MLC SCHOOL BURWOOD

YEAR 12 - 2 UNIT MATHEMATICS/ EXTENSION I

HALF YEARLY EXAMINATION March 2002

TIME ALLOWED 2 HOURS 5 MINUTES

(Includes 5 minutes reading time)

INSTRUCTIONS:

1. ALL SIX questions are to be attempted.
2. Questions are NOT of equal value.
3. Start each question in a new booklet.
4. All necessary working should be shown in every question.
5. Marks may be deducted for careless or poorly arranged work.
6. Clearly write your name and teacher on each booklet.

Question 1 (14 marks) **MARKS**

a) Given the Energy equation $\boldsymbol{E = m\,c^2}$
find the value of $\boldsymbol{c}$ correct to 2 significant figures if
$E = 32.8$ and $m = 8.1$ (1)

b) Simplify $(\sqrt{14} - \sqrt{2})^2$ giving the exact answer in simplest form (2)

c) Solve $\left| \frac{x}{2} - 3 \right| > 5$ (2)

d) Simplify $\frac{2x+1}{x} + \frac{3}{2x}$ (1)

e) Factorise fully $5x - x^2 - xy + 5y$ (2)

f) Solve simultaneously $5x - y = 19$ and $2x + 5y = -14$ (2)

g) Solve $3^{2x-3} = 27$ (2)

h) Solve $x(x-6) = 16$ (2)

Question 2 **(12 marks)** **START A NEW BOOKLET**

a) A parabola has a focus $(-4, 0)$ and directrix $x = 4$. (3)

i) Draw a neat sketch of the parabola

ii) What is the equation of the parabola?.

iii) What is the effect on the shape of the parabola if the focal length is reduced?

b) The equation of a parabola is $x^2 + 2x - 20y - 59 = 0$

i) Complete the square to find the coordinates of the vertex of the parabola. (2)

ii) Find the focal length of the parabola. (1)

iii) Hence sketch the parabola, clearly showing the vertex, focus, directrix and y-intercept. (3)

c) A point $P(x, y)$ moves so that the line PS is perpendicular to the line PT, where S is $(3, 1)$ and T is $(-2, 2)$. (3)

Show that the locus of point P is $x^2 - x + y^2 - 3y - 4 = 0$.

Question 3 (12 marks) **START A NEW BOOKLET**

a) i) Convert 7 radians to degrees and minutes (correct to nearest minute) (1)

ii) Evaluate $\lim_{x \to 0} \frac{\sin \frac{x}{5}}{x}$ (1)

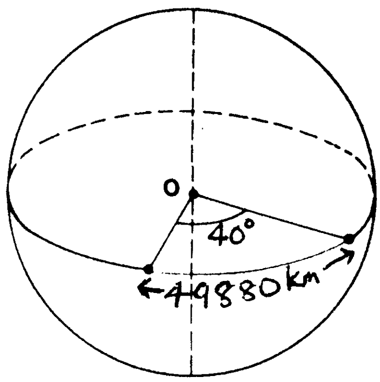
iii) Solve $\tan x = -\sqrt{3}$ for the domain $0 \le x \le 2\pi$ (2)

b) A circle has a radius of 2.5 metres. (3)

i) Find the area of a sector if the angle subtended at the centre of the circle is $\frac{3\pi}{4}$.

ii) What fraction of the circle does the sector represent?

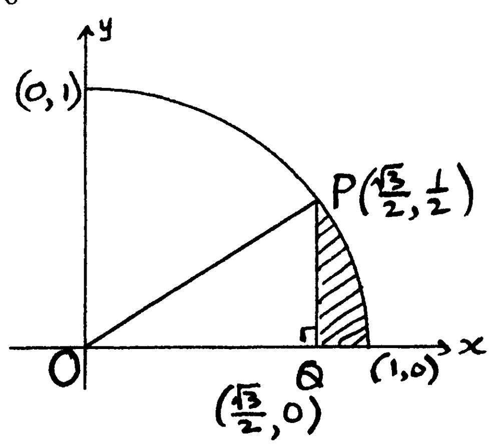
c) The planet Jupiter is known as one of the giant planets. (2)
The length of an arc on the circumference subtended by an angle of $40°$ at its centre is 49880 kilometres.
Find the radius of Jupiter. Answer correct to the nearest 100 kilometres.



d) The first quadrant of the circle $x^2 - y^2 = 1$ is shown below. (3)
Point $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ lies on the circle and line PQ is perpendicular to the x-axis.

i) Show that the exact value of $\angle POQ$ is $\frac{\pi}{6}$.

ii) Find the exact shaded area.

