Name : ____________________

Class: ______________

HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12 2011

MATHEMATICS HSC TASK 2

HALF YEARLY EXAMINATION

Examiners: P. Biczo, S. Gee, B. Morrison, D. Crancher

General Instructions

- Reading time : 5 minutes
- **Working time : 2 hours**
- Attempt **all** questions.
- **Start a new answer booklet for each question.**
- All necessary working should be shown.
- This paper contains 5 questions worth 15 marks each. Total Marks: **75 marks**
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators and mathematical templates may be used.
- This examination paper must **not** be removed from the examination room.

Marks

QUESTION 1. *Start a new answer booklet.*

(a) Find correct to 2 significant figures $\dfrac{4\cdot 23}{\sqrt{6\cdot 14-3\cdot 78}}$ **2**

(b) Find integers a and b such that $\dfrac{2}{2-\sqrt{3}}=a+\sqrt{b}$ **2**

(c) If $S=\dfrac{a}{1-r}$ find S if $a=25$ and $r=\dfrac{2}{3}$ **1**

(d) Evaluate $|-3|-|5|$ **1**

(e) Given: $(x+2)^2=12(y-3)$

(i) Find the coordinates of the vertex. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Write the equation of the directrix. **1**

(iv) Sketch $(x+2)^2=12(y-3)$ clearly labeling the focus and directrix. **1**

(f) (i) Show the locus of the point $P(x, y)$ that moves so that it is always twice the distance from the point $A(2, 1)$ as it is from the point $B(-4, -5)$ is a circle. **3**

(ii) Find the centre and the radius of the circle defined in part (i). **2**

Marks

QUESTION 2. *Start a new answer booklet.*

(a) Sketch the curve, showing the coordinates of the vertex, and any intercepts with the axes:

$$y=(x+2)^2+1$$ **2**

(b) Find the maximum value of $5+4x-x^2$ **2**

(c) (i) Write down the discriminant of $2x^2+(k-2)x+8$, where k is a constant. **1**

(ii) Hence, or otherwise, find the values of k for which the parabola $y=2x^2+kx+9$ does not intersect the line $y=2x+1$. **2**

(d) By using a suitable substitution solve for x: $(x-1)^4-11(x-1)^2+18=0$ **3**

(e) Find the values of a, b and c given that $3x^2-5x+7\equiv a(x-1)^2+b(x-1)+c$ **3**

(f) Let α and β be the roots of the equation $x^2+6x+1=0$.

(i) Find $\alpha\beta$ **1**

(ii) Hence, find $\alpha+\dfrac{1}{\alpha}$ **1**

Marks

QUESTION 3. *Start a new answer booklet.*

(a) It is given that the sequence $\log 9, \log 27, \log 81, \log 243, \ldots$

is either arithmetic or geometric. Which is it?

Justify your answer and state the common difference or ratio. **2**

(b) The first three terms of an arithmetic series are 12, 17 and 22.

(i) Find the twenty-fifth term of this series. **1**

(ii) Find the sum of the first twenty-five terms. **1**

(c) Consider the series $10 + 22 + 34 + 46 + \ldots$

How many terms are required to give a sum of 4816? **3**

(d) The first term of a geometric series is 16 and the fourth term is $\frac{1}{4}$.

(i) Find the common ratio. **2**

(ii) Find the limiting sum of the series. **1**

(iii) Explain why the limiting sum of this series exists. **1**

(e) The sequence $5, 11, 29, \ldots$ has as its nth term $3^n + 2$.

(i) Find the fourth term **1**

(ii) Evaluate $\sum_{n=1}^{4}\left(3^n + 2\right)$ **1**

(iii) Write an expression for the sum of the first n terms, S_n, and hence show that

$$S_n = \frac{3^{n+1} + 4n - 3}{2}$$ **2**

Marks

QUESTION 4. *Start a new answer booklet.*

(a) $P(-3,4)$ and $Q(3,-6)$ are the coordinates of the diameter PQ of a circle.

