KINCOPPAL–ROSE BAY
SCHOOL OF THE SACRED HEART

2010

HIGHER SCHOOL CERTIFICATE
MID COURSE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – $2\frac{1}{2}$ hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 96

- Attempt Questions 1 – 8
- All questions are of equal value

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Total Marks – 96
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Evaluate $\left(\frac{1}{\pi^{2\cdot 5}}-1\right)^2$ correct to 3 significant figures. **2**

(b) Express $\frac{\sqrt{3}}{\sqrt{7}-\sqrt{2}}$ with a rational denominator. **2**

(c) Solve $|3x-7| \le 4$ **2**

(d) State the domain of the function $y = \log_e x$ **1**

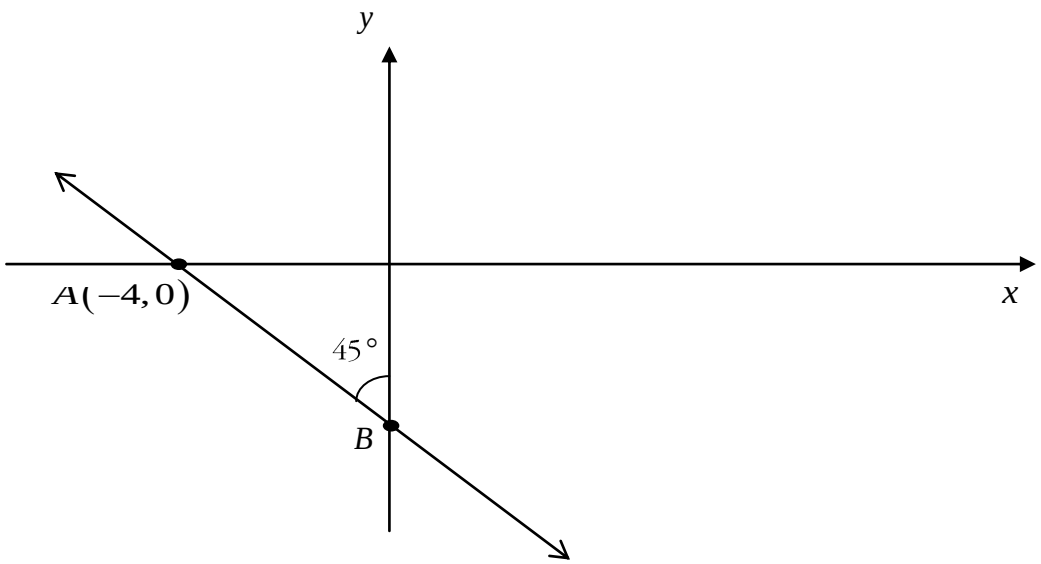
(e) If $\tan\theta = \frac{5}{12}$ and $\cos\theta > 0$, find the exact value of $\sin\theta$ **2**

(f) Solve $9^{2x-3} = 27^x$ **3**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Point *A* has coordinates $A(-4,0)$. A line is drawn through point *A* as shown in the diagram below that intersects the *y*-axis at an angle of 45°.



(i) Find the gradient of the line *AB* and state the coordinates of the point *B*. You must show working out. **2**

(ii) Find the equation of the line that is parallel to *AB* passing through $(-4,-4)$ **2**

If *AB* is the diameter of a circle, find:

(iii) The coordinates of the centre of the circle **1**

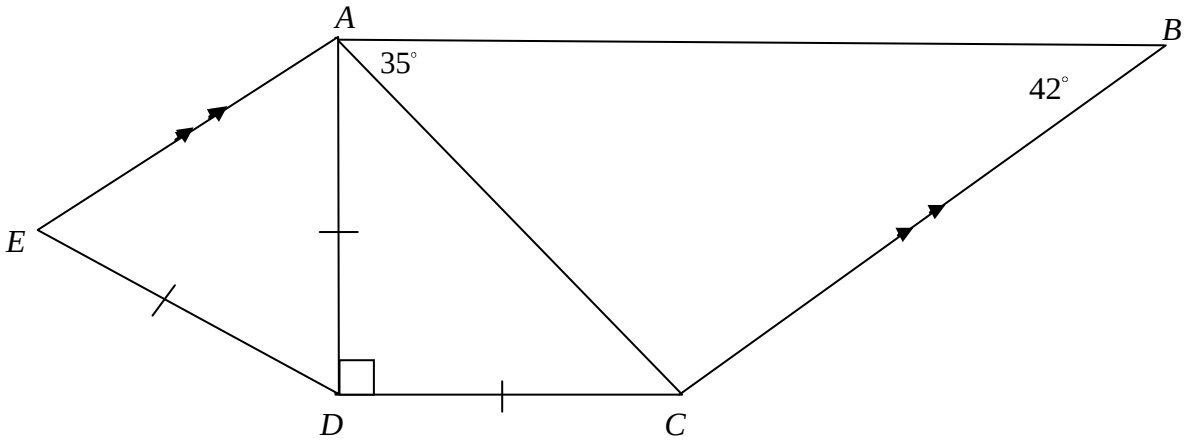
(iv) The equation of the circle with diameter *AB* **2**

