HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS

2012

YEAR 12

HALF YEARLY EXAMINATION

(ASSESSMENT TASK 2)

EXAMINERS ~ S. FAULDS, P. BICZO, S. CUPAC, G. RAWSON

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 2 hours.
- Attempt **all** questions.
- Board approved calculators and MathAids may be used.
- This examination must **NOT** be removed from the examination room
- **Section A** consists of eight (8) multiple choice questions worth 1 mark each. Fill in your answers on the multiple choice answer sheet provided.
- **Section B** requires **all** necessary working to be shown in every question. This section consists of four (4) questions worth 15 marks each. Marks may not be awarded for careless or badly arranged work. **Each question is to be started in a new answer booklet**. Additional booklets are available if required.

STUDENT NAME: ______________________________

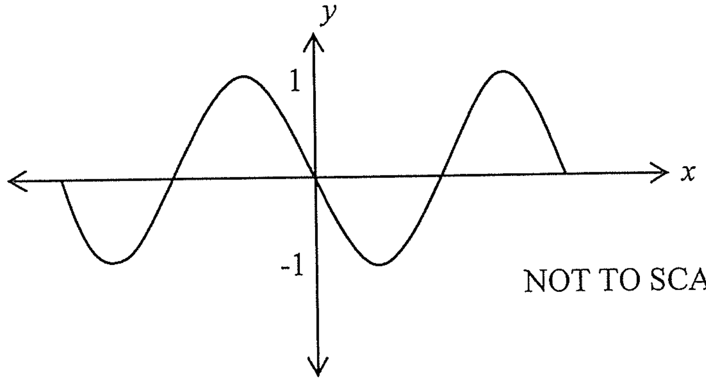
CLASS TEACHER: ______________________________

SECTION A – Multiple Choice (8 marks).

Mark the correct responses for Questions 1 – 8 on the multiple choice answer sheet provided.

QUESTION 1

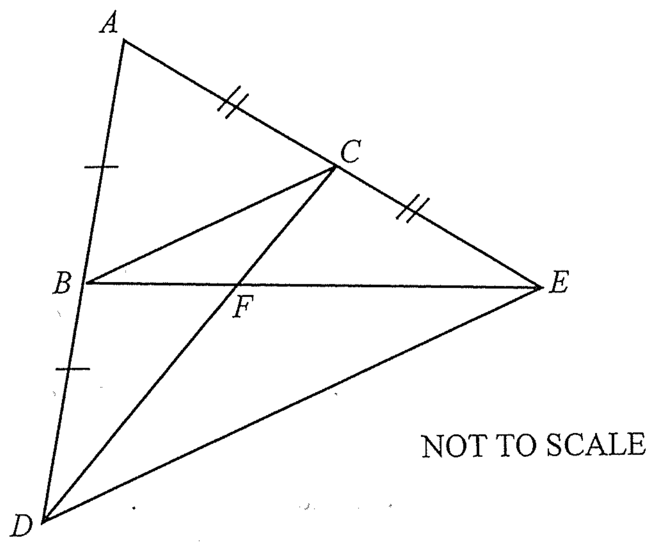
Which trigonometric function could be represented by the following curve?



A: $y = \cos x$ **B:** $y = \sin x$ **C:** $y = -\cos x$ **D:** $y = -\sin x$

QUESTION 2

Why is ΔABC similar to ΔADE?



A: sides about equal angles are in the same ratio

B: equiangular

C: three pairs of sides are in the same ratio

D: corresponding sides are in the same ratio

QUESTION 3

Which of the following is true for the equation $3x^2 - x - 2 = 0$?

A: no real roots

B: one real root

C: two real distinct roots

D: three real roots

QUESTION 4

Let α and β be the roots of the equation $3x^2 - 7x + 12 = 0$.
What is the value of $\alpha + \beta$?