(i) Find the midpoint, M, of PQ. **1**

(ii) Find the exact length of PM. **1**

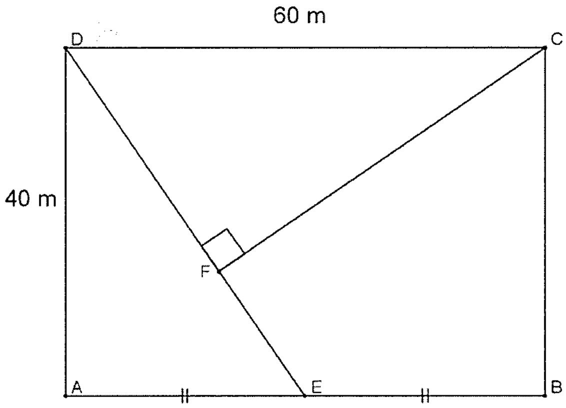
(iii) Write down the equation of the circle. **1**

(iv) Determine the gradient of the diameter PQ. **1**

(v) Find the distance of M to the line $2x - 3y - 1 = 0$ **2**

(vi) Does the line $2x - 3y - 1 = 0$ and the circle meet in 0, 1 or 2 points? Justify your answer. **1**

(b)

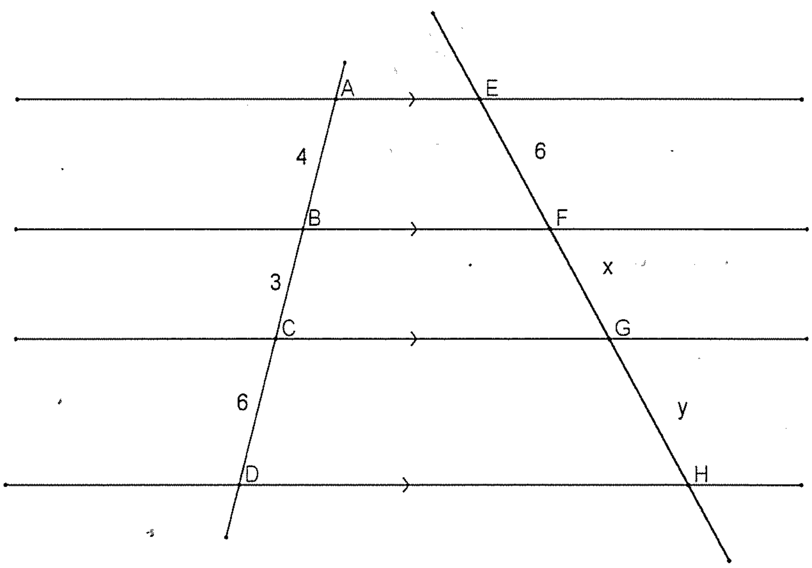


$ABCD$ is a rectangle with $CD = 60$ metres and $AD = 40$ metres.
E is the midpoint of AB and D is joined to E.
CF is drawn perpendicular to DE as shown.

(i) Prove that triangles DAE and CFD are similar. **3**

(ii) Determine the length of CF. **2**

(c)



AE, BF, CG and DH are parallel lines with transversals AD and EH.
$AB = 4$, $BC = 3$, $CD = 6$ and $EF = 6$
Find the values of x and y, giving appropriate reasoning. **3**