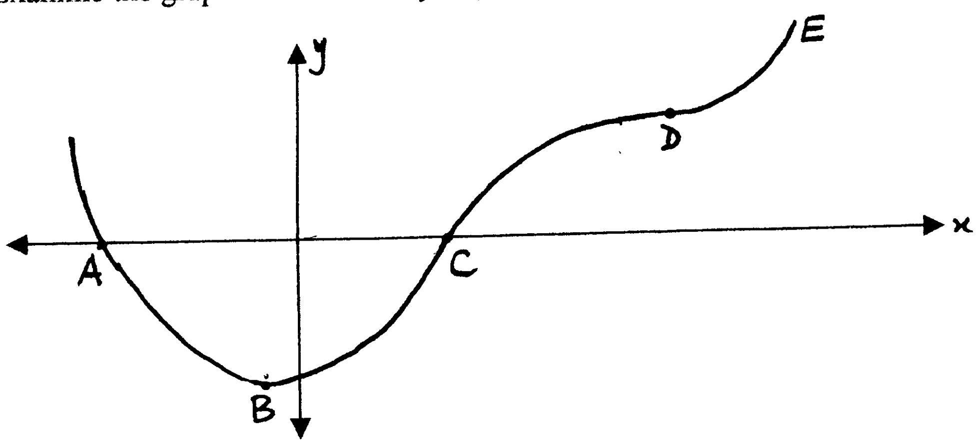


Question 4 (13 marks) **START A NEW BOOKLET** **MARKS**

a) i) Differentiate $y = (3x - 7)^{11}$ with respect to x . (1)

ii) If $f(x) = \sqrt{x}$ find the value of $f'(4)$. (2)

b) Examine the graph of the curve y= f(x) below, and answer the questions. (4)



i) At point A, $f(x) = 0$ and $f'(x) = 0$. TRUE or FALSE ?

ii) At point B, $f'(x) = 0$ and $f''(x) > 0$. TRUE or FALSE?

iii) At point C, $f(x) = 0$ and $f''(x) = 0$. TRUE or FALSE?

iv) Between which two letters is the curve increasing and $f''(x) < 0$?

c) Find the primitive of :

i) $3x^9$ (1)

ii) $\dfrac{1}{(5x-2)^2}$ (1)

d) Evaluate :

i) $\displaystyle\int_{-1}^{2} \frac{t^4}{2}\, dt$ (2)

ii) $\displaystyle\int_{0}^{3} \sqrt{x+1}\, dx$ (2)

Question 5 (13 marks) **START A NEW BOOKLET** **MARKS**

a) $y = x^3 - 3x^2 - 9x + 15$

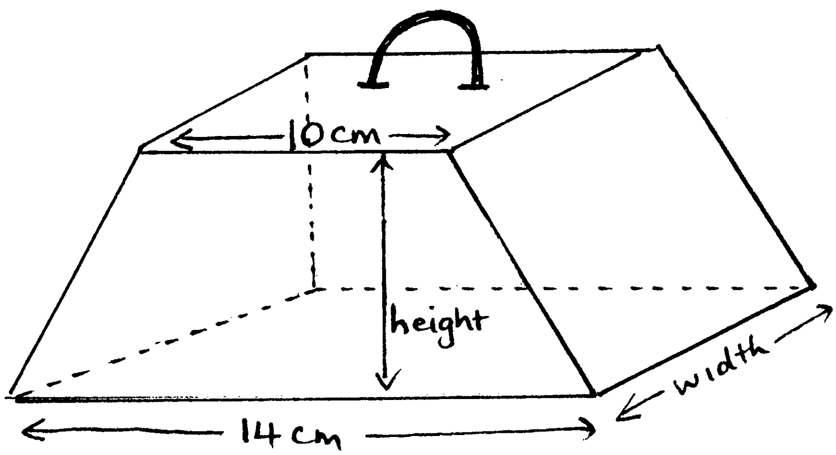
i) Find all stationary points and determine their nature. (3)