A: $-\frac{7}{3}$

B: $\frac{7}{3}$

C: 4

D: 7

QUESTION 5

The sum of n terms of a series is given by:

$$S_n = n^2 + 4n$$

The n-th term of the series will be:

A: $T_n = n^2 + 4n$

B: $T_n = 4n + 1$

C: $T_n = 3n + 2$

D: $T_n = 2n + 3$

QUESTION 6

The series:

$$a, \frac{3a^2}{2}, \frac{9a^3}{4}, \frac{27a^4}{8}, \ldots$$

will have a limiting sum provided:

A: $\frac{T_3}{T_2} < \frac{T_2}{T_1}$

B: $|a| < \frac{3}{2}$

C: $|a| < \frac{2}{3}$

D: $|a| < 1$

QUESTION 7

Ahn completed the table below to use the first derivative test to determine the nature of the stationary point on $y = f(x)$.

x	-2	-1	0
y'	$\frac{1}{3}$	0	-4

At $x = -1$ there is a:

A: maximum point

B: minimum point

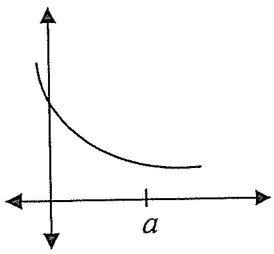
C: horizontal point of inflexion

D: point of inflexion

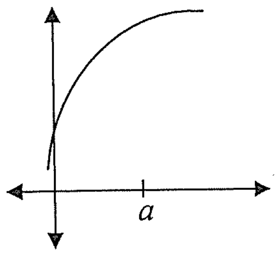
QUESTION 8

The graph which shows $f'(a) > 0$ and $f''(a) < 0$ is:

A:



B:



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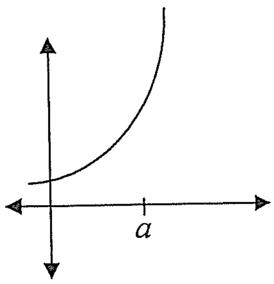
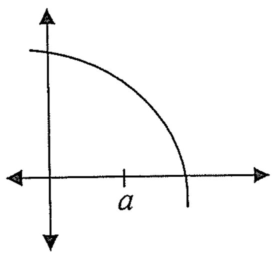


D:



END OF SECTION A

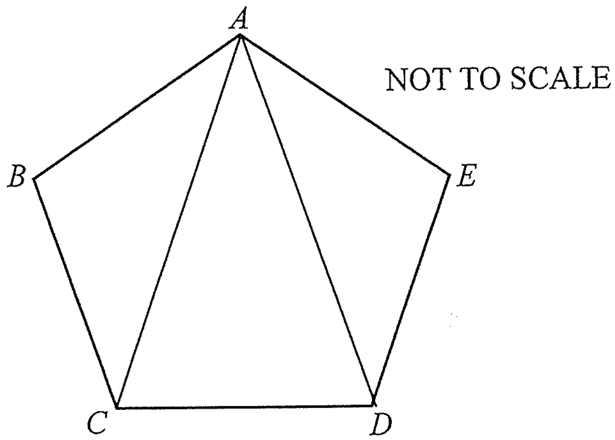
SECTION B – Four (4) free response questions of equal value.

Attempt all questions.
Show all working.
Start each question in a separate answer booklet.
At the end of the examination you must hand in four answer booklets labelled Question 1, Question 2, Question 3 and Question 4 plus your multiple choice answer sheet.

QUESTION 1 (Start a new answer booklet.) **15 marks**

Marks

(a)

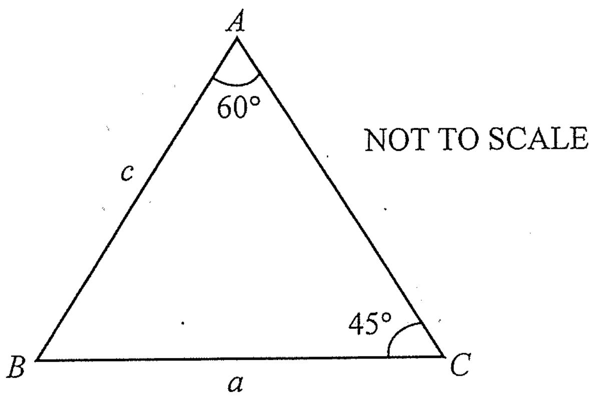


The diagram shows a regular pentagon *ABCDE*.