Marks

QUESTION 5. *Start a new answer booklet.*

(a) Find the derivative of

(i) $y = (x^2 - 9)^3$ 1

(ii) $y = \dfrac{x}{x^2 + 1}$ 2

(b) A function $f(x)$ is defined by $f(x) = 2x^2(3 - x)$

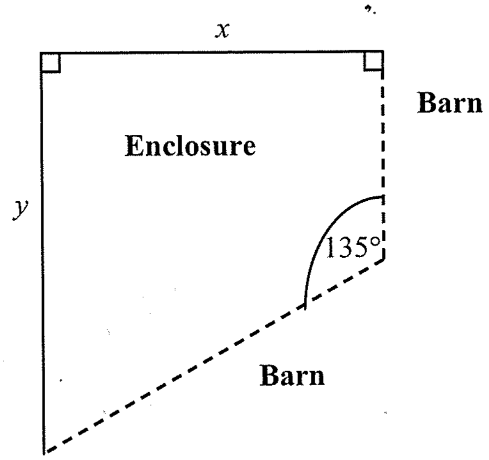
(i) Find the coordinates of the turning points of $y = f(x)$ and determine their nature. 3

(ii) Prove there is a point of inflexion at (1, 4). 1

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x axis in the domain $-1 \le x \le 4$. 2

(iv) What is the minimum value of $f(x)$ for $-1 \le x \le 4$? 1

(c) An enclosure is to be built adjoining a barn, as in the diagram. The walls of the barn meet at an angle of $135°$, and 117 metres of fencing is available for the enclosure, so that $x + y = 117$ where x and y are shown in the diagram.



(i) Show that the area of the enclosure in square metres is given by $A = 117x - \dfrac{3}{2}x^2$. 2

(ii) Use calculus to show that the largest area of the enclosure occurs when $y = 2x$. 3

END OF EXAMINATION

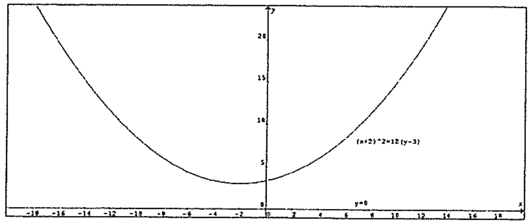
Year 12 Mathematics Task 2 Half Yearly Examination 2011

Question No. 1 Solutions and Marking Guidelines

Outcomes Addressed in this Question

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

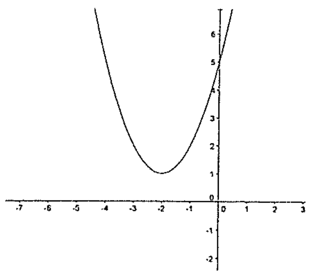
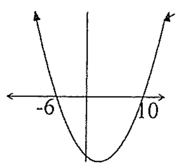
Outcome	Solutions	Marking Guidelines
P3	**a** $\frac{4\cdot 23}{\sqrt{6\cdot 14-3\cdot 78}}=2\cdot 753495467=2\cdot 8(2sig\ fig)$	2 marks correct answer 1 mark correct rounding
P3	**b** $\frac{2}{2-\sqrt{3}}=a+\sqrt{b}$ $\frac{2}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{4+2\sqrt{3}}{4-3}$ $4+\sqrt{12}=a+\sqrt{b}$ $\therefore a=4, b=12$	2 marks correct method leading to correct solutions 1marks substantially correct method
P4	**c** $S=\frac{a}{1-r}$ find S if $a=25$ and $r=\frac{2}{3}$ $S=\frac{25}{1-\frac{2}{3}}=75$	1 mark correct answer
P3	**d** $\|-3\|-\|5\|=3-5=-2$	1 mark correct answer
P4	**e** $(x+2)^2=12(y-3)$ (i)the vertex $(-2,3)$ (ii) the focus $4a=12$ $a=3$ $(-2,6)$ (iii)the equation of the directrix $y=0$	1 mark correct answer 1 mark correct answer 1 mark correct answer
P4		1 mark correct graph

