Newington College HSC Mini Examination April 2012

Answer Questions 1 to 10 (multiple choice) on the sheet attached to the end of the paper which is to be removed.

Question 1

The correct equation for a circle with centre $(-1,2)$ and radius of 5 is:

A. $(x-1)^2+(y+2)^2=25$

B. $(x-1)^2+(y+2)^2=5$

C. $(x+1)^2+(y-2)^2=25$

D. $(x+1)^2+(y-2)^2=5$

Question 2

The integral of $\int \frac{1}{x^2}+\frac{1}{x}\,dx$ is

A. $-\frac{3}{x^3}-\frac{2}{x^2}+c$

B. $-\frac{1}{x}+\log_e x+c$

C. $-\frac{1}{x}-\frac{2}{x^2}+c$

D. $-\frac{1}{3x^3}+\log_e x+c$

Question 3

If $\log_a 2=x$ and $\log_a 3=y$, then $\log_a 12$ can be written as

A. $2x+y$

B. x^2y

C. $2xy$

D. $x+2y$

Question 4

For the parabola, $(y-k)^2=4a(x-h)$, the axis of symmetry is given by:

A. $x=h$

B. $x=k$

C. $y=h$

D. $y=k$

Q5…cont/page 2

Question 5

If $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$, then for $x > 0$, $f(x)$ is

A. Increasing and concave down

B. Decreasing and concave down

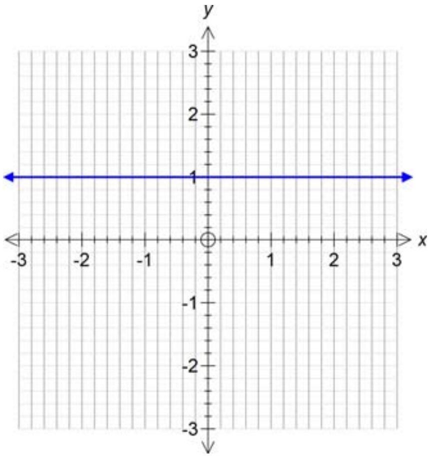
C. Increasing and concave up

D. Decreasing and concave up

Question 6

Which graph best represents the equation, $y = e^x - e^{-x}$?

A.



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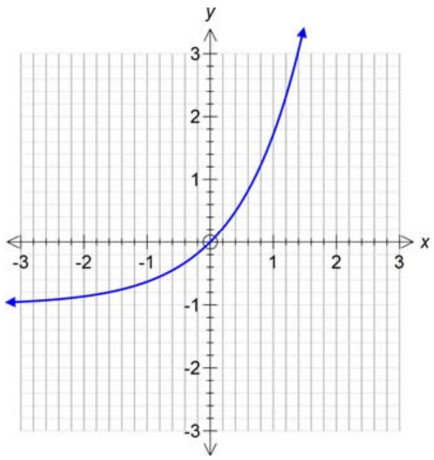

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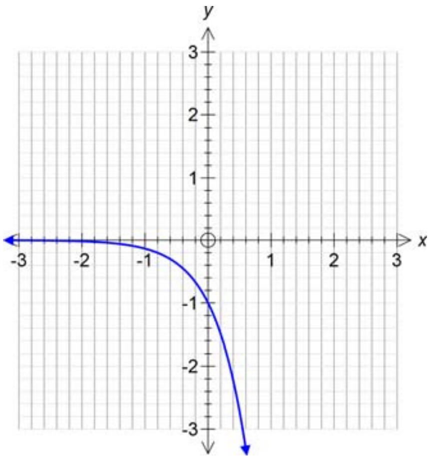

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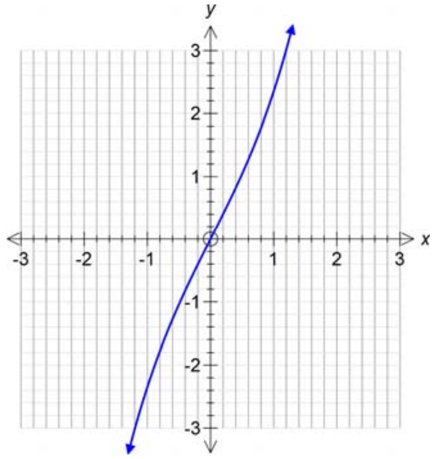


