Name:	
Class:	

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Year 12 Assessment 2 Term 1 2016

MATHEMATICS

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Data Sheet is provided.
- In Question 6 - 8, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks: 85

Section I: 5 marks

- Attempt Question 1 – 5.
- Answer on the Multiple Choice answer sheet provided.
- Allow approximately 5 minutes for this section.

Section II: 81 Marks

- Attempt Question 6 – 8
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

Section A – Multiple Choice

1. What is the solution to $2^m = 7$?

a) $m = \frac{\ln 7}{2}$

b) $m = \frac{7}{\ln 2}$

c) $m = \frac{\ln 7}{\ln 2}$

d) $m = \ln 7 - \ln 2$

2. What is the derivative of $\frac{e^x}{e^x+1}$?

a) $\ln(e^x + 1)$

b) $\frac{2e^{2x}+e^x}{(e^x+1)^2}$

c) $\ln(e^x + 1)^2$

d) $\frac{e^x}{(e^x+1)^2}$

3. The second term of an arithmetic series is 37 and the sixth term is 17. What is the sum of the first ten terms?

a) 54

b) 195

c) 280

d) 390

4. Suppose that the point $P(a, f(a))$ lies on the curve $y = f(x)$. If $f'(a) = 0$ and $f''(a) > 0$, which of the following statements describes the point P on the graph of $y = f(x)$?

a) P is a maximum turning point

b) P is a horizontal point of inflexion

c) P is a minimum turning point

d) P is a point of inflexion

5. The primitive of $tan\theta$ is:

a) $sec^2\theta$

b) $\ln(sin\theta)$

c) $tan\theta sec\theta$

d) $-\ln(cos\theta)$

Section B – Extended Response

6.

a) By writing $0.\dot{6}\dot{4}$ as the sum of a geometric series. Express it as a rational number. 2

b) For the series $2 + 6 + 18 + \cdots$

i. Find the 8th term. 1

ii. Find the sum of the first 8 terms. 1

c)

i. $\int cosx + sinx\ dx$ 2

ii. $\int_0^{\frac{\pi}{4}} sec^2x\ dx$ 2

d) Graph on a number plane $y = 3 - sin2x$ for $0 \leq x \leq 2\pi$, showing all important features. 3

e)

i. Evaluate $\log_2 8$. 1

ii. Rewrite $2\log_2 x$ in the form of $\log_a(b^n)$. 1

iii. Hence or otherwise, solve $3 + 2\log_2 x = \log_2(24x + 80)$. 3

f)

i. Show that the equation of the tangent to $y = e^{3x}$ at the point where $x = 1$ is given by $y = e^3(3x - 2)$. 3

ii. Find the exact area bounded by $y = e^{3x}$, the tangent from part i, and the y-axis. 3

g)

i. Find $\int (x - 1)(2x + 3)\ dx$ 2

ii. Evaluate $\int_1^3 (x + x\sqrt{x})^2\ dx$, leave your answer in exact form. 3

7.

a) For the series $1\frac{1}{2} + 2\frac{3}{4} + 5\frac{1}{8} + 6\frac{15}{16} + \cdots$

i. Show that the n^{th} term is given by $T_n = (2n-1) + (-1)^{n-1}(\frac{1}{2})^n$. 3

ii. Find S_7. 3

b) Find the second derivative of $xsinx$ 2

c) A circle centre O has radius 15cm. A 10cm chord AB is drawn across the circle. Find the area of the minor segment cut off by the chord AB, correct to 2 dp. 3

d) For the graph $y = \frac{3x}{x^2+1}$:

i. Find all stationary points and determine their nature. 4

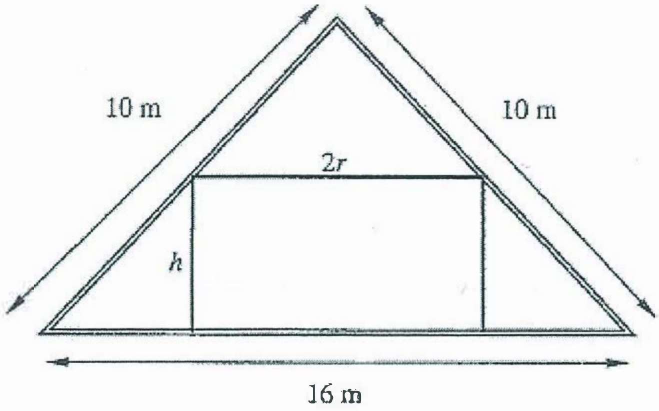
ii. Locate any intercepts. 1

iii. Sketch the graph neatly, showing all important features. 3

iv. How many solutions are there to the equation $\frac{3x}{x^2+1} = c$, where c is a non-zero constant and $\frac{-3}{2} < c < \frac{3}{2}$. 1

