HORNSBY GIRLS HIGH SCHOOL

Mathematics

Year 12 Higher School Certificate
Half-Yearly Examination 2013

STUDENT NUMBER: ____________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided
- In Questions 6 – 10, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 80

Section I Pages 2 – 4

5 marks

Attempt Questions 1 – 5

Answer on the Objective Response Answer Sheet provided

Section II Pages 5 – 11

75 marks

Attempt Questions 6 – 10. Start each question in a new writing booklet. Write your student number and fill in required information on each writing booklet.

At the end of the assessment:

- Order you solutions, starting with Objective Response answer sheet, then Questions 6 – 10
- Place the question paper on top.
- Do **NOT** staple

Question	*1-5*	*6*	*7*	*8*	*9*	*10*	*Total*
Total	*/5*	*/15*	*/15*	*/15*	*/15*	*/15*	*/80*

This assessment task constitutes 25% of the Higher School Certificate Course School Assessment

Section I

5 marks
Attempt Questions 1 – 5
Allow about 10 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 5

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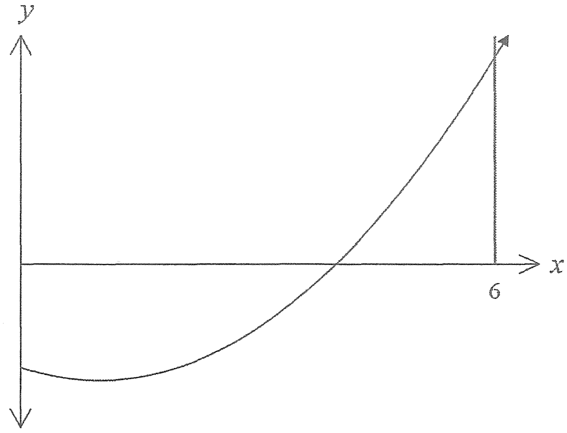
1 The diagram below shows the graph of $y = x^2 - 2x - 8$.



What is the correct expression for the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$?

(A) $A = \int_0^5 (x^2 - 2x - 8)dx + \left|\int_5^6 (x^2 - 2x - 8)dx\right|$

(B) $A = \int_0^4 (x^2 - 2x - 8)dx + \left|\int_4^6 (x^2 - 2x - 8)dx\right|$

(C) $A = \left|\int_0^5 (x^2 - 2x - 8)dx\right| + \int_5^6 (x^2 - 2x - 8)dx$

(D) $A = \left|\int_0^4 (x^2 - 2x - 8)dx\right| + \int_4^6 (x^2 - 2x - 8)dx$

2 What is the solution to the equation $\log_e(x+2) - \log_e x = \log_e 4$?

(A) $\frac{2}{5}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{2}$

3 What is the derivative of $\log_2 x$?

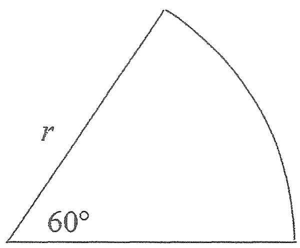
(A) $\frac{1}{x}$

(B) $\frac{1}{2x}$

(C) $\ln 2x$

(D) $\frac{1}{x\ln 2}$

4 The sector below has an area of 10π square units.



Not to scale

What is the value of r?