Outcome	Solutions	Marking Guidelines
P4	**f** (i) $PA=2PB$ $\sqrt{(x-2)^2+(y-1)^2}=2\sqrt{(x+4)^2+(y+5)^2}$ $(x-2)^2+(y-1)^2=4\left[(x+4)^2+(y+5)^2\right]$ $x^2-4x+4+y^2-2y+1=4\left[x^2+8x+16+y^2+10y+25\right]$ $3x^2+36x+3y^2+42y+159=0$ $x^2+12x+36+y^2+14y+49=-\frac{159}{3}+36+49$ $(x+6)^2+(y+7)^2=32$ which is a circle.	3 marks correct method leading to correct solutions 2 marks substantially correct method 1 mark elementary progress towards correct solution
P4	(ii) circle centre $(-6,-7)$ radius $\sqrt{32}=4\sqrt{2}$	2 marks correct answer from previous working 1 marks one correct answer

Year 12 Half Yearly | Mathematics | Examination 2011
Question No. 2 | Solutions and Marking Guidelines

Outcomes Addressed in this Question

P4 Chooses and applies appropriate algebraic and graphical techniques
P5 Understands the concept of a function and the relationship between a function and its graph
H9 Communicates using mathematical language, notation, diagrams and graphs

Outcome	Solutions	Marking Guidelines
P 5	(a)  The parabola $y=(x+2)^2$ touches the x axis at $x=-2$ and is concave up. $y=(x+2)^2+1$ is the graph of $y=(x+2)^2$ shifted up 1 unit. $\therefore$ Vertex is $(-2,1)$. $y=(x+2)^2+1$ cuts the y axis when $x=0$, i.e. at $y=5$.	2 marks : correct graph, with vertex and y intercept marked correctly. 1 mark : substantial progress toward correct graph
P 4	(b) If $y=5+4x-x^2$ which is a concave down parabola, the maximum value occurs at the axis of symmetry $x=\frac{-b}{2a}=\frac{-4}{2\times -1}$ i.e. at $x=2$ When $x=2$, $y=5+8-4=9$ $\therefore$ maximum value is 9.	2 marks : correct solution 1 mark : substantial progress toward correct graph
P 4	(c) (i) $\Delta=b^2-4ac$ $\Delta=(x-2)^2-4\times 2\times 8$ $\Delta=x^2-4x+4-64$ $\therefore \Delta=x^2-4x-60$	1 mark : correctly finds Δ
P 4	(ii) $y=2x^2+kx+9$ and $y=2x+1$ intersect when $2x^2+kx+9=2x+1$ i.e. when $2x^2+(k-2)x+8=0$ As there is no solution (don't intersect), $\Delta<0$. From (i), $x^2-4x-60<0$ $(k+6)(k-10)<0$ From the graph $-6<k<10$ 	2 marks : correct solution 1 mark : substantial progress toward correct solution
P 4	(d) $(x-1)^4-11(x-1)^2+18=0$ Let $m=(x-1)^2$, then $m^2-11m+18=0$ $(m-2)(m-9)=0$ $\therefore m=2$ or 9 $\therefore (x-1)^2=2$ or $(x-1)^2=9$ $\therefore x-1=\pm\sqrt{2}$ or $x-1=\pm 3$ $\therefore x=\pm\sqrt{2}+1$ or $x=\pm 3+1$ $\therefore x=\pm\sqrt{2}+1, -2$ or 4.	3 marks : correct solution 2 marks : substantial progress toward correct solution 1 mark : finds suitable substitution
P 4	(e) $3x^2-5x+7\equiv a(x-1)^2+b(x-1)+c$ $\equiv a(x^2-2x+1)+bx-b+c$ $\equiv ax^2-2ax+a+bx-b+c$ $\equiv ax^2+(-2a+b)x+(a-b+c)$ Equating like coefficients, $a=3$ $-5=-2a+b$ $7=a-b+c$ $\therefore -5=-2\times 3+b$ $\therefore 7=3-1+c$ $\therefore b=1$ $\therefore c=5$ $\therefore a=3,\ b=1,\ c=5$.	3 marks : correct solution 2 marks : substantial progress toward correct solution 1 mark : uses an appropriate method to correctly evaluate one of the pronumerals.
P 4	(f) For $x^2+6x+1=0$, $a=1$, $b=6$, $c=1$. (i) $\alpha\beta=\frac{c}{a}=\frac{1}{1}$. $\therefore \alpha\beta=1$	1 mark : correct answer
	(ii) From (i), $\alpha\beta=1$, $\therefore \beta=\frac{1}{\alpha}$. $\alpha+\frac{1}{\alpha}=\alpha+\beta=\frac{-b}{a}=\frac{-6}{1}$. $\therefore \alpha+\frac{1}{\alpha}=-6$.	1 mark : correct solution