Question 7 continues on next page

e) In some rural areas, hot water tanks are installed in the roofs of large homesteads. The diagram below shows the cross-section of a cylindrical tank in such a homestead's roof. The cylindrical tank fits exactly into the roof with diameter 2r metres and height h metres. The cross-section of the roof is an isosceles triangle with dimensions shown.



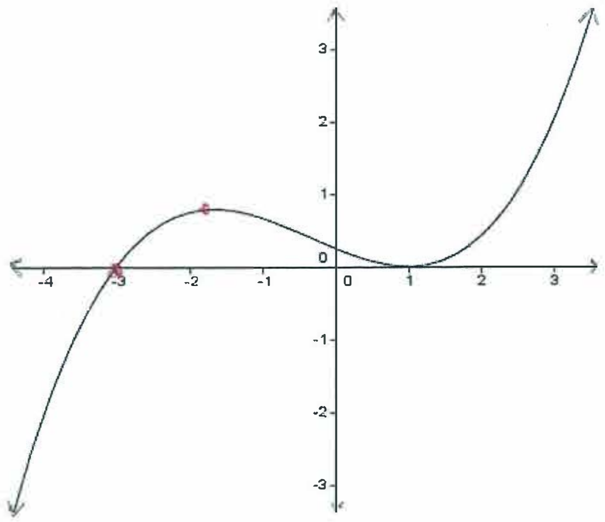
i. Show that the height of the roof is 6 metres. 1

ii. Show that the volume of the cylindrical tank is given by $V = \frac{3\pi}{4}(8r^2 - r^3)$. 3

iii. Find the value of r which gives the hot water tank its greatest volume. Hence find the exact volume. 3

8.

a) $f'(x)$ is shown on the diagram below, neatly Sketch $y = f(x)$ 2



b) If $x = (1+t)e^{5t}$, prove that:

$$\frac{d^2x}{dt^2}\cancel{\frac{d^2y}{dx^2}} - 10\frac{dx}{dt} + 25x = 0$$ 3

c) Find the integer b such that $\int_1^4 \frac{2x+7}{x^2+7x+10}\,dx = \ln b$ 3

d) Solve $4sinxcosx = sinx$ to the nearest degree. $(0° \le x \le 360°)$ 3

e) Find the volume of the solid formed when $y = \sqrt{sinx}$ is rotated about the x-axis between $x = 0$ and $x = \pi$. 2

f)

i. Given that $S_n = 3n^2 - 4$. Show that $T_n = 6n - 3$. 2

ii. Find the first value of n such that T_n is a perfect square. 1

Question 8 continues on the next page

g) Nat just retired with $1 000 000 in her bank account. This account attracts interest at a rate of 5% p.a. compounded annually. She intends to withdraw money in equal amounts every year immediately after interest is compounded.

Let A_n be the amount of money in her account after the n^{th} withdrawal of $\$W$.

i. Write down an expression for the amount left after two withdrawals. 1

ii. Show that $W = 80242.59$, if Nat wants the money to last exactly 20 withdrawals. 3

iii. After her 10th withdrawal, the bank changes its interest rate to 3% p.a. on the remaining balance. How many further withdrawals can Nat take if W stays the same? 4

iv. What should Nat's withdrawal amount be right from the beginning if she wishes to still make the money last 20 withdrawals, taking into account the change in interest rate after 10 years? 3

END OF EXAMINATION

MATHEMATICS: Question		
Suggested Solutions	**Marks**	**Marker's Comments**

Multiple Choice

1. $2^m = 7$

$\ln 2^m = \ln 7$

$m \ln 2 = \ln 7$

$m = \dfrac{\ln 7}{\ln 2}$ (C)

2. $f(x) = \dfrac{e^x}{e^x+1}$

$f'(x) = \dfrac{e^x(e^x+1) - e^x(e^x)}{(e^x+1)^2}$

$= \dfrac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2}$

$= \dfrac{e^x}{(e^x+1)^2}$ (D)