(i) Which congruence test could be used to prove $\Delta ABC \equiv \Delta AED$? **1**

(ii) Find the size of $\angle ABC$. **1**

(b)



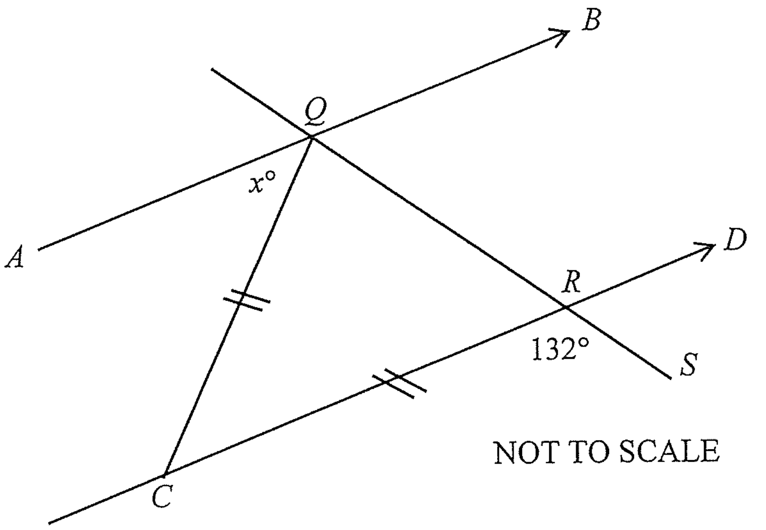
In the diagram, *ABC* is a triangle where $\angle ACB = 45°$ and $\angle BAC = 60°$. **2**
Find the exact value for the ratio $\frac{a}{c}$.

Question 1 continues on next page…

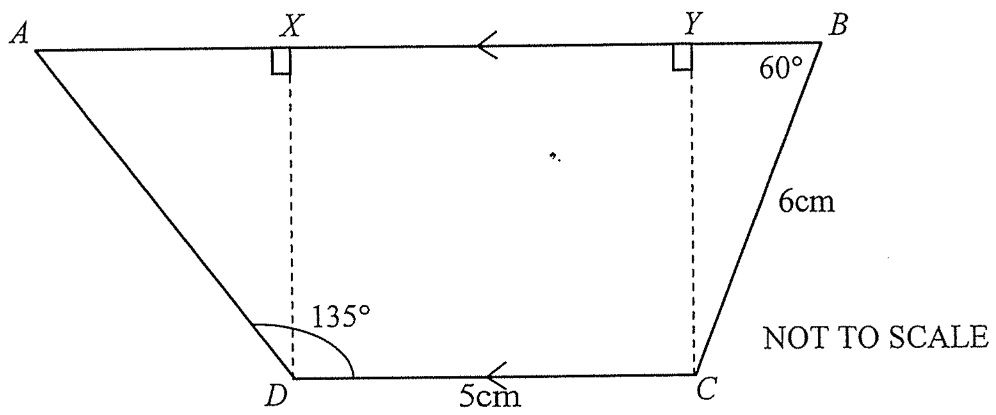
Question 1 (continued) **Marks**

(c) In the diagram, AB is parallel to CD, $QC = RC$, $\angle CRS = 132°$ and $\angle AQC = x°$. **3**

Find the value of x, giving complete reasons.



(d)



The diagram shows a trapezium $ABCD$ in which AB is parallel to DC. $DC = 5\text{cm}$, $BC = 6\text{cm}$, $\angle ABC = 60°$ and $\angle ADC = 135°$.

Perpendiculars are drawn from D and C to meet AB at X and Y.

(i) Why is $BY = 3\text{cm}$. **2**

(ii) Show that $CY = 3\sqrt{3}$ cm. **2**

(iii) Find the exact value of AB. **2**

(iv) Find the area of the trapezium $ABCD$ (leave your answer in surd form). **2**

Marks

QUESTION 2 (Start a new answer booklet.) **15 marks**

(a) Consider the parabola $x^2 = 8(y-3)$

(i) Write down the coordinates of the vertex. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Sketch the parabola showing the vertex and focus. **1**

(b) A point, P(x, y), moves on the number plane so that PA = PB where A and B are the points (3, 3) and (–2, 1) respectively. What is the equation of the locus of P? **2**

(c) Find values for A, B and C if $x^2 - 3x + 7$ is to be written in the form $A(x+2)^2 + B(x+2) + C$. **3**

(d) Find the values of k for which the quadratic equation $3x^2 + 2x + k = 0$ has no real roots. **2**

(e) (i) Show that for all values of m, the line $y = mx - 3m^2$ touches the parabola $x^2 = 12y$. **2**

(ii) Find the values of m for which this line passes through the point (5, 2). **2**

(iii) Hence determine the equations of the two tangents to the parabola $x^2 = 12y$ from the point (5, 2). **1**

QUESTION 3 (Start a new answer booklet.) **15 marks**

Marks

(a) The seventh term of an arithmetic series is 5 and the fourteenth term is –23.