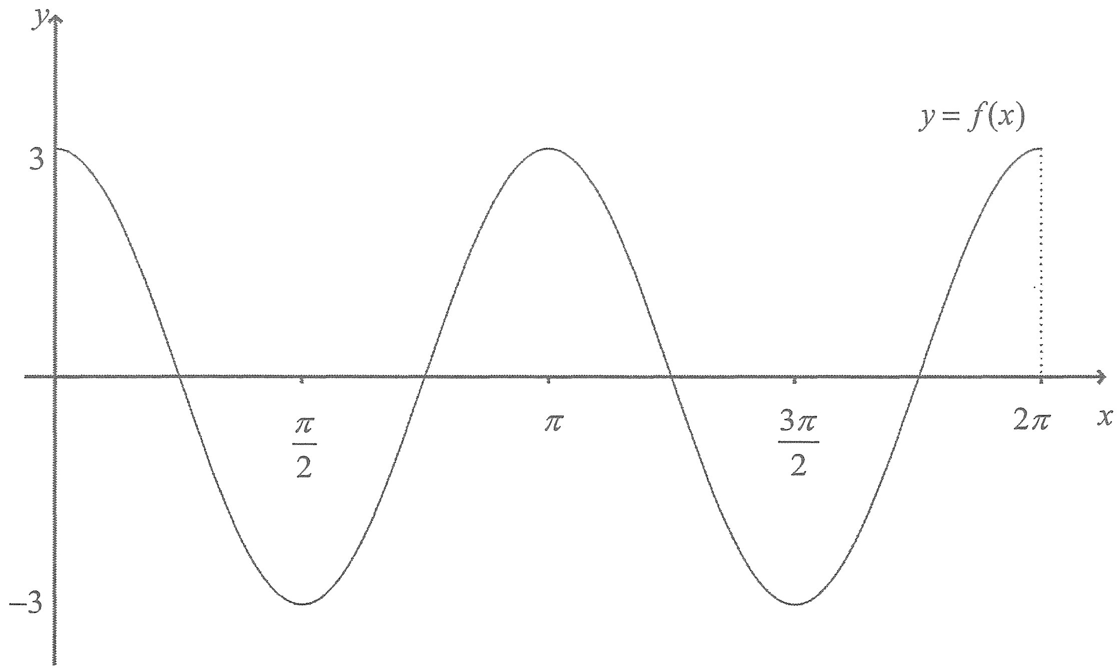
(A) $\sqrt{60}$

(B) $\sqrt{60\pi}$

(C) $\sqrt{\frac{\pi}{3}}$

(D) $\sqrt{\frac{1}{3}}$

5 The diagram shows a sketch of the graph $y = f(x)$, for $0 \le x \le 2\pi$.



The function $f(x)$ is:

(A) $f(x) = 3\cos 2x$

(B) $f(x) = 2\cos 3x$

(C) $f(x) = 3\cos\dfrac{x}{2}$

(D) $f(x) = 2\cos\dfrac{x}{3}$

End of Section I

Section II

75 marks
Attempt Questions 6 – 10
Allow about 1 hour and 50 minutes for this section

Begin each question in a new writing booklet, indicating the question number.
Extra writing booklets are available.

Question 6 (15 marks) Start a new writing booklet.

(a) Solve $5^{4x-3} = 1$ **2**

(b) Find $\log_7 5$ correct to 2 decimal places. **2**

(c) Simplify $3\log 5 - \log 25$, leaving your answer in exact form. **2**

(d) Solve $\log_3 x = 4$ **1**

(e) (i) Sketch the curve $y = \log_{10}(x+1)$ **2**

(ii) State the domain of the function in (i). **1**

(f) Evaluate e^{-5} correct to 2 significant figures. **2**

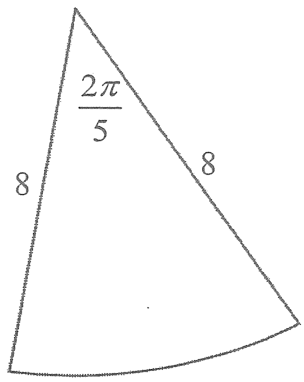
(g) Differentiate $y = e^{2x}$. **1**

(h) Find the equation of the tangent to the curve $y = 2e^x$ at the point where $x = -1$. **2**

Question 7 (15 marks) Start a new writing booklet.