3. $T_2 = 37$ $T_6 = 17$

$= a + d$, $= a + 5d$

$a = 37 - d$

$\therefore 17 = 37 - d + 5d$ (B)

$-20 = 4d$

$d = -5$, $a = 42$

$S_{10} = \frac{10}{2}(2(42) + (10-1)(-5))$

$= 195$

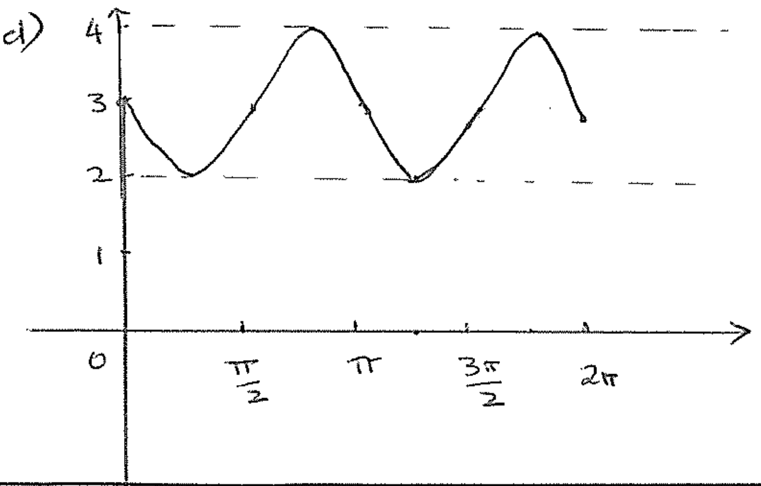
4. (C)

5. $\int \tan\theta \, d\theta = \int \dfrac{\sin\theta}{\cos\theta} \, d\theta$ (D)

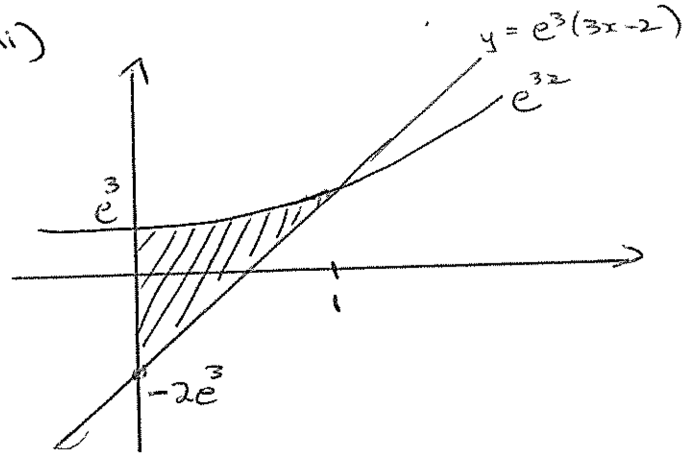
$= -\ln(\cos\theta)$

Question 6 - 2 unit

a) $0.\dot{6}\dot{4} = 0.64 + 0.0064 + 0.000064 \ldots$ $a = 0.64$ $r = 0.01$ since $\lvert r \rvert < 1$ $0.\dot{6}\dot{4}$ has a limiting sum $S_\infty = \frac{a}{1-r} = \frac{0.64}{1-0.01}$ $= \frac{0.64}{0.99}$ $= \frac{64}{99}$	✓ ✓	One had to write $0.\ddot{64}$ as a geometric series
b(i) $2 + 6 + 18 + \ldots\ldots$ $a = 2$ and $r = 3$ $T(8) = ar^{n-1}$ $= 2.3^{8-1}$ $= 2 \times 3^7$ $= 4374$ ✓ (ii) $S(8) = \frac{a(r^n-1)}{r-1}$ $= \frac{2(3^8-1)}{3-1}$ $= 6560$	✓ ✓	well done
c (i) $\int (\cos x + \sin x)dx$ $= \sin x - \cos x + C$ (ii) $\int_0^{\frac{\pi}{4}} \sec^2 x \, . dx$ $= \left[\tan x\right]_0^{\pi/4}$ $= \tan\frac{\pi}{4} - \tan 0$ $= 1$	✓✓ ✓ ✓	$\int \cos x = \sin x$ ✓ $\int \sin x = -\cos x$ ✓ take away 1 if no 'C' well done