(i) Find the first term and common difference of the series. **2**

(ii) What is the sum of the first fourteen terms of the series? **1**

(b) Evaluate:

(i) $\sum_{k=1}^{3} 2k^2 + 1$ **1**

(ii) $\sum_{n=1}^{\infty} 5.\left(\frac{4}{5}\right)^{n-1}$ **2**

(c) A small manufacturer makes components for automatic transmissions. Currently, maximum production is 750 units per week. With an investment of \$1 000 000 production can be increased to 1 350 units per week.

(i) A bank agrees to lend the manufacturer the required amount over 5 years at an interest rate of 12%p.a. compounded monthly. It is agreed that 20 equal quarterly repayments of \$R will be made, with the final repayment being made at the end of 5 years.

(α) Write down an expression for the amount still owing after three months, immediately after the first quarterly repayment of \$R has been made. **1**

(β) Develop an expression, in terms of R, showing that the amount owing after the full 20 quarters is: **2**

$$A = 1000000(1.01)^{60} - R\left(\frac{1.01^{60} - 1}{1.01^{3} - 1}\right)$$

(γ) Hence, find the value of R that will have the manufacturer's loan completely repaid after 5 years. Give your answer to the nearest dollar. **2**

(ii) To preserve quality, the manufacturer decides to phase in the increase in production by 24 units per week.

(α) How many weeks will it take the manufacturer to reach the new, full level of production if increases start at the beginning of the first week? **2**

(β) What will be the total number of units produced during the time that production is being increased? **2**

QUESTION 4 (Start a new answer booklet.) **15 marks** **Marks**

(a) Consider the curve $f(x) = x^3 - 6x^2 + 9x + 4$.

(i) Find any stationary points and determine their nature. **3**

(ii) Find any points of inflexion. **2**

(iii) What is the maximum value of $y = f(x)$ in the domain $-1 \leq x \leq 3$? **1**

(iv) Sketch the curve for $-1 \leq x \leq 3$, showing the co-ordinates of any stationary points and endpoints. **2**

(v) For what values of x is $y = f(x)$ concave down over the given domain? **1**

(vi) For what values of x is $y = f(x)$ decreasing? **1**

(b) By using calculus, show that the curve $y = \dfrac{x}{x+1}$ is increasing for all values of x, for which it is defined. Justify your answer. **2**

(c) Use calculus to show that the curve $y = (3x+2)^3$ has a horizontal point of inflexion. **3**

END OF EXAMINATION

Year 12 Mathematics Half-yearly Examination 2012

Question No. 1 Solutions and Marking Guidelines

Outcomes Addressed in this Question

H2 constructs arguments to prove and justify results

H5 applies appropriate techniques from the study of geometry and trigonometry to solve problems

Outcome	Solutions	Marking Guidelines
H5	(a) (i) SAS	**1 mark** Correct solution
H5	(ii) $\frac{180\times(3)}{5}=108°$ (angle in regular pentagon)	**1 mark** Correct solution
H5	(b) $\frac{a}{\sin 60°}=\frac{c}{\sin 45°}$ $\frac{a}{c}=\frac{\sin 60°}{\sin 45°}$ $\frac{a}{c}=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{1}$ $\therefore \frac{a}{c}=\frac{\sqrt{6}}{2}$	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution
H2, H5	(c) $\angle CRQ=48°$ (angle sum of a straight line is $180°$) $\angle QCR=180°-48°-48°$ (angle sum of isosceles ΔQRC) $=84°$ $\therefore x=84°$ (alternate angles are equal, $AB \parallel CD$)	**3 marks** Correct solution with full reasons provided **2 marks** Correct solution with some reasons provided **1 mark** Correct solution
H2, H5	(d) (i) $\cos 60°=\frac{BY}{6}$ $BY=6\times\cos 60°$ $=6\times\frac{1}{2}$ $=3$	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution
H2, H5	(ii) By Pythagoras' Theorem $CY^2=6^2-3^2$ $=27$ $CY=\sqrt{27}$ $\therefore CY=3\sqrt{3}$	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution
H5	(iii) $AB=AX+XY+YB$, $(XY=5,\ YB=3)$ $AX=3\sqrt{3}\times\tan 45°$ $AX=3\sqrt{3}$ $\therefore AB=3\sqrt{3}+5+3$ $=8+3\sqrt{3}$	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution
H5	(iv) $Area=\frac{1}{2}h(a+b)$ $=\frac{1}{2}\times 3\sqrt{3}\times(5+8+3\sqrt{3})$ $=\frac{3\sqrt{3}}{2}(3\sqrt{3}+8)\text{ cm}^2$ or $\frac{39\sqrt{3}+27}{2}\text{ cm}^2$	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution

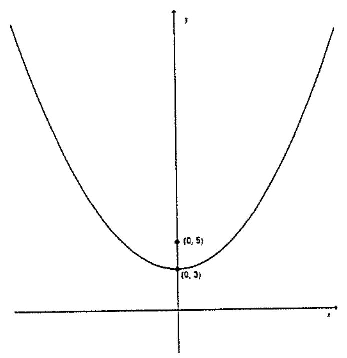
Year 12	Mathematics 2012	TASK 2
Question No. 2	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P5 - understands the concept of a function and the relationship between a function and its graph

H4 - expresses practical problems in mathematical terms based on simple given models

P4 - chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

Outcome	Solutions	Marking Guidelines
P5	(a) $x^2 = 8(y-3)$ $4a = 8$ $a = 2$	
	(i) Vertex is (0, 3)	**1 mark:** correct answer
	(ii) Focus is (0, 5)	**1 mark:** correct answer
	(iii) 	**1 mark:** correct answer
H4	(b) $PA^2 = PB^2$ $(x-3)^2+(y-3)^2=(x+2)^2+(y-1)^2$ $x^2-6x+9+y^2-6y+9=x^2+4x+4+y^2-2y+1$ $10x+4y-13=0$	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution
P4	(c) $x^2-3x+7=A(x+2)^2+B(x+2)+C$ let $x=-2$: $4+6+7=0+0+C$ $C=17$ let $x=-1$: $1+3+7=A+B+17$ $A+B=-6$ ...(1) let $x=0$: $0-0+7=4A+2B+17$ $2A+B=-5$ ...(2) solving (1) & (2): $A+B=-6$ ...(1) $2A+B=-5$ ...(2) $A=1$ $B=-7$ $\therefore\ A=1,\ B=-7,\ C=17$	**3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

P4	(d) no real roots ... $\Delta = 0$ $\Delta = b^2 - 4ac$ $= 4 - 4.3.k$ $= 4 - 12k < 0$ $12k > 4$ $k > \frac{1}{3}$	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution
	(e) (i) $x^2 = 12y \quad ...(1)$ $y = mx - 3m^2 \quad ...(2)$ sub (2) into (1) $x^2 = 12(mx - 3m^2)$ $x^2 = 12mx - 36m^2$ $x^2 - 12mx + 36m^2 = 0$ $\Delta = 144m^2 - 4.36m^2$ $= 144m^2 + 144m^2$ $= 0$ $\therefore$ one real root $\therefore$ tangent	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution *(must include either attempt to find Δ, or show one solution via perfect square)*
	(e) (ii) $y = mx - 3m^2$ sub in $(5,2)$... $2 = 5m - 3m^2$ $3m^2 - 5m + 2 = 0$ $(3m-2)(m-1) = 0$ $m = \frac{2}{3}, 1$	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution
	(e) (iii) sub into $y = mx - 3m^2$ when $m = \frac{2}{3}$, $y = \frac{2}{3}x - \frac{4}{3}$ and when $m = 1$, $y = x - 3$	**1 mark:** correct answer