ii) For what values of x is the curve concave upwards? (1)

iii) Draw a neat sketch of the curve in the domain $-2 \le x \le 6$ (2)

iv) State the absolute maximum and absolute minimum of the curve in this domain. (2)

b) A fancy gift box for lollies is made in the shape of a trapezoidal right prism seen below. The length of the top and bottom faces are 10 cm and 14 cm as shown. The manufacturer must ensure that the sum of the height and width totals 30 cm.



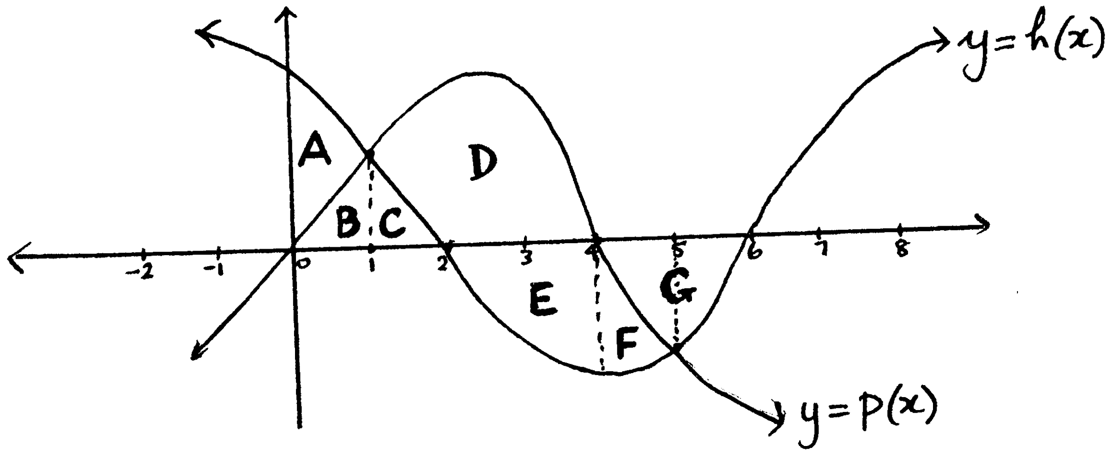
i) If h is the perpendicular height of the prism, show that the volume of the gift box is given by $V = 360h - 12h^2$ (2)

ii) Find the maximum volume of the gift box. (3)

Question 6 (14 marks) **START A NEW BOOKLET** **MARKS**

a) The two curves $y = p(x)$ and $y = h(x)$ are sketched below. (3)

Different areas enclosed by the curves and the axes are labelled A to G



The integrals below represent the sum of which areas.

i) $\int_0^4 p(x)\,dx$

ii) $\int_1^5 \{p(x) - h(x)\}\,dx$

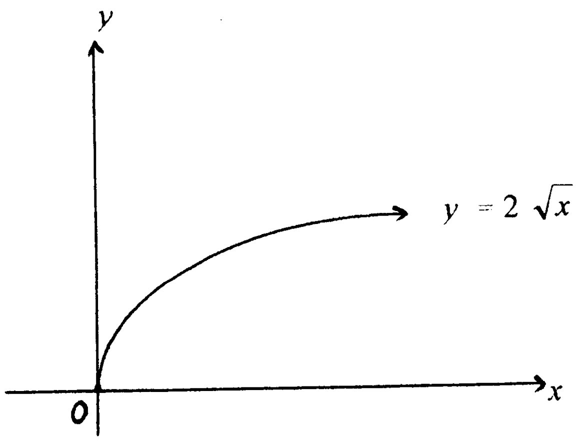
iii) Give an integral which would define the area denoted by letter G.

b) i) Find the points of intersection of the curve $y = x^2 - 2x - 3$ and the line $y = 3x - 3$. (2)

ii) Sketch the graph of $y = x^2 - 2x - 3$ and $y = 3x - 3$ on the same number plane. (1)

ii) Calculate the area of the region enclosed by the graphs . (2)

c) The region enclosed by the curve $y = 2\sqrt{x}$ and the y-axis between $y = 1$ and $y = 3$ is rotated about the y-axis. Find the exact volume of the solid of revolution formed . (3)



Question 6 continued....