d) 	✓ ✓ ✓	1 mark - for amplitude and Scale 1 mark - inverted sine graph 1 mark - two sine graphs in domain ($0 \le x \le 2\pi$)
e) (i) $2^x = 8$ $\therefore x = 3$ (ii) $\log_2(x^2)$ (iii) $\log_2 8 + \log_2 x^2 = \log_2(24x+80)$ $\log(8x^2) = \log_2(24x+80)$ $\therefore 8x^2 = 24x+80$ $8x^2 - 24x - 80 = 0$ $x^2 - 3x - 10 = 0$ $(x-5)(x+2) = 0$ $\therefore x = 5$ or $x = -2$ Since $x > 0$ $x = 5$ only	✓ ✓ ✓ ✓ ✓	 recognise that 3 is same as $\log_2 8$
(f) $y = e^{3x}$ $\frac{dy}{dx} = 3e^{3x}$ when $x=1$, $y = e^3$ and $\frac{dy}{dx} = 3e^3$ which is the gradient $\therefore$ tangent is given by $y - e^3 = 3e^3(x-1)$ $y = 3e^3(x-1) + e^3$ $= e^3(3x-2)$	✓ ✓ ✓	correct differentiation getting the y value when $x=1$ and the gradient substituting in equation and showing that it is $e^3(3x-2)$

f (ii)



- Poorly done.
- Drawing graphs would have helped.

$$A = \int_0^1 e^{3x}\,dx - \int_0^1 e^3(3x-2)\,dx$$ ✓

Correct area regions with limits 0 and 1

$$= \left[\frac{e^{3x}}{3}\right]_0^1 - \left[e^3\left(\frac{3x^2}{2} - 2x\right)\right]_0^1$$ ✓

integration

$$= \left(\frac{1}{3}e^3 - \frac{1}{3}\right) - \left(\frac{3e^3}{2} - 2e^3\right)$$

$$= \left(\frac{5e^3}{6} - \frac{1}{3}\right) u^3$$ ✓

substitution

Few students got x the subject of the formula

$y = e^{3x}$
$\ln y = 3x$
$x = \frac{\ln y}{3}$

and

$y = 3x e^3 - 2e^3$
$x = \frac{y + 2e^3}{3e^3}$

- students could not integrate $(\ln y)$ and hence could not go further

One student:

$\frac{d}{dx}(x\ln x) = \ln x + 1$

$\therefore x\ln x = \int \ln x + x$

$\therefore \int \ln x = x\ln x - x$ ✓

Hence $\frac{1}{3}\int_1^{e^3} \ln y = \frac{1}{3}\left[y\ln y - y\right]_1^{e^3}$

$= \frac{1}{3}(2e^3 + 1)$

$\therefore$ Area is $\triangle - \frac{1}{3}(2e^3+1)$

$\frac{1 \times 3e^3}{2} - \frac{2}{3}e^3 - \frac{1}{3}$

$= \frac{5}{6}e^3 - \frac{1}{3}$

g(i) $\int (x-1)(2x+3)\,dx$

$= \int 2x^2 + x - 3)\,dx$ ✓ Expand

$= \frac{2x^3}{3} + \frac{x^2}{2} - 3x + C$ ✓ integration

(ii) $\int_1^3 (x + x\sqrt{x})^2\,dx$

$\int_1^3 (x^2 + 2x^2\sqrt{x} + x^3)\,dx$ ✓ correct expansion

$\left[\frac{x^3}{3} + \frac{2x^{7/2}}{7/2} + \frac{x^4}{4}\right]_1^3$ ✓ Correct integration

$\left(9 + \frac{108\sqrt{3}}{7} + \frac{81}{4}\right) - \left(\frac{1}{3} + \frac{4}{7} + \frac{1}{4}\right)$

$= 28\frac{2}{21} + \frac{108\sqrt{3}}{7}$ ✓ correct substitution and answer.