Year 12 Mathematics Half Yearly Examination 2012		
Question No. 3	Solutions and Marking Guidelines	
Outcomes Addressed in this Question		
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and **series to solve problems**		
Outcome	**Solutions**	**Marking Guidelines**
H5	(a) (i) $T_7 \quad a+6d=5 \quad \ldots 1$ $T_{14} \quad a+13d=-23 \quad \ldots 2$ $2-1 \quad 7d=-28$ $d=-4$ Sub. in 1 $\quad a-24=5$ $a=29$ ie. First term $a=29$, common difference $d=-4$	**2 marks** Correct solution, stating both the first term and common difference. **1 mark** Demonstrates knowledge of and correctly applies the formula for general term of an arithmetic series
H5	(ii) $S_n=\frac{n}{2}(a+l)$ $S_{14}=\frac{14}{2}(29-23)$ $=7\times 6$ $=42$	**1 mark** Correct solution
H5	(b) (i) $\sum_{k=1}^{3} 2k^2+1=(2\times 1+1)+(2\times 4+1)+(2\times 9+1)$ $=3+9+19$ $=31$	**1 mark** Correct solution
H5	(ii) $\sum_{n=1}^{\infty} 5.\left(\frac{4}{5}\right)^{n-1}=\frac{a}{1-r}$ $=\frac{5}{1-\frac{4}{5}}$ $=\frac{5}{\frac{1}{5}}$ $=25$	**2 marks** Correct solution, stating both the first term, common difference and sum. **1 mark** Demonstrates knowledge of and correctly applies the formula for sum to infinity of a geometric series.
H5	(c) (i) (α) $A_3=1000000(1.01)^3-R$	**1 mark** Correct expression given
H5	(β) $A_6=[1000000(1.01)^3-R](1.01)^3-R$ $=1000000(1.01)^6-R(1.01)^3-R$ $=1000000(1.01)^6-R(1+1.01^3)$ $A_9=[1000000(1.01)^6-R(1+1.01^3)](1.01)^3-R$ $=1000000(1.01)^9-R(1+1.01^3+1.01^6)-R$ $\vdots$ $A_{60}=1000000(1.01)^{60}-R(1+1.01^3+1.01^6+\ldots+1.01^{54}+1.01^{57})$ G.P with $a=1$ $r=1.01^3$ $n=20$ $S_{20}=\frac{a(r^n-1)}{r-1}$ $=\frac{[(1.01^3)^{20}-1]}{1.01^3-1}$ ≈ 26.9528 $\therefore A_{60}=1000000(1.01)^{60}-R\left(\frac{(1.01^{60})-1}{1.01^3-1}\right)$	**2 marks** Correct solution, showing how the final expression is developed. **1 mark** Substantial progress in developing the required expression.

H5	(γ) After 60 months $A_{60} = 0$ ie. $\left(\dfrac{1.01^{60}-1}{1.01^{3}-1}\right)R = 1000000(1.01)^{60}$ $R = \dfrac{1000000(1.01)^{60}}{26.9528}$ $= \$67402$	**2 marks** Correctly evaluates the value of R **1 mark** Demonstrates knowledge of the condition required to evaluate R. ie. $A_{60} = 0$.
H5	(ii) (α) $a = 774 \qquad d = 24 \qquad T_n = 1350$ $T_n = a + (n-1)d$ $1350 = 774 + (n-1)\times 24$ $576 = 24n - 24$ $600 = 24n$ $n = 25$ ie. The full level of production is reached in 25 weeks.	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution including knowledge of the formula for general term of an arithmetic series.
H5	(β) $S_n = ? \qquad a = 774 \qquad d = 24 \qquad n = 25$ $S_n = \frac{n}{2}[2a + (n-1)d)$ $= \frac{25}{2}(1548 + 24\times 24)$ $= 26550$ ie. A total of 26550 units are produced during the time that production is increasing.	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution including knowledge of the formula for finding the sum of an arithmetic series.