(v) Show that the line in part (ii) is a tangent to the circle with diameter *AB*. **2**

Question 2 continues on page 3

Question 2 (continued)

(b) In the diagram below $ED = AD = DC$, $\angle ADC = 90^\circ$ and $AE \parallel BC$.
$\angle ABC = 42^\circ$, $\angle CAB = 35^\circ$



Find the size of $\angle AED$. Give reasons for your answer. **3**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Differentiate with respect to x

(i) $\dfrac{3}{x^5}$ **1**

(ii) $\left(5-x^2\right)^7$ **2**

(ii) x^3e^{4x} **2**

(b) Use Simpson's Rule with five function values to approximate

$$\int_0^4 f(x)dx \text{ where } f(x)=\sqrt{x}$$

Give your answer correct to 1 decimal place. **3**

(c) Evaluate

$$\int_0^1 (e^{5x}-e^{-x})\,dx$$

Give your answer in exact form. **2**

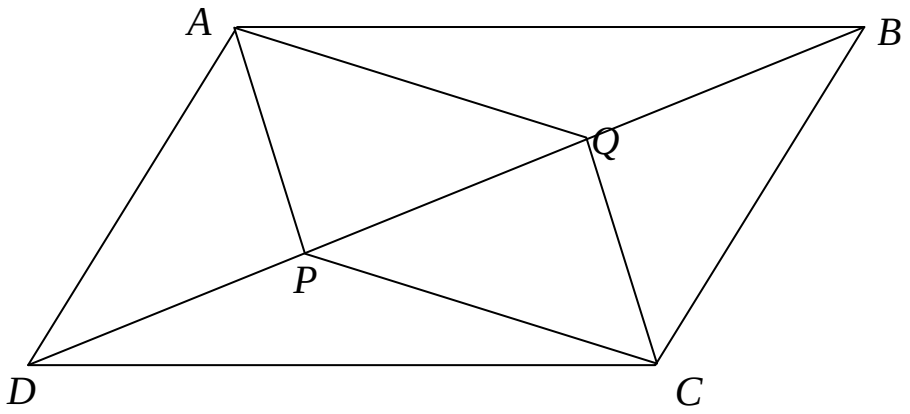
(d) Find

$$\int \frac{x^7-x}{3x^5}dx$$ **2**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



ABCD is a parallelogram. Points *P* and *Q* lie on the diagonal *BD* such that *PD* = *QB*.

(i) Copy or trace the diagram into your answer booklet and indicate all the information given on your diagram. **1**

(ii) Prove that $\Delta ABQ \equiv \Delta CDP$ stating the congruency test used. **3**

(iii) Hence show that $AQ = CP$ **1**

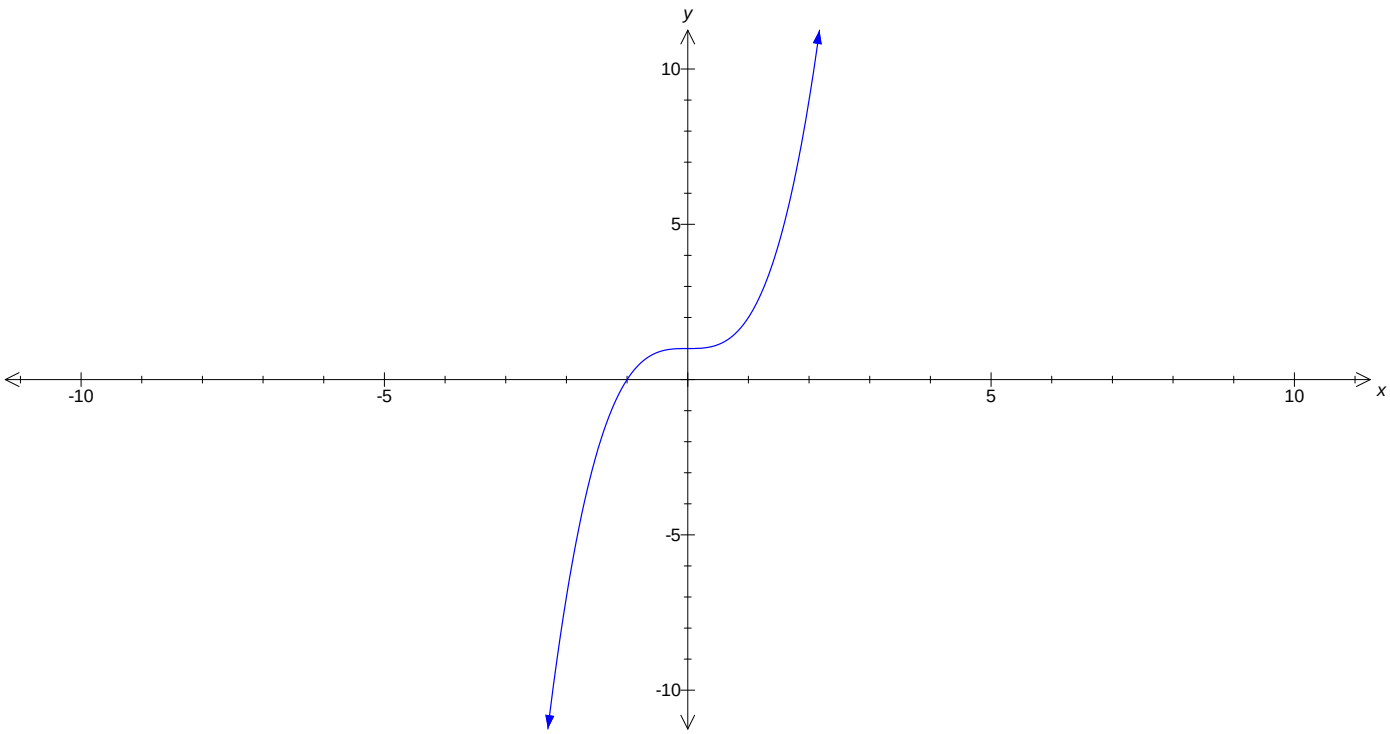
(b) Find the values of *k* if the roots of the equation $2x^2 - 5x + k = 0$ are real and different. **2**