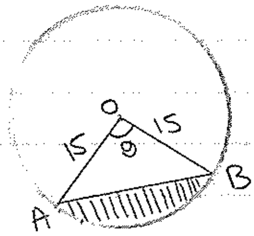
accepted

$\frac{590}{21} + \frac{324\sqrt{3}}{21}$

or $\frac{2360}{84} + \frac{1296\sqrt{3}}{84}$

4

MATHEMATICS: Question 7

Suggested Solutions	Marks	Marker's Comments
a) i) $(1+\frac{1}{2})+(3-\frac{1}{4})+(5+\frac{1}{8})+(7-\frac{1}{16})+\ldots$ $=\underbrace{(1+3+5+7+\ldots)}_{\text{arithmetic}}+\underbrace{(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots)}_{\text{geometric}}$	1	Recognising $2\frac{3}{4}=3-\frac{1}{4}$
$a=1,\ d=2$ $\quad a=\frac{1}{2},\ r=-\frac{1}{2}$ $T_n=[1+(n-1)2]+(\frac{1}{2})(-\frac{1}{2})^{n-1}$ $=(2n-1)+(-1)^{n-1}(\frac{1}{2})^n$	2	Need to <u>show</u> the two formulas for terms.
ii) $S_7=\frac{7}{2}(2(1)+(7-1)(2))+\dfrac{\frac{1}{2}(1-(-\frac{1}{2})^7)}{1-(-\frac{1}{2})}$	2	
$=49\frac{43}{128}$	1	
b) $y=x\sin x$ $y'=x\cos x+\sin x$	1	
$y''=-x\sin x+\cos x+\cos x$ $=2\cos x-x\sin x$	1	
c)  $\theta=\cos^{-1}\left(\dfrac{15^2+15^2-10^2}{2\times15\times15}\right)$ $=\cos^{-1}(\frac{7}{9})$	1	
$A=\frac{1}{2}r^2(\theta-\sin\theta)$ $=\frac{1}{2}(15)^2(\cos^{-1}(\frac{7}{9})-\sin(\cos^{-1}(\frac{7}{9})))$	1	
$=5.75262\ldots$ (calc.) $=5.75\text{ cm}^2$	1	radians used
d) i) $y=\dfrac{3x}{x^2+1}$ $y'=\dfrac{3(x^2+1)-(3x)(2x)}{(x^2+1)^2}$	1	
stationary points when $y'=0$ $3(x^2+1)-(3x)(2x)=0$ $3x^2+3-6x^2=0$ $3-3x^2=0$		

MATHEMATICS: Question 7

Suggested Solutions	Marks	Marker's Comments

$3(1-x^2)=0$

$\therefore 1-x^2=0$

$x^2=1$

$x=\pm 1$ — 1

when $x=1$, $y=\frac{3}{2}$

x	0.9	1	1.1
y'	0.17	0	-0.13

$\therefore (1, \frac{3}{2})$ is a maximum — 1

when $x=-1$, $y=-\frac{3}{2}$

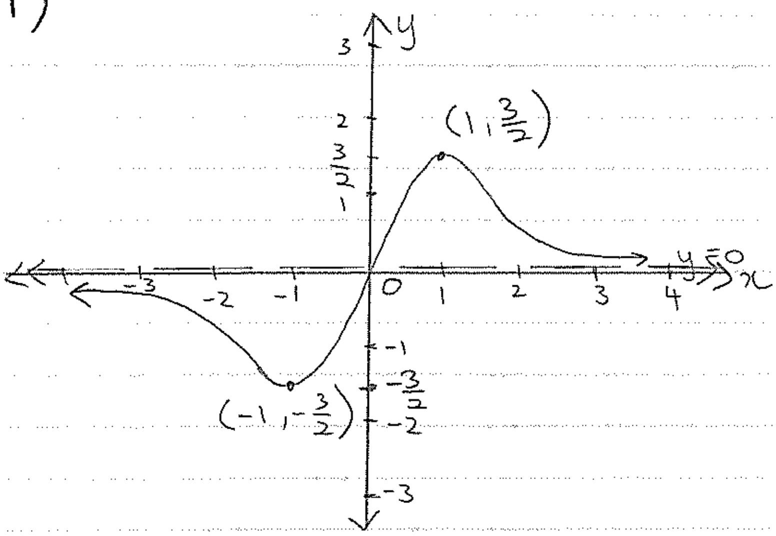
x	-1.1	-1	-0.9
y'	-0.13	0	0.17

$\therefore (-1, -\frac{3}{2})$ is a minimum — 1

ii) when $x=0$, $y=0$ — 1

$\therefore$ intercept at $(0,0)$

iii)



1 horizontal asymptote

1 shape

1 turning pts + scale.