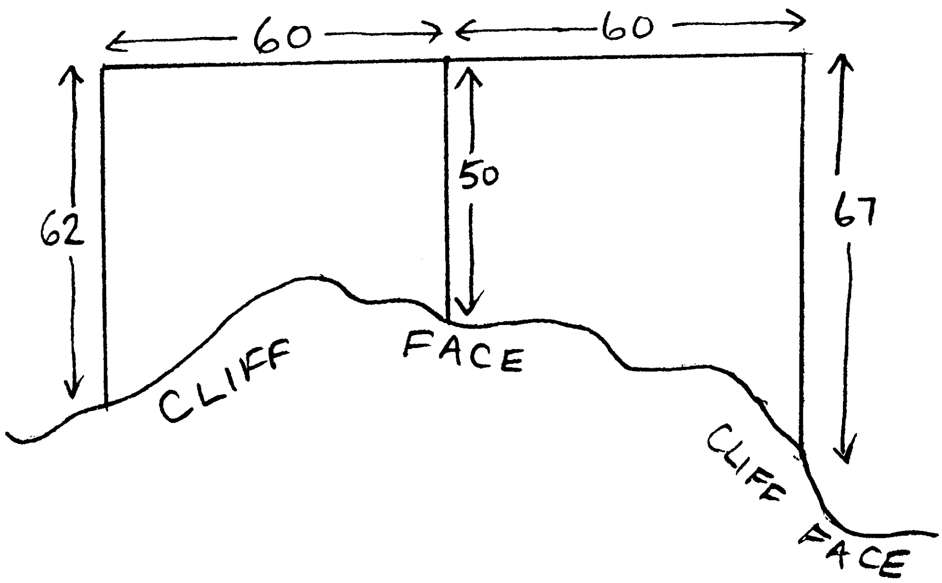
d) A farmer has a field area that is bounded by fences on three straight sides.

The fourth boundary is a steep cliff. He needs to determine the area for seed sowing.

Distances within the field are given on the diagram.

All distances are in metres

Diagram not to scale.



i) Use the trapezoidal rule to determine the approximate area of the field. (2)

ii) Is this estimate greater or less than the true area? Justify your answer. (1)

Q1 - SL   Q4 JM
Q2 SW   Q5 KN
Q3 WC   Q6 CB

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Question 1 /14 marks

a) $E = mc^2$ (1)

$32.8 = 8.1\, c^2$

$c = \sqrt{\frac{32.8}{8.1}}$

$= 2.0$ (2 sig fig)

b) $(\sqrt{14} - \sqrt{2})^2 = 14 - 2\sqrt{14}\sqrt{2} + 2$ (2)

$= 16 - 2\sqrt{28}$

$= 16 - 4\sqrt{7}$

c) $\left|\frac{x}{2} - 3\right| > 5$ (2)

$\frac{x}{2} - 3 > 5$ or $-\left(\frac{x}{2} - 3\right) > 5$

$\frac{x}{2} > 8$ $-\frac{x}{2} > 2$

$x > 16$ or $x < -4$

d) $\frac{2x+1}{x} + \frac{3}{2x} = \frac{(4x+2)+3}{2x}$ (1)

$= \frac{4x+5}{2x}$

e) $5x - x^2 - xy + 5y$ (2)

$= x(5-x) + y(5-x)$

$= (x+y)(5-x)$

f) $5x - y = 19$ (1)

$2x + 5y = -14$ (2)   (2)

Subst y in (2): $2x + 5(5x - 19) = -14$

$27x - 95 = -14$

$x = \frac{81}{27}$

$= 3$

$\therefore y = 5(3) - 19 = -4$

Soln $(x, y) = (3, -4)$

g) $3^{2x-3} = 27 = 3^3$ (2)

$\therefore 2x - 3 = 3$

$2x = 6$

$x = 3$

h) $x(x-6) = 16$ (2)