Year 12 Mathematics Half Yearly Examination 2011

Question No. 3 Solutions and Marking Guidelines

Outcomes Addressed in this Question

H5- Applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems.

H9 communicates using mathematical language, notation, diagrams and graphs

Outcome	Solutions	Marking Guidelines
H5	**3.** **a)** $T_1 = \log 9 = \log(3^2) = 2\log 3$ $T_2 = \log 27 = \log(3^3) = 3\log 3$ $T_3 = \log 81 = \log(3^4) = 4\log 3$ $T_4 = \log 243 = \log(3^5) = 5\log 3$ <u>Test if arithmetic</u> $T_4 - T_3 = 5\log 3 - 4\log 3 = \log 3$ $T_3 - T_2 = 4\log 3 - 3\log 3 = \log 3$ Since $T_4 - T = T_3 - T_2 = \log 3$, then this sequence is an arithmetic sequence with common difference of $d = \log 3$.	2 marks for complete correct solution [must show common difference is $d = \log 3$]. 1 mark for completing the correct solution but did not have $d = \log 3$ as the common difference.
H5	**b)** **i)** $T_{25} = 12 + (25-1)5 = 132$	1 mark for correct answer
H5	**ii)** $S_{25} = \frac{25}{2}(12+132) = 1800$	1 mark for correct answer
H5	**c)** $\frac{n}{2}\left[2(10)+(n-1)12\right] = 4816$ $20n + 12n^2 - 12n = 9632$ $3n^2 + 2n - 2408 = 0$ $(3n+86)(n-28) = 0$ n must be a positive integer $\therefore n = 28$	3 marks for complete correct solution 2 marks for partial correct solution 1 mark for substituting correctly i.e. $\frac{n}{2}\left[2(10)+(n-1)12\right] = 4816$
H5	**d)** **i)** $16r^3 = \frac{1}{4}$ $r^3 = \frac{1}{64}$ $r = \frac{1}{4}$	2 marks for complete correct solution 1 marks for partial correct solution
H5	**ii)** $S_\infty = \frac{16}{1-\frac{1}{4}} = \frac{64}{3}$	1 mark for correct answer
H5	**iii)** The limiting sum of this series exists since $\lvert r \rvert < 1$	1 mark for correct answer
H5	**e)** **i)** $T_4 = 3^4 + 2 = 83$	1 mark for correct answer
H5	**ii)** $\sum_{n=1}^{4}(3^n+2)$ $= (3^1+2)+(3^2+2)+(3^3+2)+(3^4+2)$ $= 5+11+29+83$ $= 128$	1 mark for correct answer
H5 H9	**iii)** $S_n = \sum_{n=1}^{4}(3^n+2)$ $= (3^1+2)+(3^2+2)+(3^3+2)+(3^4+2)$ $= (3+2)+(9+2)+(27+2)+(81+2)$ $= \underbrace{(3+9+27+81)}_{\text{This is a geometric series with } r=3 \text{ and } a=3} + 2n$ $= \frac{3(1-3^n)}{1-3} + 2n$ $= \frac{3-3^{n+1}}{-2} + 2n$ $= \frac{3^{n+1}-3}{2} + 2n$ $= \frac{3^{n+1}+4n-3}{2}$	2 marks for complete correct solution 1 marks for partial correct solution

Year 12 **Mathematics Advanced** **Half Yearly Exam 2011**

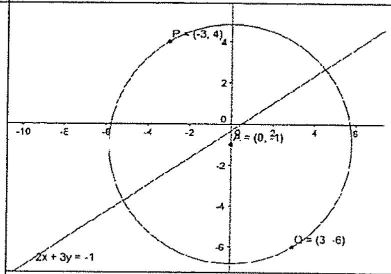
Question No. 4 **Solutions and Marking Guidelines**

Outcomes Addressed in this Question

P2 provides reasoning to support conclusions which are appropriate to the context

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities.