iv) 2 solutions — 1

e) i) $h^2=10^2-8^2$ (by pythagoras) — 1

$h=\sqrt{36}$

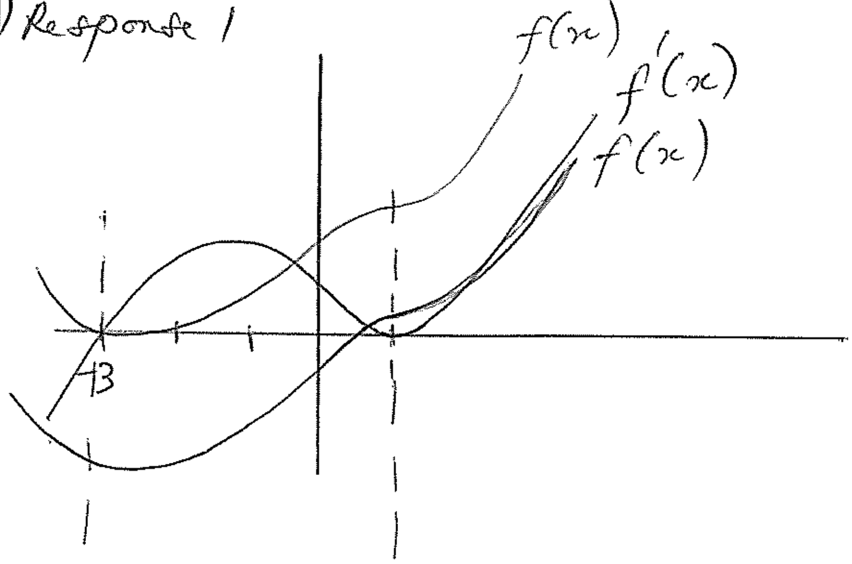
$=6$, $h>0$

MATHEMATICS: Question 7

Suggested Solutions	Marks	Marker's Comments
ii) $\frac{r}{8} = \frac{6-h}{6}$ (corresponding sides of similar triangles are in the same ratio)	1	stating similar triangles
$6r = 48 - 8h$ $8h = 48 - 6r$ $h = 6 - \frac{3}{4}r$		
$V = \pi r^2 h$ $\therefore V = \pi r^2\left(6 - \frac{3}{4}r\right)$ $= \frac{3}{4}\pi\left(8r^2 - r^3\right)$	1	
iii) $V' = \frac{3}{4}\pi\left(16r - 3r^2\right)$ for stationary pts $V' = 0$ $0 = 16r - 3r^2$ $0 = r(16 - 3r)$	1	
$\therefore r = 0,\ r = \frac{16}{3}$ $V'' = \frac{3}{4}\pi(16 - 6r)$ for $r = 0$ $V'' = \frac{3}{4}\pi(16 - 6(0))$ $= 12\pi > 0$ $\therefore r = 0$ is a minimum for $r = \frac{16}{3}$, $V'' = \frac{3}{4}\pi\left(16 - 6\left(\frac{16}{3}\right)\right)$ $= -12\pi < 0$ $\therefore r = \frac{16}{3}$ is a maximum	1	$r = \frac{16}{3}$ + test for maximum
when $r = \frac{16}{3}$ $V = \frac{3}{4}\pi\left(8\left(\frac{16}{3}\right)^2 - \left(\frac{16}{3}\right)^3\right)$ $= \frac{512\pi}{9}\ u^3$	1	

MATHEMATICS: Question........ 8 1/5

Suggested Solutions	Marks	Marker's Comments
8a) Response 1	2	shape - 1mark change in direction at point $x = 1$ and -3 1 mark



Response 2

x	-4	-3	-2	-1	0	1	2	3
$f'(x)$	−	0	+	+	+	0	+	+
$f(x)$	↘	→	↗	↗	↗	→	↗	↗

b) $x = (1+t)e^{5t}$ — Marks: 3 — 1 mark for $\frac{dx}{dt}$

using product rule

$$x' = e^{5t} \cdot 1 + (1+t)5e^{5t} \qquad u = 1+t$$
$$u' = 1$$
$$= e^{5t} + 5e^{5t} + 5te^{5t} \qquad v = e^{5t}$$
$$v' = 5e^{5t}$$
$$= 6e^{5t} + 5te^{5t}$$
$$= e^{5t}(6+5t)$$

$$u = e^{5t}$$
$$u' = 5e^{5t}$$
$$x'' = (6+5t) \cdot 5e^{5t} + e^{5t} \cdot 5$$
$$= 30e^{5t} + 25te^{5t} + 5e^{5t} \qquad v = 6+5t$$
$$= 35e^{5t} + 25te^{5t} \qquad v' = 5$$
$$= 5e^{5t}(7+5t)$$