(a) Convert $\frac{7\pi}{9}$ radians to degrees. **1**

(b)



The diagram above shows a sector of a circle. Find the perimeter of the circle, correct to one decimal place. **2**

(c) Find $\tan(2 \cdot 4)$ correct to two decimal places. **1**

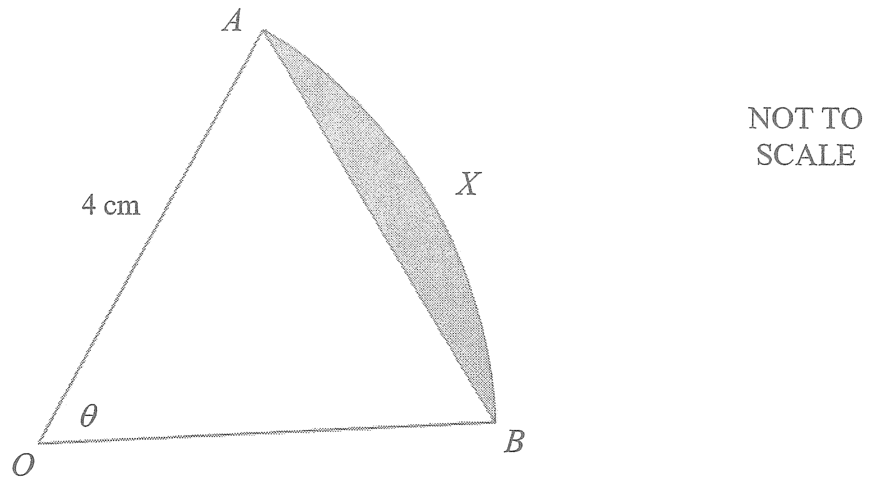
(d) Write down the exact value of $\tan\frac{11\pi}{6}$. **2**

(e) Solve $2\sin\theta - \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$. **2**

Question 6 continues on page 7

Question 7 (continued)

(f)



The area of the sector shown above is 8π cm^2.

(i) Find θ in radians. **2**

(ii) Find the exact area of the shaded segment. **2**

(g) For the function $y = 5\sin\dfrac{x}{2}$,

(i) State the amplitude. **1**

(ii) State the period. **1**

(iii) Sketch its graph in the domain $0 \le x \le 2\pi$. **1**

End of Question 7

Question 8 (15 marks) Start a new writing booklet

(a) Differentiate the following with respect to x:

(i) $y = \log_e \sqrt{2x+1}$. **2**

(ii) $y = 3xe^{4x+1}$. **2**

(iii) $y = \dfrac{\ln 3x}{x^3}$. **2**

(b) (i) Write $\log_e \dfrac{3x+2}{2x-1}$ in an equivalent form using logarithm laws. **1**

(ii) Hence, or otherwise, find the derivative of $y = \log_e \dfrac{3x+2}{2x-1}$ in simplest form **2**

(c) For the function $f(x) = e^{2x}(2-x)$:

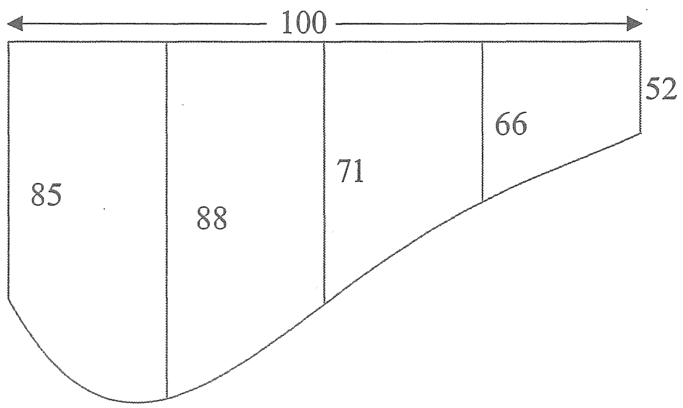
(i) Show that $f'(x) = e^{2x}(3-2x)$ **1**

(ii) Find the stationary point of the curve and determine its nature. **3**

(d) The diagram shows a paddock, bounded on one side by a river. **2**

Use the given measurements and Simpson's rule to approximate the area of the paddock.

Give the answer to the nearest m^2.



NOT TO SCALE

Question 9 (15 marks) Start a new booklet.