(c) Find the values of $a, b \text{ and } c$ for which $3x^2 + 5x - 10 \equiv ax(x+3) + bx^2 + c(x+1)$ **2**

Question 4 continues on page 6

Question 4 (continued)

(d) (i) State the equation of the function shown below. **1**



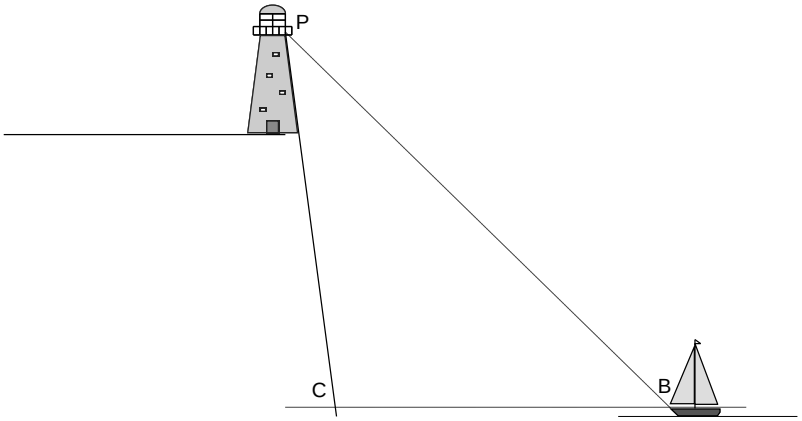
(ii) Use the answer sheet page to accurately sketch on the same graph, $y = |x-1|$ **1**

(iii) Hence solve $x^3 \leq |x-1| - 1$ **1**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A lighthouse sits at the top of a cliff which slopes at 105° from the horizontal. From the bow of the ship (B) the angle of elevation of the platform of the lighthouse (P) is 46°. Using radio signals the ship determines the distance PB is 800 metres.



(i) Copy the diagram and mark all of the relevant information above on it. **1**

(ii) Calculate the distance *BC* of the bow from the base of the cliff **2**
Give your answer correct to the nearest metre.

(b) Find the equation of the normal to the curve $y = e^{5x-1}$ at the point $x = 1$ **3**

(c) Show that $\dfrac{\left(1+\tan^2\theta\right)\cot\theta}{\operatorname{cosec}^2\theta} = \tan\theta$ **3**

(d) Solve $3\sin^2\theta + 2\sin\theta = 1$ for $0 \le \theta \le 360^\circ$ **3**
Give your answers in degrees and minutes.

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) If α, β are the roots of the equation $2x^2 - 14x - 1 = 0$, find the value of

(i) $\alpha\beta$ **1**

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$. **2**

(iii) $\alpha^2 + \beta^2$ **1**

(b) Given the quadratic polynomial $(x-4)^2 = -8(y+1)$, find the coordinates of the:

(i) Vertex **1**

(ii) Focus (hint draw a diagram) **2**

(iii) Find the coordinates of the end points of the latus rectum **2**

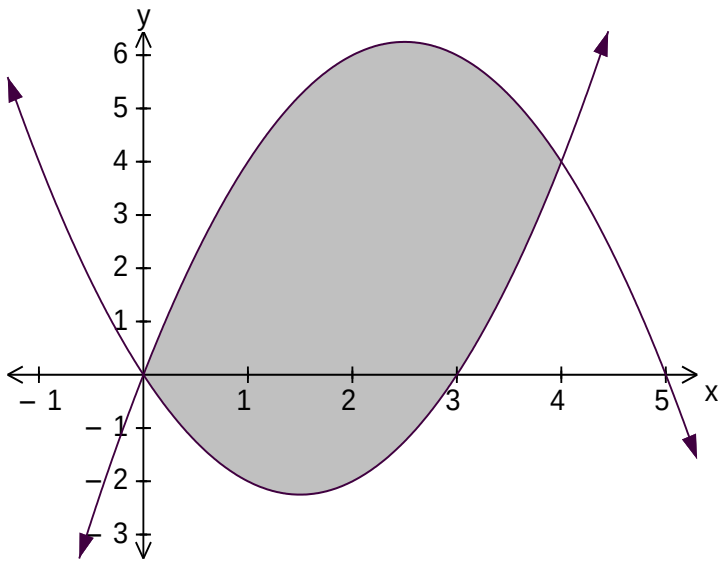
(c) The area bounded by the parabola $y = 1 - x^2$, and the x and y axes in the first quadrant is rotated about the x-axis. Find the exact volume of the solid formed. **3**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) If $\log_a 2 + 2\log_a x - \log_a 6 = \log_a 3$ find the value of x. **2**

