_1^2 \frac{x^2}{3} - \frac{1}{3} + x^{-2}\,dx$

$= \left[\frac{x^3}{9} - \frac{x}{3} - x^{-1}\right]_1^2$

$= \left(\frac{8}{9} - \frac{2}{3} - \frac{1}{2}\right) - \left(\frac{1}{9} - \frac{1}{3} - 1\right)$

$= \frac{17}{18}$

(ii) $\int_{-1}^{0} \sqrt{1-2x}\,dx$

$= \int_{-1}^{0} (1-2x)^{\frac{1}{2}}\,dx$

$= \left[\frac{(1-2x)^{\frac{3}{2}}}{\frac{3}{2}\times -2}\right]_{-1}^{0}$

$= -\frac{1}{3}\left[\sqrt{(1-2x)^3}\right]_{-1}^{0}$

$= -\frac{1}{3}\left(1 - \sqrt{27}\right)$

$= -\frac{1}{3} + \sqrt{3}$

d) Using $\frac{h}{2}\left[y_0 + y_3 + 2(y_1 + y_2)\right]$

$= \frac{1}{2}\left[2 + 1.37 + 2(1.5 + 1.41)\right]$

$= 4.595$

$= 4.6$ (1d.p)

e) 4 strips → 5 function values

x	0	0.5	1	1.5	2
y	4	3.2	2	1.231	0.8
	y_0	y_1	y_2	y_3	y_4

$\int_0^2 \frac{4}{x^2+1}\,dx \doteq \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2y_2\right]$

$= \frac{0.5}{3}\left[4 + 0.8 + 4(3.2 + 1.231) + 2\times 2\right]$

$= 4.42$ (3 s.f)

Q10

a) $\frac{dy}{dx} = 2x - 3$

$y = x^2 - 3x + c$

Sub $(1,2)$: $2 = 1 - 3 + c$

$2 = -2 + c$

$4 = c$

$\therefore$ Eqn of curve is :

$x^2 - 3x + 4$

b) $V = \pi \int_1^3 (x-1)^4 \, dx$

$= \frac{\pi}{5}\left[(x-1)^5\right]_1^3$

$= \frac{\pi}{5}(32 - 0)$

$= \frac{32\pi}{5}$ units^3

c) (i) $y = x^2 - 2x - 3$ — (1)

$y = x + 7$ — (2)

Sub (1) in (2)

$x^2 - 2x - 3 = x + 7$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x = 5, -2$

$y = 12, 5$

$\therefore A = (-2, 5)$ $B = (5, 12)$

(ii) Area $= \int_{-2}^{5} (x+7) - (x^2 - 2x - 3) \, dx$

$= \int_{-2}^{5} 3x - x^2 + 10 \, dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3} + 10x\right]_{-2}^{5}$

$= \left(\frac{75}{2} - \frac{125}{3} + 50\right) - \left(6 + \frac{8}{3} - 20\right)$

$= 45\frac{5}{6} - \left(-11\frac{1}{3}\right)$

$= 57\frac{1}{6}$ units^2

d) (i) $V = \pi \int_0^k x^2 \, dy$

$= \pi \int_0^k \frac{y^4}{900} \, dy$

$= \frac{\pi}{900} \int_0^k y^4 \, dy$

$\therefore 50\pi = \frac{\pi}{900} \int_0^k y^4 \, dy$

$\therefore \int_0^k y^4 \, dy = 45000$ as req.

(ii) $\left[\frac{y^5}{5}\right]_0^k = 45000$

$\frac{k^5}{5} - 0 = 45000$

$k^5 = 225000$

$k = 11.8$ (1d.p)

Q11

a) Area $= \int_0^{\frac{\pi}{6}} \sin 2x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx$

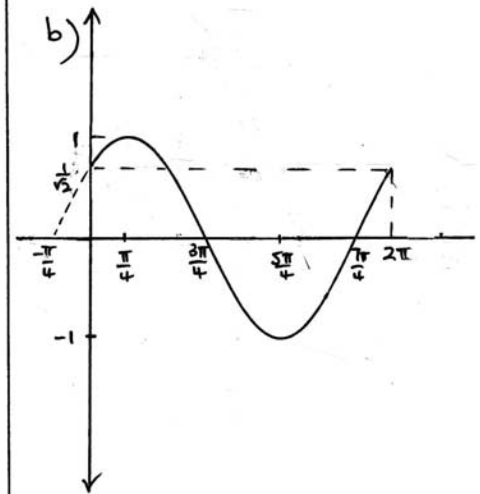
$= -\frac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= -\frac{1}{2}\left(\frac{1}{2} - 1\right) + \left(1 - \frac{1}{2}\right)$

$= \frac{1}{4} + \frac{1}{2}$

$= \frac{3}{4}$ units^3

b)



c) (i) LHS $= \sin\theta \tan\theta$

$= \sin\theta \cdot \frac{\sin\theta}{\cos\theta}$

$= \frac{\sin^2\theta}{\cos\theta}$

RHS $= \sec\theta - \cos\theta$

$= \frac{1}{\cos\theta} - \cos\theta$

$= \frac{1 - \cos^2\theta}{\cos\theta}$

$= \frac{\sin^2\theta}{\cos\theta}$

$=$ LHS as req.

(ii) $\sin\theta \tan\theta = 0$

$\therefore \sec\theta - \cos\theta = 0$

$\frac{1}{\cos\theta} = \cos\theta$

$1 = \cos^2\theta$

$\therefore \cos\theta = \pm 1$

$\theta = 0, \pi, 2\pi$

d) (i) Let $y = \sin^3 4x$

$y' = 3\sin^2 4x \times 4\cos 4x$

$= 12\sin^2 4x \cos 4x$

(ii) $= 12(1 - \cos^2 4x)\cos 4x$

$= (12 - 12\cos^2 4x)\cos 4x$

$= 12(\cos 4x - \cos^3 4x)$

$\therefore \int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \cos 4x - \cos^3 4x \, dx = \frac{1}{12}\left[\sin^3 4x\right]_{\frac{\pi}{8}}^{\frac{\pi}{6}}$

$= \frac{1}{12}\left(\sin^3\frac{2\pi}{3} - \sin^3\frac{\pi}{2}\right) = \frac{1}{12}\left(\frac{3\sqrt{3}}{8} - 1\right)$