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

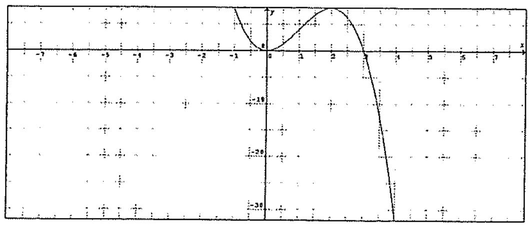
Outcome	Solutions	Marking Guidelines
P2 P3 P4		Diagram provided for general reference only. Not required for answer.
	(a) (i) $M=\{\frac{(-3+3)}{2},\frac{(4+-6)}{2}\}$ $M=(0,-1)$	**1 mark:** Both coordinates must be found correctly.
	(a) (ii) $\|PM\|=\sqrt{(-3-0)^2+(4--1)^2}$ $\|PM\|=\sqrt{34}$	**1 mark:** Correct solution.
	(a) (iii) $Radius=\|PM\|, centre=(0,-1)$ $Equation\ of\ circle\ x^2+(y+1)^2=34$	**1 mark:** Correct solution.
	(a) (iv) $Gradient\ of\ line\ PQ=\frac{4--6}{-3--3}$ $=\frac{-5}{3}$	**1 mark:** Correct solution.
	(a) (v) $Distance=\frac{\|2\times0-3\times-1-1\|}{\sqrt{2^2+(-3)^2}}$ $=\frac{2}{\sqrt{13}}$ $=\frac{2\sqrt{13}}{13}$	**2 marks:** Correct solution, did not need to be rationalised. **1 mark:** Correct substitution into distance formula
	(a) (vi) The distance from the centre of the circle to the secant is less than the radius, which means that the secant must cut the circle in exactly two places. (Note: if this distance had equalled the radius we would have had a tangent which would mean one point of contact; while if the distance had been greater than the radius there would have been no points of contact.) `	**1 mark:** Statement that the distance from the centre to the line is less than the radius. Some students drew the graph and argued from that. Other students proved that there were 2 valid points of intersection by solving the 2 equations simultaneously

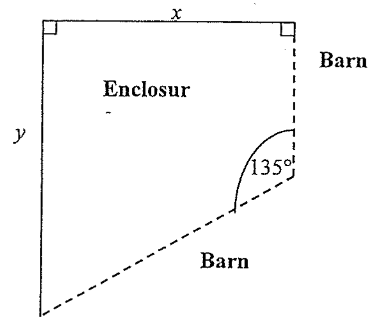
	(b) (i) Prove triangle DAE is similar to triangle CFD Proof: angle DEA = angle CDF (alternate angles are equal as AB is parallel to DC) angle DAE = angle CFD (both 90^0, given in information) angle ADE = angle FCD (angle sum of triangle equals 180^0) Hence triangle DAE is similar to triangle CFD (equiangular)	**3 marks:** Correct solution. Note: The statement of the equality of the third pair of angles is not required. **2 marks:** Substantial progress. **1 mark:** Some progress Note: Abbreviations such as AAA or AA were not acceptable.
	(b) (ii) In right angled triangle DAE, AE = 30 m (E bisects AB), hence, by Pythagoras, DE = 50 m. As triangles DAE and CFD are similar $\frac{CF}{DA}=\frac{CD}{DE}$ (*corresponding sides of similar triangles*) $\frac{CF}{40}=\frac{60}{50}\ \therefore\ CF=48m$	**2 marks:** Correct solution. **1 mark:** Correct answer but not giving a reason why the ratios were equal. Note: A number of students formed incorrect ratios.
	(a) 4:3:6 = 6:x:y (intercepts on parallel lines are in the same ratio) $\frac{3}{4}=\frac{x}{6}\ and\ \frac{6}{4}=\frac{y}{6}$ $Hence\ x=4\frac{1}{2}\ and\ y=9$	**3 marks:** For correct answers and the statement about intercepts and parallel lines. **2 marks:** For the correct answers and a poor statement about the property. Note: A number of students did not mention the 3 key words of intercepts, parallel and ratio