(b)



(i) The curves $y = x^2 - 3x$ and $y = 5x - x^2$ intersect at the point (0, 0). **1**
Show that they also intersect at the point (4, 4).

(ii) Find the exact shaded area enclosed between the two curves. **3**

(c) A closed cylindrical can is made from a sheet of metal with an area of 600π cm^2
Given that the surface area of a closed cylinder is $SA = 2\pi r^2 + 2\pi rh$ and the volume is $V = \pi r^2 h$:

(i) Show that the volume of the can is given by **2**
$V = 300\pi r - \pi r^3$

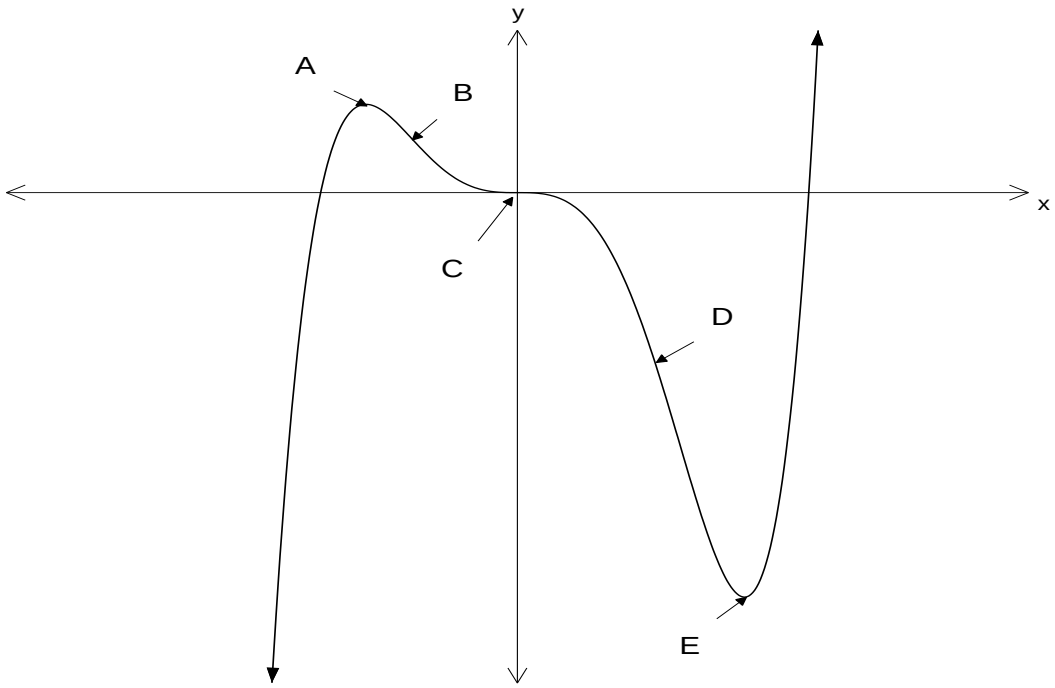
(ii) Find the exact maximum volume of the cylinder. **3**

(iii) Justify your answer to part (ii). **1**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The graph of the curve $y = f(x)$ is drawn below.



(i) Name the points of inflexion. **1**

(ii) When is the graph decreasing? **1**

(iii) Sketch the gradient function. **2**

(b) Consider the curve $y = x^3 - 6x^2 + 8$

(i) Find any stationary points and determine their nature. **3**

(ii) Show that there is one point of inflexion and find its co-ordinates. **2**

(iii) Hence sketch the curve for the domain $-1 \le x \le 5$ indicating important features. **2**

(iv) Find the maximum value of the function in the domain $-1 \le x \le 5$. **1**

End of Examination

KINCOPPAL–ROSE BAY
SCHOOL OF THE SACRED HEART

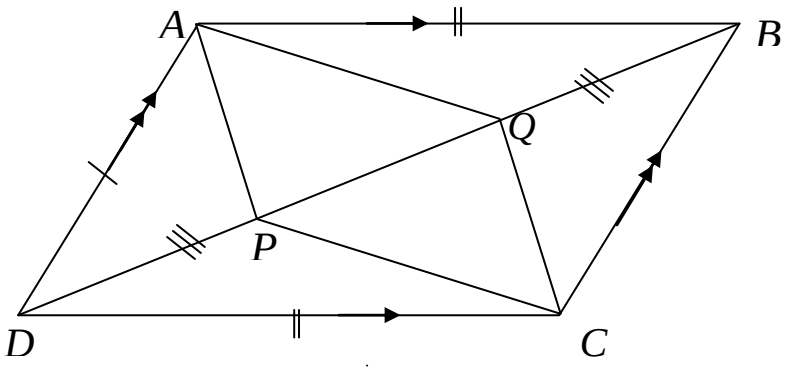
2010
HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