(a) Differentiate the following with respect to x:

(i) $\tan(1-2x)$ **2**

(ii) $\sin 3x^2$ **2**

(iii) $\dfrac{x}{\cos x}$ **2**

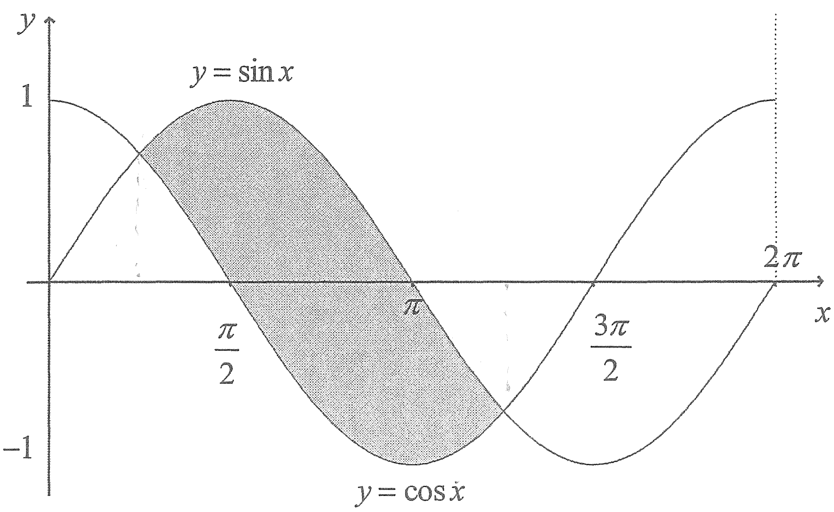
(b) Find:

(i) $\int \cos\frac{x}{3}\,dx$ **1**

(ii) $\int_0^{\frac{\pi}{2}} 4\sin 2x\,dx$ **3**

(c) The gradient function of a curve $y = f(x)$ is given by $f'(x) = 4\cos 4x$. **2**
Find the equation of the curve if it passes through the point $(0,-2)$.

(d) The diagram below shows the curves $y = \cos x$ and $y = \sin x$.



NOT TO SCALE

(i) Show that the curves intersect at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. **1**

(ii) Find the shaded area . **2**

Question 10 (15 marks) Start a new writing booklet.

(a) (i) Differentiate $y = x\cos x$ **1**

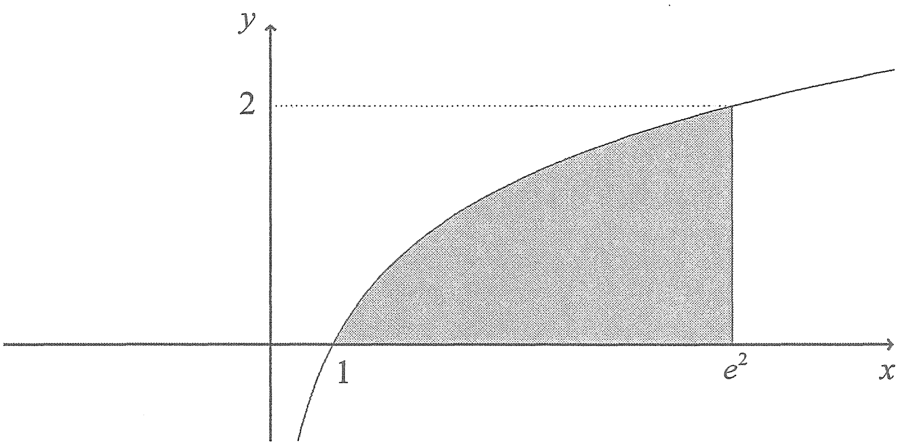
(ii) Hence, or otherwise, find $\int x\sin x dx$. **2**

(b) Evaluate:

(i) $\int_0^1 \frac{x}{x^2+1} dx$. **2**

(ii) $\int_{-1}^{3} e^{\frac{-x}{2}} dx$. **2**

(c) By considering the area against the y-axis, find the area enclosed by the graph of $y = \log_e x$, the x-axis and $x = e^2$. **3**