Year 12	Mathematics	Task 2 Half Yearly Examination 2011
Question No. 5	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P5	understands the concept of a function and the relationship between a function and its graph
P7	determines the derivative of a function through routine application of the rules of differentiation
H4	expresses practical problems in mathematical terms based on simple given models
H5	applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6	uses the derivative to determine the features of the graph of a function
H9	communicates using mathematical language, notation, diagrams and graphs

Outcome	Solutions	Marking Guidelines
P7 H9	**A(*i*)** $y=\left(x^2-9\right)^3$ $\frac{dy}{dx}=3\left(x^2-9\right)^2\times 2x=6x\left(x^2-9\right)^2$	1 mark correct answer
	(*ii*) $y=\frac{x}{x^2+1}$ $y'=\frac{x^2+1-x\times 2x}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$	2 marks correct method leading to correct solution 1marks substantially correct method
P5 H9	**b** (*i*) $f(x)=2x^2(3-x)$ $f(x)=6x^2-2x^3$ $f'(x)=12x-6x^2$ $f''(x)=12-12x$ For stationary points $f'(x)=0$ $12x-6x^2=0$ $6x(2-x)=0$ $x=0, x=2$ $f''(0)=12-12\times 0=12>0, \therefore$ relative minima at $(0,0)$ $f''(2)=12-12\times 2=-12<0, \therefore$ relative maxima at $(2,8)$	3 marks correct method leading to correct solution including proof of the nature of the turning points. 2 marks substantially correct method 1 mark correct differentiations
H6	(*ii*) $f''(x)=12-12x$ $f''(1)=12-12\times 1=0$, possible point of inflexion. Test concavity change. $f''(0)=12-12\times 0=12>0, \therefore$ concave up. $f''(2)=12-12\times 2=-12<0, \therefore \therefore$ concave down and concavity changes. $\therefore$ inflexional point at $(1,4)$.	1 mark must test change of concavity.
	(*iii*) 	2 mark correct graph illustrating intercepts, end points and turning and inflexional points. Note inflexional point must not look like a horizontal point of inflexion. 1marks substantially correct graph

	(*iv*) $) = 2x^2(3-x)$ $f(4)=2(4)^2(3-4)=-32$	1 mark correct answer
H4 H5	**(c)**  Total Area= Area rectangle plus area right angled isosceles triangle equal sides x units $A=x\left(y-x\right)+\frac{1}{2}x^2$ but $x+y=117 \therefore y=117-x$ $A=x\left(117-x-x\right)+\frac{1}{2}x^2$ $A=117x-2x^2+\frac{1}{2}x^2$ $A=117x-\frac{3}{2}x^2$	2 marks correct method leading to correct solution 1 marks substantially correct method
H9	(*ii*) $A=117x-\frac{3}{2}x^2$ $\frac{dA}{dx}=117-3x$ $\frac{d^2A}{dx^2}=-3 \therefore$ relative maxima for relative maximum or minimum $\frac{dA}{dx}=0$. $117-3x=0$ $x=\frac{117}{3}=39$ but $y=117-x$ $y=117-39=78$ $78=2\times 39$ $\therefore y=2x$ gives relative maxima.	3 marks correct method leading to correct solution including proof of the nature of the turning points. 2 marks substantially correct method 1 mark elementary progress towards solution