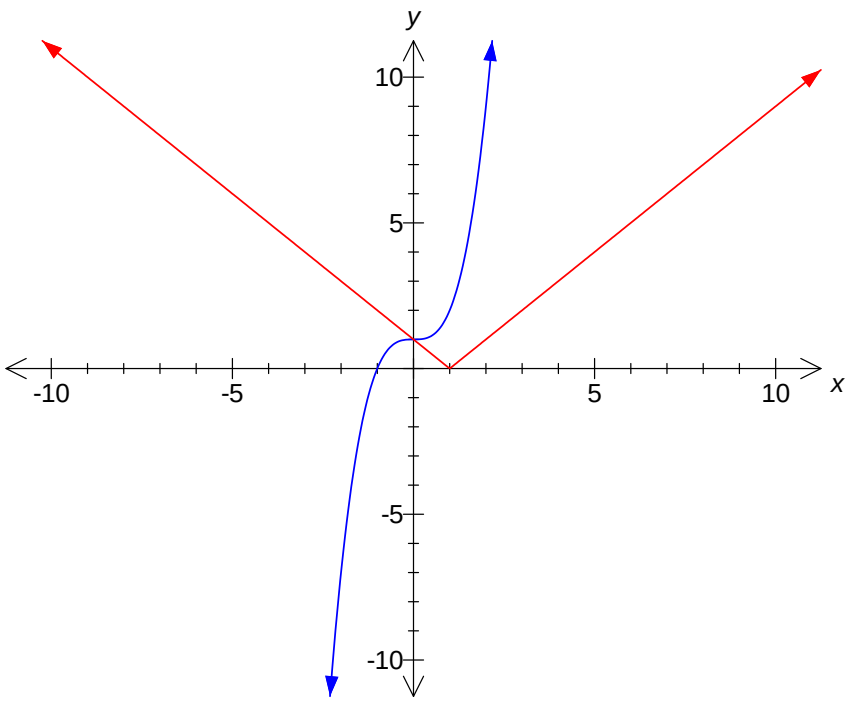
Mathematics
SOLUTIONS

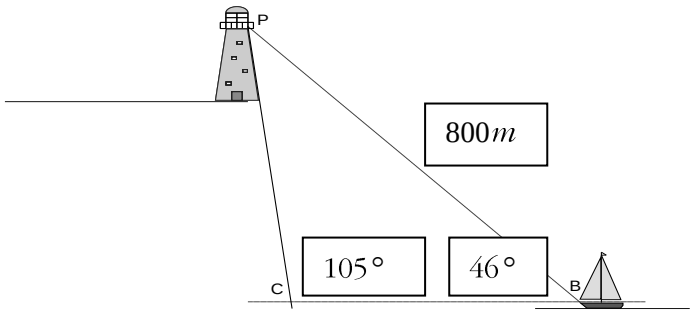
Question 1	Criteria	Marks
1(a)	0.88893905 $= 0.889$	**1** **1**
1(b)	$\dfrac{\sqrt{3}\left[\sqrt{7}+\sqrt{2}\right]}{7-2}$ $= \dfrac{\sqrt{21}+\sqrt{6}}{5}$	**1** **1**
1(c)	$x \le \dfrac{11}{3} = 3\dfrac{2}{3}$ *and* $x \ge 1$ or $1 \le x \le \dfrac{11}{3}$	**1** **1**
1(d)	Domain : $x > 0$, for all real x	**1**
1(e)	$H = \sqrt{5^2 + 12^2} = 13$ $\therefore \sin\theta = \dfrac{5}{13}$	**1** **1**
1(f)	$3^{2(2x-3)} = 3^{3x}$ $4x - 6 = 3x$ $x = 6$	**1** **1** **1**