Q7…cont./Page 3

Question 7

If $\sin\theta = \cos\left(\frac{3\pi}{4}\right)$, then θ could be equal to

A. $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

B. $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$

C. $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$

D. $\frac{\pi}{4}$ and $\frac{7\pi}{4}$

Question 8

$2^{-\log_2 5}$ is equal to

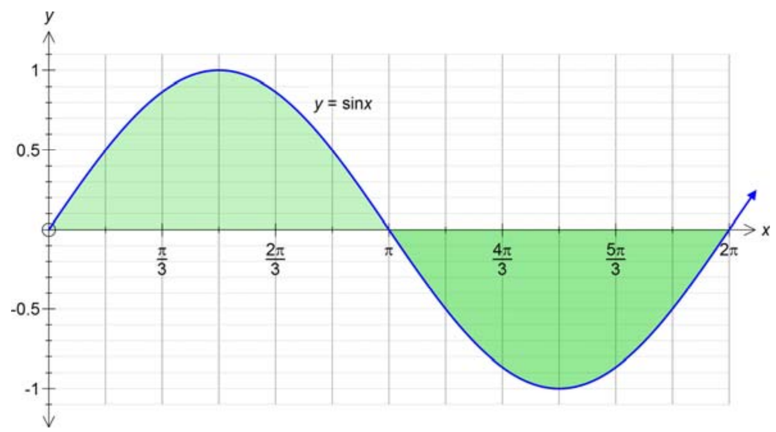
A. $\frac{1}{5}$

B. $-\frac{1}{5}$

C. 5

D. -5

Question 9



Using the graph above, or otherwise, if $\int_0^{\pi} \sin x \, dx = 1$ then the area shaded above is equal to:

A. 2

B. 0

C. π

D. 2π

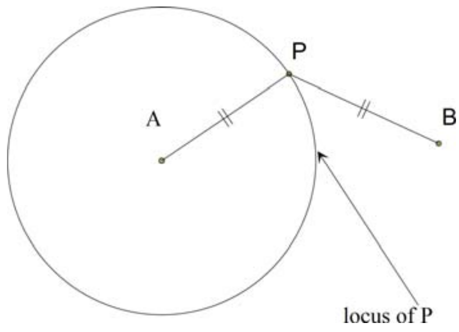
Q10…cont/page 4

Question 10

Choose the locus from the diagrams below that is best described by the information given;

"A point P moves such that it is equidistant from two fixed points A and B."

A.



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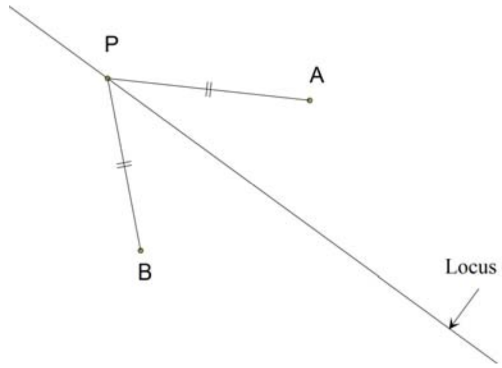
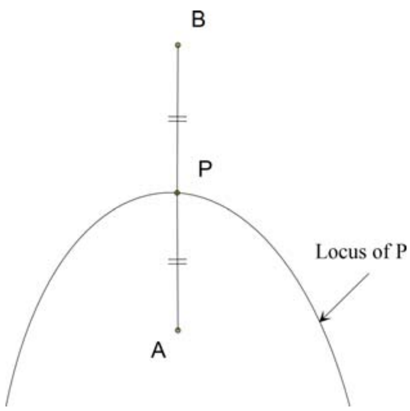
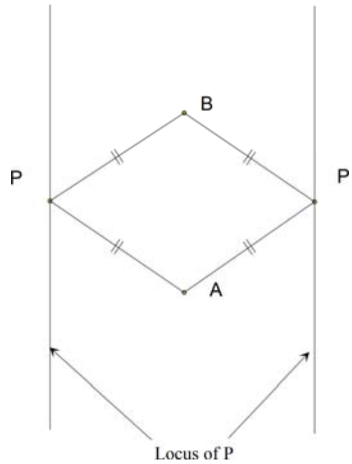


C.