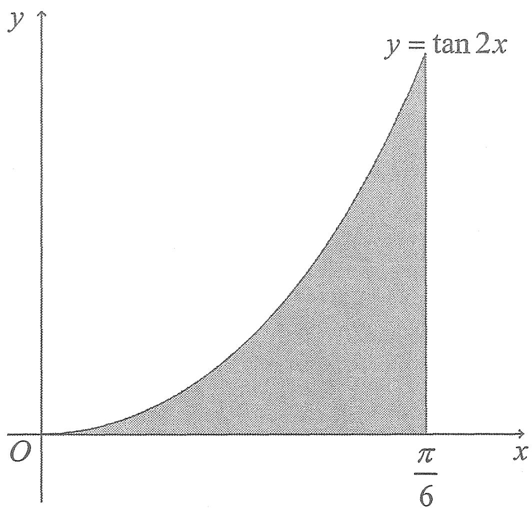
NOT TO SCALE

Question 10 continues on page 11

Question 10 (continued)

(d) (i) By expressing secθ and tanθ in terms of sinθ and cosθ, show that $\sec^2\theta - \tan^2\theta = 1$. 1

(ii)



The diagram shows part of the function $y = \tan 2x$. The shaded region is bounded by the curve, the x-axis, and the line $x = \frac{\pi}{6}$. The region is rotated about the x-axis to form a solid.

(α) Show that the volume of the solid is given by $V = \pi\int_0^{\frac{\pi}{6}} \left(\sec^2 2x - 1\right) dx$ 2

You may use your result from part (i).

(β) Find the exact volume of the solid. 2

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Mathematics 2013 Year 12 Half Yearly

Section 1

1. $x^2 - 2x - 8 = 0$ $\begin{matrix} x & -4 \\ x & 2 \end{matrix}$

$(x-4)(x+2) = 0$

$\therefore x = 4, -2$

$\therefore$ (D)

2. $\log_e\left(\frac{x+2}{x}\right) = \log_e 4$

$\therefore \frac{x+2}{x} = 4$

$x + 2 = 4x$

$2 = 3x$

$\therefore x = \frac{2}{3}$ (B)

3. $\frac{\log_e x}{\log_e 2} = \frac{1}{\log_e 2} \times \frac{1}{x}$

(D)

4. $A = \frac{1}{2} r^2 \theta$

$10\pi = \frac{1}{2} r^2 \times \frac{\pi}{3}$

$60 = r^2$

$\therefore r = \sqrt{60}$ (A)

5. $a = 3$, Period $= \pi$

$\therefore n = 2$

$\therefore f(x) = 3\cos 2x$

(A)

6 a) $5^{4x-3} = 5^0$

$\therefore 4x - 3 = 0$

$4x = 3$

$x = 3/4$

b) $\log_7 5 = \frac{\log_e 5}{\log_e 7}$

$= 0.827087...$

$= 0.83$

c) $3\log 5 - \log 25 = \log 5^3 - \log 25$

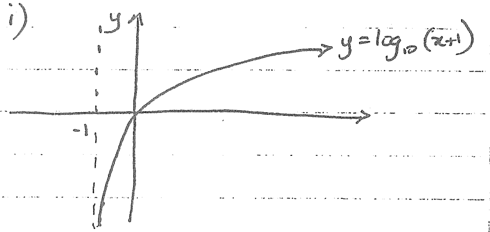
$= \log \frac{125}{25}$

$= \log 5$

d) $3^4 = x$

$\therefore x = 81$

e) i)



ii) domain : $x > -1$

f) $e^{-5} = 6.7379... \times 10^{-3}$

$= 6.74 \times 10^{-3}$

g) $y' = 2e^{2x}$

h) $y = 2e^x$

$y' = 2e^x$

when $x = -1$, $y = 2e^{-1}$

$y' = 2e^{-1}$

$\therefore y - \frac{2}{e} = \frac{2}{e}(x - -1)$

$ey - 2 = 2x + 2$

$\therefore 0 = 2x - ey + 4$

or $\left(y = \frac{2}{e}x + \frac{4}{e}\right)$

7a) $\frac{7\pi}{9} \times \frac{180}{\pi} = 140^\circ$

b) $P = r\theta + 8 + 8$

$= 8 \times \frac{2\pi}{5} + 16$

$= 26.05309...$

$= 26.1$ u.