1 mark for $\frac{d^2x}{dt^2}$

MATHEMATICS: Question 8		2/5
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
to prove $x'' - 10x' + 25x = 0$ $35e^{5t} + 25te^{5t} - 10(6e^{5t} + 5te^{5t}) + 25(e^{5t} + te^{5t})$ $= 0$ LHS = RHS		1 mark to show equals 0
c) $\int_1^4 \frac{2x+7}{x^2+7x+10}\,dx = \ln b$ $f(x) = x^2+7x+10$ $f'(x) = 2x+7$ $\therefore \int_1^4 \frac{2x+7}{x^2+7x+10} = \left[\ln(x^2+7x+10)\right]_1^4$ — 1 $= \ln 54 - \ln 18$ — 1 $= \ln \frac{54}{18} = \ln 3$ $\therefore b = 3$ — 1		
d) $4\sin x \cos x = \sin x$ $4\sin x\cos x - \sin x = 0$ — 1 $\sin x(4\cos x - 1) = 0$ $\therefore \sin x = 0$; $x = 0, 180, 360$ $4\cos x = 1$; $\cos x = 1/4$; $x = 75°31', 284°29'$ 1 To the nearest degree. $x = 0°, 76°, 180°, 284°, 360°$	1	

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MATHEMATICS	: Question 8	3/5
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
e) $y = \sqrt{\sin x}$ find V rotated abt x axis. $y^2 = \sin x$ $\therefore \int y^2 = \pi \int_0^\pi \sin x$ — 1 $= \pi [-\cos x]_0^\pi$ $= \pi [-\cos\pi - (-\cos 0)]$ $= \pi (1+1)$ — 1 $= 2\pi$ unit^3		didnt write π didnt have -ve when integrating sin
e) i) $S_n = 3n^2 - 4$ $S_{n-1} = 3(n-1)^2 - 4$ $= 3n^2 - 6n + 3 - 4$ — 1 $= 3n^2 - 6n - 1$ $T_n = S_n - S_{n-1}$ — 1 $= 3n^2 - 4 - (3n^2 - 6n - 1)$ $= 6n - 3$		
ii) T_n - perfect square. sub values. $T_1 = 3$ ($\neq$ sq no) $T_2 = 9$ (= sq no) — 1 $\therefore n = 2$		

MATHEMATICS: Question 8		4/5
Suggested Solutions	**Marks Awarded**	**Marker's Comments**

9 i) $A_1 = 1000000 \times 1.05 - W$

$A_2 = (1000000 \times 1.05 - W) \times 1.05 - W$

$= 1000000(1.05)^2 - 1.05W - W$ — 1

ii) $A_{20} = 1000000(1.05)^{20} - 1.05^{19} - \dots - W$

$= 1000000(1.05)^{20} - W(1 + 1.05^1 \dots + 1.05^{19})$ — 1 GP stated

$= 1000000 \times (1.05)^{20} - W\left[\dfrac{1(1.05^{20} - 1)}{1.05 - 1}\right]$ — 1 a, r shown and put in formula

$W = \dfrac{1000000 \times 1.05^{20} \times 0.05}{(1.05^{20} - 1)}$ — 1

$= \$80242.59$ (shown

iii) first 10 withdrawals

$A_{10} = 1000000 \times 1.05^{10} - \dfrac{W(1.05^{10} - 1)}{0.05}$

$= 619611.95$ (balance) — 1

Next n years $r - 1.03$ $W - 80240.59$ $A_{10} - 619611.95$

$\therefore A_n = A_{10} \times 1.03^n - \dfrac{W(1.03^n - 1)}{0.03}$

$A_{10} \times 1.03^n \times 0.03 = W(1.03)^n - W$

$A_{10} \times 1.03^n \times 0.03 - W(1.03)^n = -W$

$1.03^n(A_{10} \times 0.03 - W) = -W$ — 1

MATHEMATICS: Question……..		5/5
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
$1.03^n = \dfrac{-W}{A_{10} \times 0.03 - W}$ $= \dfrac{-80242.59}{619611.95 \times 0.03 - 80242.59}$ $1.03^n = 1.301$ $n \lg_{10} 1.03 = \lg_{10} 1.301$ $n = \dfrac{\lg_{10} 1.301}{\lg_{10} 1.03}$ $= 8.9$ Further withdrawals will be 9	1 1	