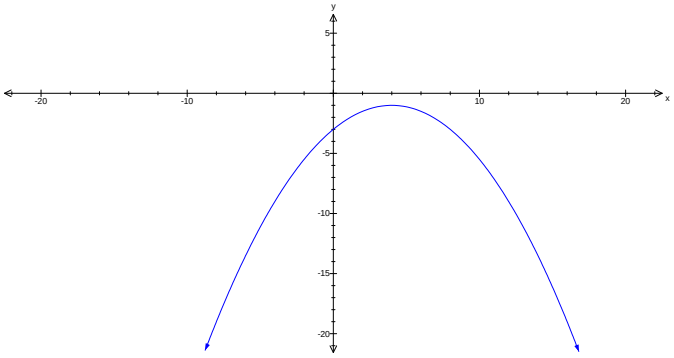
Question 2	Criteria	Marks
2(a)(i)	$m = \tan 135^\circ = -1$	**1**
2(a)(ii)	$y + 4 = -1(x + 4)$ $y = -x - 8$ *or* $x + y + 8 = 0$	**1** **1**
2(a)(iii)	$A(-4, 0)$ $B(0, -4)$ midpoint AB = $(-2, -2)$	**1**
2(a)(iv)	distance centre to A = $\sqrt{(-2+4)^2 + (-2+0)^2}$ $= \sqrt{8}$ *or* $2\sqrt{2}$ *Circle*: $(x+2)^2 + (y+2)^2 = 8$	**1** **1**
2(a)(v)	$\frac{\lvert x + y + 8 \rvert}{\sqrt{1^2 + 1^2}}$ *to* $(-2, -2) = \frac{\lvert -2 - 2 + 8 \rvert}{\sqrt{2}}$ $= \frac{4}{\sqrt{2}}$ *or* $2\sqrt{2}$ Since the distance from the line to the centre of the circle is equal to the radius, the line must be a tangent to the circle	**1** **1**
2(b)	$\angle EAB = 180^\circ - 42^\circ$ (cointerior angles $\angle EAB$ and $\angle ABC$; $EA \parallel CB$) $= 138^\circ$ $\angle DAC = 45^\circ$ (equal angles of isosceles ΔADC; angle sum ΔADC) $\angle DAE = 138^\circ - (45^\circ + 35^\circ)$ (sum of adjacent angles) $= 58^\circ$ $\angle AED = 58^\circ$ (equal angles of isosceles ΔADE)	**1** **1** **1**

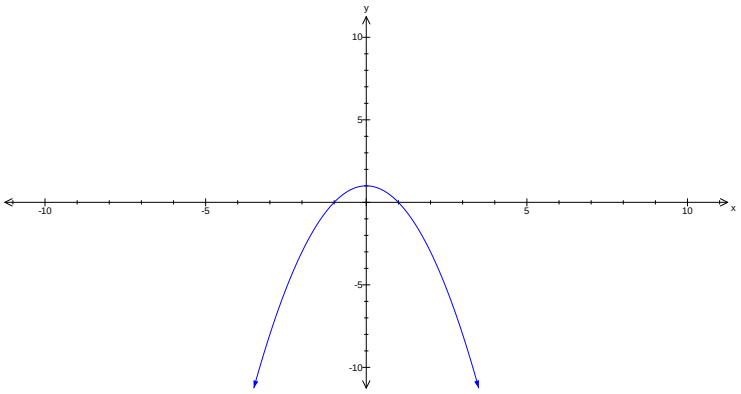
Question 3	Criteria	Marks
3(a)(i)	$-15x^{-6} = \dfrac{-15}{x^6}$	**1**
3(a)(ii)	$7(5-x^2)^6 \times -2x$ $= -14x(5-x^2)^6$	**1** **1**
3(a)(iii)	$3x^2e^{4x} + 4x^3e^{4x}$ $= x^2e^{4x}(3+4x)$	**1** **1**
3(b)	<table><tr><td>a= 0</td><td>1</td><td>2</td><td>3</td><td>b=4</td></tr><tr><td>0</td><td>1</td><td>$\sqrt{2}$</td><td>$\sqrt{3}$</td><td>2</td></tr><tr><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td></tr></table> $n = 4, h = \dfrac{4-0}{4} = 1$ $\int_0^4 f(x)\,dx = \dfrac{1}{3}\left[(0+2)+4\left[1+\sqrt{3}\right]+2\sqrt{2}\right]$ $= 5.2522.....$ $= 5.3$	**1** **1** **1**
3(c)	$\left[\dfrac{1}{5}e^{5x} + e^{-x}\right]_0^1$ $= \left(\dfrac{1}{5}e^5 + e^{-1}\right) - \left(\dfrac{1}{5}e^0 + e^0\right)$ $= \dfrac{1}{5}e^5 + e^{-1} - 1\dfrac{1}{5}$	**1** **1**
3(d)	$\dfrac{1}{3}\int (x^2 - x^{-4})dx$ $= \dfrac{1}{3}\left[\dfrac{x^3}{3} + \dfrac{x^{-3}}{3}\right] + C$ $= \dfrac{1}{9}\left[x^3 + \dfrac{1}{x^3}\right] + C$ $= \dfrac{1}{9}x^3 + \dfrac{1}{9x^3} + C$	**1** **1**

Question 4	Criteria	Marks
4(a)(i)		**1**
4(a)(ii)	$DC = AB$ (opposite sides of parallelogram $ABCD$ are equal) $DP = QB$ (given) $\angle PDC = \angle ABQ$ (alternate $\angle$'s are equal; $AB \parallel DC$) $\therefore \Delta ABQ \equiv \Delta CDP$ (two sides and the included angle equal SAS)	**1** **1** **1**
4(a)(iii)	$AQ = CP$ (corresponding sides of congruent triangles are equal)	**1**
4(b)	$\Delta = (-5)^2 - 4 \times 2 \times k = 25 - 8k$ for real and different roots $\Delta > 0$ $\therefore 25 - 8k > 0$ $k < \frac{25}{8}$ *or* $k < 3\frac{1}{8}$	**1** **1**
4(c)	$3x^2 + 5x - 10 \equiv ax^2 + 3ax + bx^2 + cx + c$ $\equiv x^2(a+b) + x(3a+c) + c$ $c = -10$ $3a + c = 5 \Rightarrow a = 5$ $a + b = 3 \Rightarrow b = -2$ $a = 5, b = -2, c = -10$	**1** **1**