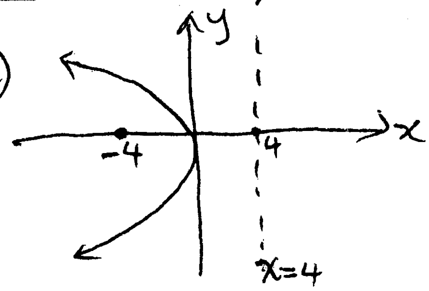
$x^2 - 6x - 16 = 0$

$(x-8)(x+2) = 0$

$x = 8$ or -2

Question 2 /12 marks

a) (3)



ii) $y^2 = -4ax$

$y^2 = -16x$

iii) $a = 4$ $a = 2$ (For Example)

$y^2 = -16x$ $y^2 = -8x$

$x = -\frac{1}{16}y^2$ $x = -\frac{1}{8}y^2$

Larger dilation factor.

Curve becomes steeper when focal length is reduced.

b) i) $x^2 + 2x = 20y + 59$ (2)

$(x+1)^2 = 20y + 59 + 1^2$

$(x+1)^2 = 20(y+3)$

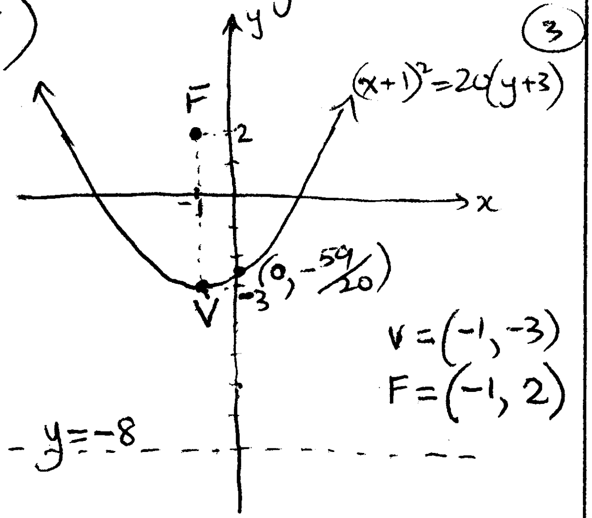
Vertex $(-1, -3)$

Q2 b)

ii) $(x+1)^2 = 20(y+3)$ (1)

Focal length 5

iii) (3)



$V = (-1, -3)$

$F = (-1, 2)$

c) $S(3,1)$ $T(-2,2)$ (3)

$m_{PS} = \frac{y-1}{x-3}$ $m_{PT} = \frac{y-2}{x+2}$

$PT \perp PS \therefore m_{PS} \times m_{PT} = -1$

$\frac{y-1}{x-3} \times \frac{y-2}{x+2} = -1$

$\frac{y^2-3y+2}{x^2-x-6} = -1$

$y^2 - 3y + 2 = -(x^2 - x - 6)$

$\therefore x^2 - x + y^2 - 3y - 4 = 0$

QED.

Question 3 / 12 marks

a) i) $7^c = 7 \times \frac{180}{\pi}$ (1)

$= 401°4'$ (near minute)

ii) $\lim_{x \to 0} \frac{\sin \frac{x}{5}}{x}$ (1)

$= \lim_{x \to 0} \frac{\sin \frac{x}{5}}{5 \times \frac{x}{5}}$

$= \frac{1}{5} \lim_{x \to 0} \frac{\sin \frac{x}{5}}{\frac{x}{5}} = \frac{1}{5}$

iii) $\tan x = -\sqrt{3}$ $0 \le x \le 2\pi$ (2)

$\tan^{-1}\sqrt{3} = (60°)\ \pi/3$ Quad 2 & 4

$\therefore x = \pi - \pi/3,\ 2\pi - \pi/3$

$= 2\pi/3,\ 5\pi/3$

b) $r = 2.5m$ $\theta = 3\pi/4$

i) $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times \frac{25}{4} \times \frac{3\pi}{4}$

$= \frac{75}{32}\pi\ m^2$ (7.363 3dp)

ii) Fraction $= \frac{3\pi/4}{2\pi} = 3/8$

c) $40° = \frac{40\pi}{180} = \frac{2\pi}{9}$ (2)