c) $\tan(2.4)$

$= -0.9160142...$

$= -0.92$ (2 dp)

d) $\tan \frac{11\pi}{6}$ S | A / T | C

$= -\tan\left(2\pi - \frac{\pi}{6}\right)$

$= -\tan\frac{\pi}{6}$

$= -\frac{1}{\sqrt{3}}$

e) $2\sin\theta = \sqrt{3}$ S | A / T | C

$\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{2\pi}{3}$

f) i) $A = \frac{1}{2} r^2 \theta$

$2\pi = \frac{1}{2} \times 4^2 \theta$

$2\pi = 8\theta$

$\therefore \theta = \frac{\pi}{4}$

ii) $A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \times 4^2 \left(\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$

$= 8\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$

$= 8\left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)$

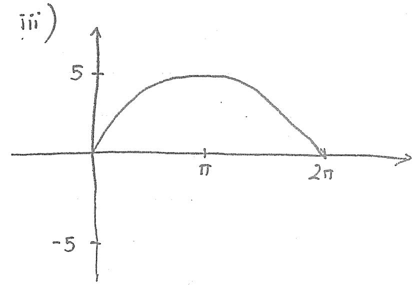
$= 2\pi - 4\sqrt{2}$ u^2

g) i) $a = 5$

ii) $T = \frac{2\pi}{\frac{1}{2}}$

$= 4\pi$

iii)



8 a) i) $y = \log_e (2x+1)^{\frac{1}{2}}$

$\therefore y = \frac{1}{2} \log_e (2x+1)$

$y' = \frac{1}{2} \times \frac{2}{2x+1}$

$= \frac{1}{2x+1}$

ii) $y = 3x e^{4x+1}$

$y' = 3 \times e^{4x+1} + 3x \times 4e^{4x+1}$

$= 3e^{4x+1}(1+4x)$

iii) $y = \frac{\ln 3x}{x^3}$

$y' = \frac{x^{3} \times \frac{3}{3x} - \ln 3x \times 3x^2}{(x^3)^2}$

$= \frac{x^2 - 3x^2 \ln 3x}{x^6}$

$= \frac{x^2(1-3\ln 3x)}{x^{6}}$

$= \frac{1-3\ln 3x}{x^4}$

b) i) $\log_e \frac{3x+2}{2x-1} = \log_e (3x+2) - \log_e (2x-1)$

ii) $y' = \frac{3}{3x+2} - \frac{2}{2x-1}$

$= \frac{3(2x-1) - 2(3x+2)}{(3x+2)(2x-1)}$

$= \frac{-7}{(3x+2)(2x-1)}$

c) $f(x) = e^{2x}(2-x)$

i) $f'(x) = e^{2x} \times -1 + 2e^{2x}(2-x)$

$= e^{2x}(-1 + 2(2-x))$

$= e^{2x}(3-2x)$

ii) stat. pt when $f'(x) = 0$

ie. $e^{2x}(3-2x) = 0$

$3 - 2x = 0$

$3 = 2x$

$\therefore x = \frac{3}{2}$

when $x = \frac{3}{2}$ $y = e^{2 \times \frac{3}{2}}(2 - \frac{3}{2})$

$= \frac{e^3}{2}$

x	1·4	1·5	1·6
$f'(x)$	3·288	0	-83·4

$\therefore$ max.