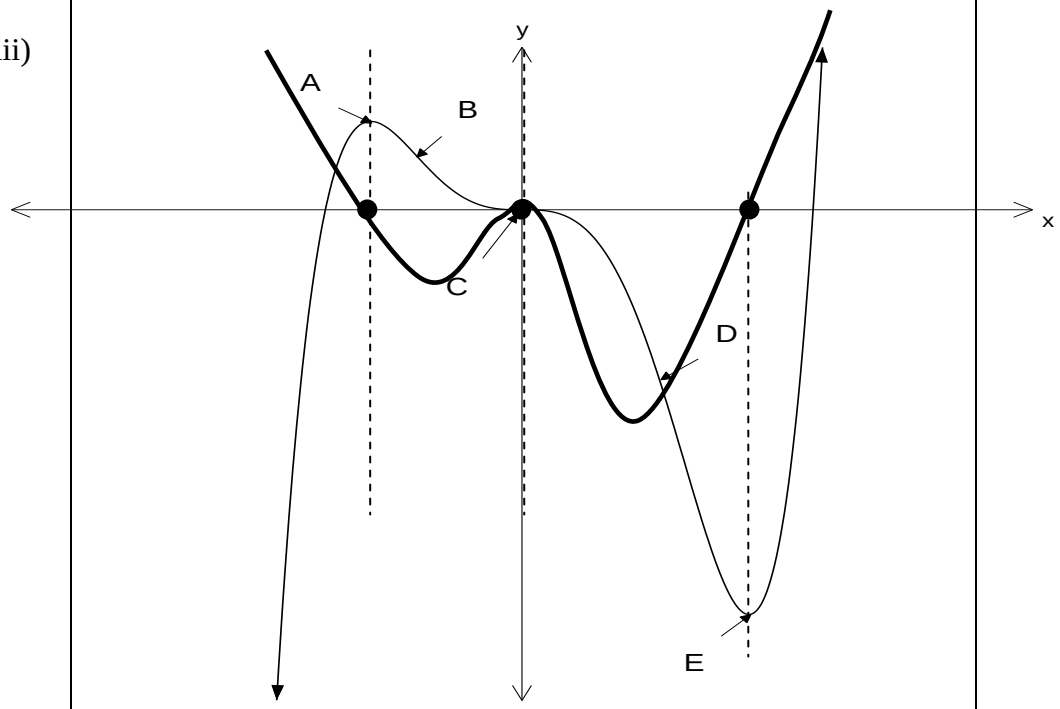
4(d)(i)	$y = x^3 + 1$	**1**
4(d)(ii)		
4(d)(iii)	$x^3 + 1 \leq \lvert x - 1 \rvert$ from the graph the cubic is below the absolute value for: $x \leq 0$	**1**

Question 5	Criteria	Marks
5(a)(i)		**1**
5(a)(ii)	$\angle CBP = 29°$ $\frac{BC}{\sin 29°} = \frac{800}{\sin 105°}$ $BC = 401.52948....metres$ $BC = 402 metres$	**1** **1**
5(b)	$m = \frac{dy}{dx} = 5e^{5x-1}; x = 1, m_1 = 5e^4$ $\therefore m_2 = \frac{-1}{5e^4}$ $x = 1, y = e^4$ $\therefore y - e^4 = \frac{-1}{5e^4}[x-1]$ $y = \frac{-1}{5e^4}[x-1] + e^4 = \frac{1}{5e^4} + e^4 - \frac{x}{5e^4}$	 **1** **1** **1**
5(c)	$\frac{\left(\sec^2\theta\right)\frac{1}{\tan\theta}}{\sin^2\theta} = \frac{\left(\frac{1}{\cos^2\theta}\right)\times\frac{\cos\theta}{\sin\theta}}{\sin^2\theta}$ $= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\times\frac{\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$	**1 , 1** **1**
5(d)	$3\sin^2\theta + 2\sin\theta - 1 = 0$ $(3\sin\theta - 1)(\sin\theta + 1) = 0$ $\sin\theta = \frac{1}{3}, \sin\theta = -1$ $\theta = 19°28', 160°32', 270°00'$	**1** **1** **1**