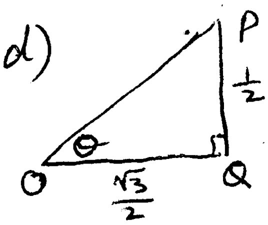
$l = r\theta$

$49880 = r \times \frac{2\pi}{9}$ ✓

$r = 71447.83\ldots$

Radius 71400 km (nearest 100 km)

d)



i) $\tan\theta = \frac{1/2}{\sqrt{3}/2}$ (1)

$= \frac{1}{\sqrt{3}}$

$\therefore \theta = \pi/6$ (30°)

ii) Shaded Area = Sector − Triangle (2)

OR $\frac{1}{2}$ (Minor segment of $\pi/3^c$)

$r = \sqrt{3/4 + 1/4}$

$= 1$

Area $= \left(\frac{1}{2} 1^2 \frac{\pi}{6}\right) - \left(\frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{1}{2}\right)$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

OR $A = \frac{1}{2}\left\{\frac{1}{2}r^2(\alpha - \sin\alpha)\right\}$

$= \frac{1}{4}(\pi/3 - \sin\pi/3)$

$= \pi/12 - \frac{\sqrt{3}}{8}$

Question 4 /13 marks

a) i) $y=(3x-7)^{11}$ (1)

$\frac{dy}{dx} = 11\times3(3x-7)^{10}$

$= 33(3x-7)^{10}$

ii) $f(x)=\sqrt{x} = x^{1/2}$ (2)

$f'(x) = \frac{1}{2}x^{-1/2}$ ✓

$= \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{4}$ ✓

b) (4)

i) FALSE

ii) TRUE

iii) TRUE

iv) C to D

c) i) $\int 3x^9\,dx = \frac{3x^{10}}{10} + C$ (1)

ii) $\int (5x-2)^{-2}dx$

$= \frac{(5x-2)^{-1}}{5(-1)} + C$ (1)

$= -\frac{1}{5(5x-2)} + C$

d) i) $\int_{-1}^{2} \frac{t^4}{2}\,dt = \left[\frac{t^5}{10}\right]_{-1}^{2}$

$= \frac{32}{10} - \left(-\frac{1}{10}\right) = \frac{33}{10}$ (2)

ii) $\int_0^3 (x+1)^{\frac{1}{2}}dx$ (2)

$= \left[\frac{2}{3}(x+1)^{3/2}\right]_0^3$

$= \frac{2}{3}\times8 - \frac{2}{3}\times1$

$= \frac{14}{3}$

Question 5 /13 marks

a) $y = x^3-3x^2-9x+15$

$y' = 3x^2-6x-9$

$y'' = 6x-6$

i) Test $y'=0$: (3)

$3x^2-6x-9=0$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3$ or -1

* At $x=3$, $y=27-27-27+15$

$=-12$

$y''=12>0$ Concave up.

$\therefore (3,-12)$ is a Minimum Turning Pt.

* At $x=-1$, $y=-1-3+9+15$

$=20$

$y''=-12<0$ Concave down

$\therefore (-1,20)$ is a Maximum T.P.

ii) Curve concave up when $\frac{d^2y}{dx^2}>0$ (1)

$6x-6>0$

$\therefore x>1$

Curve concave upwards for all $x>1$

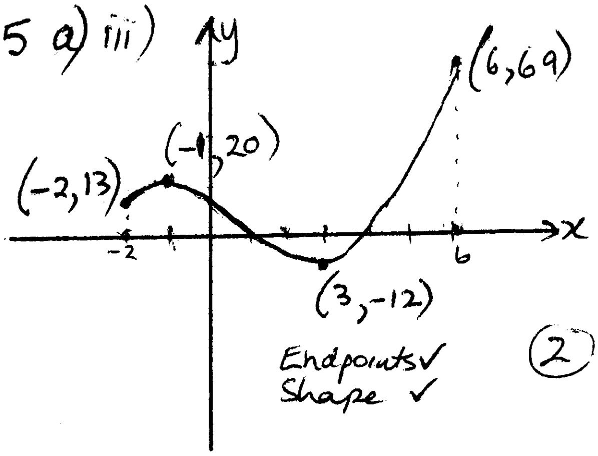
iii) At $x=-2$, $y=-8-12+18+15$

$=13$

At $x=6$, $y=216-108-54+15$

$=69$

Q5 a) iii)