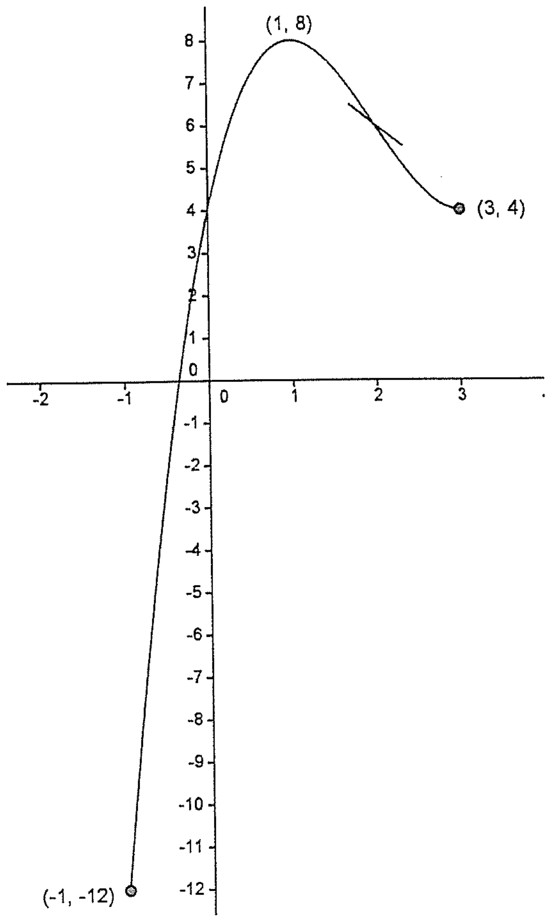
Year 12Half Yearly Question No.4	Mathematics Solutions and Marking Guidelines	Examination 2012

Outcomes Addressed in this Question

H6 Uses the derivative to determine features of the graph of a function

H7 Uses the features of a graph to deduce information about the derivative

Outcome	Solutions	Marking Guidelines
H6	a)(i) $f(x)=x^3-6x^2+9x+4$ $f'(x)=3x^2-12x+9=0$ for stationary points $\therefore x^2-4x+3=0$ $\therefore (x-3)(x-1)=0$ $\therefore$ stationary points at $x=1$ and $x=3$. $f''(x)=6x-12$ When $f''(1)=6-12=-6$, $f''(x)<0$ $\therefore$concave down & so relative maximum at $x=1$. When $f''(3)=18-12=6.$, $f''(x)>0$ $\therefore$concave up & so relative minimum at $x=3$.	3 marks: correct solution 2 marks: substantially correct solution 1 mark: partially correct solution
H6	(ii)Possible points of inflexion at $f''(x)=0$ $\therefore 6x-12=0$ $\therefore x=2$ is a possible point of inflexion. x: 1, 2, 3 y'': -6, 0, 6 Since the concavity changes, $x=2$ is a point of Inflexion.	2 marks: correct solution 1 mark: substantially correct solution
H6	(iii) When $x=1$, $y=1-6+9+4=8$ (relative maximum) and when $x=3$, $y=27-54+27+4=4$. (relative minimum) At the endpoints: When $x=-1$, $y=-1-6-9+4=-12$ and when $x=3$, $y=4$. $\therefore$ Maximum value of y for $-1\le x\le 3$ is 8. Note: $(1,8)$ is not the maximum value for y.	1 mark: correct answer
		2 marks: correct graph

H6	(iv) 	1 mark: substantially correct solution
H7	(v) From the graph, concave down when $-1 \le x < 2$ i.e. to the left of the point of inflexion.	1 mark: correct answer
H7	(vi) From the graph, decreasing when $1 < x < 3$.	1 mark: correct answer
H6	(b) $y = \dfrac{x}{x+1}$ $y' = \dfrac{(x+1).1 - x.1}{(x+1)^2}$ using the quotient rule $\therefore y' = \dfrac{1}{(x+1)^2}$ Since $(x+1)^2$ is positive for all values of x for which the function is defined (undefined at $x = -1$), y' is positive. $\therefore$ function is increasing for all x for which defined.	2 marks: correct solution 1 mark: substantially correct solution
H6	(c) $y = (3x+2)^3$ $\therefore y' = 3(3x+2)^2 \times 3$	3 marks: correct solution 2 marks: substantially correct solution 1 mark: partially correct solution

$\therefore y' = 9(3x+2)^2$

$y' = 0$ for stationary points.

Solving $9(3x+2)^2 = 0$

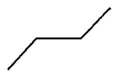
$(3x+2)^2 = 0$

$3x+2=0$

$\therefore x = -\frac{2}{3}$ is a stationary point.

Using the first derivative test with $y' = 9(3x+2)^2$ ::

x	-1	$-2/3$	0
y'	9	0	36



$\therefore$ horizontal point of inflexion at $x = -\frac{2}{3}$.

Note: Alternatively show both $y' = 0$ and $y'' = 0$ at $x = -\frac{2}{3}$ **and** change in concavity (as, for example, $y = x^4$ has both $y' = 0$ and $y'' = 0$ at $x = 0$, but does not have a horizontal point of inflexion there.