$\therefore$ maximum turning point at $\left(\frac{3}{2}, \frac{e^3}{2}\right)$

d) $A \doteq \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$

0	25	50	75	100
85	88	71	66	52

$= \frac{50-0}{6}\left\{85 + 4 \times 88 + 2 \times 71 + 4 \times 66 + 52\right\}$

$= 7458\frac{1}{3}$ m²

≈ 7458 m²

1a) i) $\frac{d}{dx}\left(\tan(1-2x)\right)$

$= -2\sec^2(1-2x)$

ii) $\frac{d}{dx}\left(\sin 3x^2\right)$

$= 6x\cos 3x^2$

iii) $\frac{d}{dx}\left(\frac{x}{\cos x}\right)$

$= \frac{\cos x \times 1 - x \times -\sin x}{(\cos x)^2}$

$= \frac{\cos x + x\sin x}{\cos^2 x}$

b) i) $\int \cos\frac{x}{3}\,dx$

$= 3\sin\frac{x}{3} + c$

ii) $\int_0^{\frac{\pi}{2}} 4\sin 2x\,dx$

$= \left[-\frac{4}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left(-2\cos\frac{2\pi}{2} - -2\cos 0\right)$

$= 2 + 2$

$= 4$

c) $f'(x) = 4\cos 4x$

$f(x) = \frac{4}{4}\sin 4x + c$

when $x = 0$, $f(x) = -2$

ie. $-2 = \sin 0 + c$

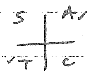
$\therefore c = -2$

$\therefore f(x) = \sin 4x - 2$

d) $y = \cos x$ $\quad y = \sin x$

i) $\therefore \cos x = \sin x$

$\therefore 1 = \frac{\sin x}{\cos x}$



ie. $\tan x = 1$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$= \frac{\pi}{4}, \frac{5\pi}{4}$

ii) $A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$

$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$

$= \frac{4}{\sqrt{2}}$ or $2\sqrt{2}\ u^2$

10a) i) $y = x\cos x$

$y' = x \times -\sin x + 1 \times \cos x$

$= \cos x - x\sin x.$

ii) $\int x\sin x\,dx = -\int(\cos x - x\sin x)\,dx + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

b) i) $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$

$= \frac{1}{2}\left[\ln 2 - \ln 1\right]$

$= \frac{1}{2}\ln 2$

ii) $\int_{-1}^{3} e^{-\frac{x}{2}}\,dx = \left[-2e^{-\frac{x}{2}}\right]_{-1}^{3}$

$= -2e^{-\frac{3}{2}} - -2e^{\frac{1}{2}}$

$= -2e^{-\frac{3}{2}} + 2e^{\frac{1}{2}}$ $\quad$ (or $2\left(e^{\frac{1}{2}} - e^{-\frac{3}{2}}\right)$) etc.

c) If $y = \log_e x \Rightarrow x = e^y$

$\therefore$ Area against y-axis : $\quad A = \int_0^2 e^y\,dy$

$= \left[e^y\right]_0^2$

$= e^2 - e^0$

$= e^2 - 1$

Now enclosed area is rectangle - area against y-axis

ie. $2e^2 - (e^2 - 1)$

$= e^2 + 1\ u^2$

) i). $\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\therefore \sec^2\theta - \tan^2\theta = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{1-\sin^2\theta}{\cos^2\theta}$

$= \frac{\cos^2\theta}{\cos^2\theta}$

$= 1$

ii) α) $V = \pi \int_a^b y^2 \, dx$

$y = \tan 2x$

$\therefore y^2 = \tan^2 2x$

$= \pi \int_0^{\frac{\pi}{6}} (\sec^2 2x - 1) \, dx$

from (i) $\sec^2\theta - \tan^2\theta = 1$

$\therefore \tan^2\theta = \sec^2\theta - 1$

hence

$\tan^2 2x = \sec^2 2x - 1$

β) $V = \pi \left[\tan^2 2x - x \right]_0^{\frac{\pi}{6}}$

$= \pi \left[\left(\tan^2 \frac{2\pi}{6} - \frac{\pi}{6} \right) - \left(\tan^2 0 - 0 \right) \right]$

$= \pi \left[\sqrt{3} - \frac{\pi}{6} \right]$

$= \sqrt{3}\pi - \frac{\pi^2}{6} \; u^3$ or $\frac{6\sqrt{3}\pi - \pi^2}{6}$