(2)

iv) Absolute Maximum is 69 (2)

Absolute Minimum −12

b) i) (2)

$h + w = 30$

$\therefore w = 30 - h$

$V = \frac{1}{2}(x+y)h \times w$

$= \frac{1}{2} \times 24 \times h \times (30-h)$

$= 12h \times 30 - 12h^2$

$= 360h - 12h^2$ QED

ii) $\frac{dV}{dh} = 360 - 24h$ (3)

$\frac{d^2V}{dh^2} = -24$

$\frac{dV}{dh} = 0$ when $360 = 24h$

$h = 15$ ✓

At $h = 15$, $\frac{d^2V}{dh^2} < 0$

$\therefore$ ∩ a maximum ✓ will occur

Sub $h=15$, $V = 360 \times 15 - 12 \times 15^2$

$= 2700$ ✓

Max. Volume $2700\ cm^3$

Question 6 / 14 marks.

a) i) B + C + D (3)

ii) D + E + F

iii) $\left|\int_4^5 p(x)\,dx\right| + \left|\int_5^6 h(x)\,dx\right|$

b) i) $y = x^2 - 2x - 3$, $y = 3x - 3$ $\quad y = (x-3)(x+1)$ (2)

$x^2 - 2x - 3 = 3x - 3$

$x^2 - 5x = 0$

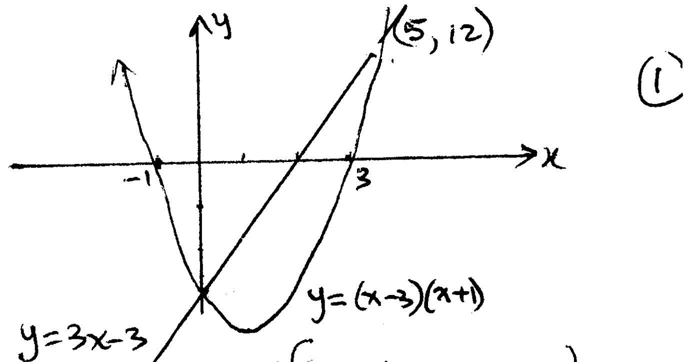
$x(x-5) = 0 \Rightarrow x = 0, 5$

At $x = 0$, $y = -3$

At $x = 5$, $y = 12$

Intersection at $(0,-3)$ & $(5,12)$

ii) (1)



iii) $A = \int_0^5 \left[(3x-3) - (x^2-2x-3)\right] dx$

$= \int_0^5 (-x^2 + 5x)\,dx$ ✓

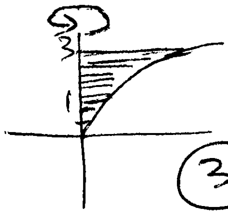
$= \left[-\frac{x^3}{3} + \frac{5x^2}{2}\right]_0^5$

$= -\frac{125}{3} + \frac{125}{2}$

Area $= \frac{125}{6}$ units^2

$= 20\frac{5}{6}\ u^2$

Q6 c)



(3)

$y = 2\sqrt{x}$

$\frac{y}{2} = \sqrt{x}$

$x^2 = \frac{y^4}{16}$ ✓

Sue Q2
Kestie Q5
Caroline Q6
Julie Q4
Will Q3
Scott Q1

$V = \pi \int_1^3 x^2 \, dy$

$= \pi \int_1^3 \frac{y^4}{16} \, dy$

$= \pi \left[\frac{y^5}{5 \times 16} \right]_1^3$ ✓

$= \frac{\pi}{80}(243 - 1)$

$= \frac{121\pi}{40}$ ✓

Volume is $\frac{121\pi}{40}$ units^3

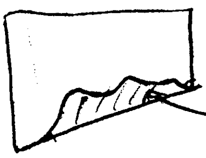
d) i) $A \doteqdot \frac{60}{2}(62+50) + \frac{60}{2}(50+67)$ (2)

$= 3360 + 3510$

$= 6870$

Approx. area of field 6870 m^2

ii) Estimate is greater than true area since line for trapezium is definitely outside thee area (1)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