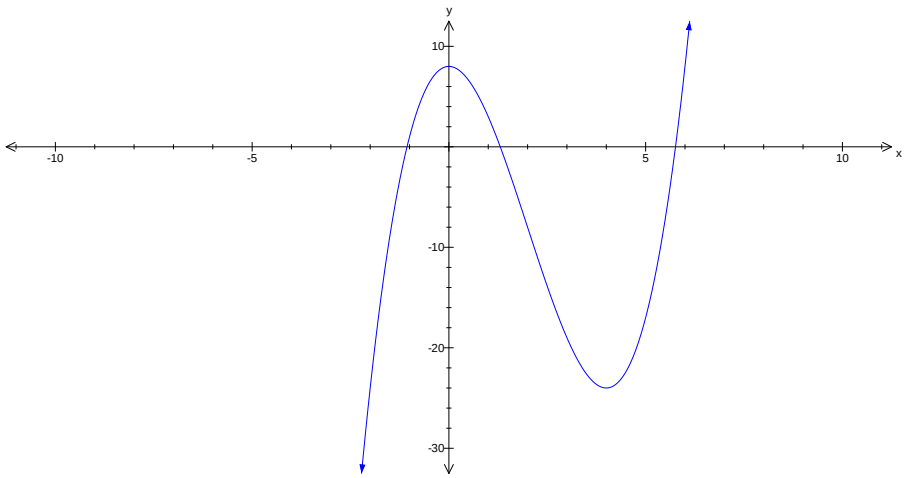
Question 6	Criteria	Marks
6(a)(i)	$\alpha\beta = -\frac{1}{2}$	**1**
6(a)(ii)	$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}$ $=\frac{7}{-\frac{1}{2}}=-14$	**1** **1**
6(a)(iii)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ $=7^2-2\left(-\frac{1}{2}\right)=50$	**1**
6(b)(i)	$(4,-1)$	**1**
6(b)(ii)	$a=2;\ (4,-3)$ 	**2**
6(b)(iii)	length of latus rectum = 8; 4 left of focus and 4 right of focus $(8,-3)$ *and* $(0,-3)$	**2**

6(c)	 $V = \pi\int_0^1 \left(1-x^2\right)^2\, dx$ $= \pi\int_0^1 1-2x^2+x^4\ dx$ $= \pi\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_0^1$ $= \pi\left[1-\frac{2}{3}+\frac{1}{5}-0\right]$ $\frac{8}{15}\pi$	**1** **1** **1**

Question 7	Criteria	Marks
7(a)	$\log_a\left[\frac{2x^2}{6}\right]=\log_a 3$ $\frac{x^2}{3}=3$ $x^2=9$ $x=\pm 3$ $x\neq -3$ $\therefore x=3$	**1** **1**
7(b)(i)	$4^2-3(4)=4, 5(4)-4^2=4$	**1**
7(b)(ii)	$A=\int_0^4 (5x-x^2)-(x^2-3x)dx$ $=\int_0^4 (8x-2x^2)dx=\left[4x^2-\frac{2x^3}{3}\right]_0^4$ $=\left\{4(4)^2-\frac{2(4)^3}{3}\right\}-\left\{4(0)^2-\frac{2(0)}{3}\right\}$ $=64-\frac{2(64)}{3}-0=21\frac{1}{3}units^2$	**1** **1** **1**
7(c)(i)	$600\pi=2\pi r^2+2\pi rh$ $h=\frac{300-r^2}{r}$ $V=\pi r^2\left[\frac{300-r^2}{r}\right]=300\pi r-\pi r^3$	 **1** **1**
7(c)(ii)	$\frac{dy}{dx}=300\pi-3\pi r^2=0$ $r^2=100, \therefore r=10cm, \quad reject \quad r=-10cm$ $\therefore V=3000\pi-1000\pi=2000\pi cm^3$	**1** **1** **1**
7(c)(ii)	$\frac{d^2y}{dx^2}=-6\pi r=-6\pi(10)=-60\pi<0$ So V is a maximum.	**1**

Question 8	Criteria	Marks
8(a)(i)	*B, C, D*	**1**
8(a)(ii)	Between *A* and *C* and *C* and *E*	**1**
8(a)(iii)	 1 correct zeros 1 correct shape	**2**
8(b)(i)	$\frac{dy}{dx} = 3x^2 - 12x$ $\frac{dy}{dx} = 0 \Rightarrow 3x(x-4) = 0$ $\therefore x = 0, 4 \qquad y = 8, -24$ $\frac{d^2y}{dx^2} = 6x - 12$ $at\ x = 0\ \frac{d^2y}{dx^2} < 0 \qquad \Rightarrow (0,8)\ is\ \max$ $at\ x = 4\ \frac{d^2y}{dx^2} > 0 \qquad \Rightarrow (4,-24)\ is\ \min$	**1** **1** **1**

<table>
<tr>
<td>8(b)(ii)</td>
<td>
$\frac{d^2y}{dx^2} = 0$
$6x - 12 = 0 \Rightarrow x = 2$
Test:
<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$\frac{d^2y}{dx^2}$</td><td><0</td><td>0</td><td>>0</td></tr>
</table>
There is a change in concavity and hence $(2,-8)$ is a point of inflexion
</td>
<td>1

1</td>
</tr>
<tr>
<td>8(b)(iii)</td>
<td>


1 for endpoints: $(-1,1)$ and $(5,-17)$
1 for sketch showing important features turning point, y-intercept
</td>
<td>1

1</td>
</tr>
<tr>
<td>8(b)(iv)</td>
<td>Max value in the domain is 8</td>
<td>1</td>
</tr>
</table>