D.



Q11…cont/page 5

Question 11 Start this question in a new booklet (15 Marks) **Marks**

(a) If α and β are the roots of the quadratic equation, $y = x^2 - 5x + 6$, find:

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\alpha^2 + \beta^2$

(iv) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ (6)

(b) Show that $y = -\left(x^2 - 3x + 6\right)$ is negative definite for all values of x. (2)

(c) For the curve, $y = x^2 - 4x$, rewrite in the form, $(x-h)^2 = 4a(y-k)$, and hence, or otherwise, find:

(i) the vertex

(ii) the focal length

(iii) the focus

(iv) the equation of the directrix (7)

Question 12 Start this question in a new booklet (15 marks)

(a) Find the following limits:

(i) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

(ii) $\lim\limits_{x \to \infty} \dfrac{1 - x}{x}$ (3)

Q12…cont./page 6

Question 12 (cont.) **Marks**

(b) Differentiate the following:

(i) $\left(x^3 - 5\right)^7$

(ii) $x^3\left(2 - x\right)^4$

(iii) $\dfrac{x}{x^2 - 4}$ (8)

(c) (i) Find the roots of the curve, $y = x^2 - 3x + 2$.

(ii) Find the equation of the tangent for the root with the greatest value of *x*. (4)

Question 13 Start this question in a new booklet (15 marks)

(a) For the curve, $y = x^3 + x^2 - x + 5$,

(i) find any stationary points and their nature.

(ii) find any points of inflexion.

(iii) Sketch the curve showing all the above and the *y* intercept. (8)

(b) A square-based prism has a total surface area of 96 cm^2.

(i) Using *x* cm as the base length and *y* cm as the height draw a diagram of the prism.

(ii) Show that $y = \dfrac{\left(48 - x^2\right)}{2x}$

(iii) Hence, write an equation for the volume of the prism, in terms of *x* **only**.

(iv) Hence, or otherwise, find the maximum volume of the prism and the values of *x* and *y* when this occurs. (7)

Q14…cont./page 7

Question 14 Start this question in a new booklet (15 marks) Marks

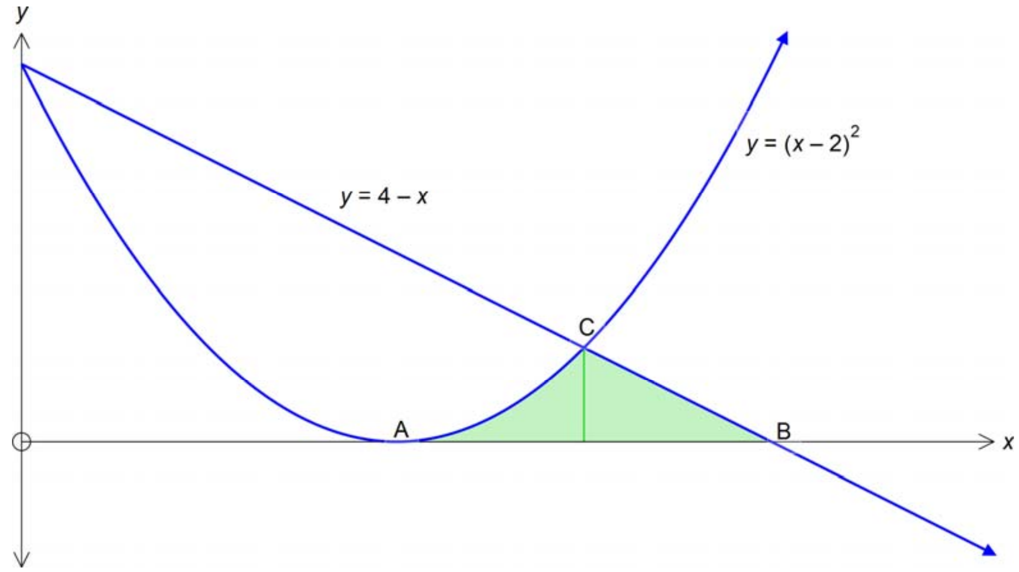
(a) Find the integral of

(i) $\int 3x^2 + \frac{2}{x^2}\, dx$

(ii) $\int_0^1 (4-x)^5\, dx$ (4)

(b) Find the area bounded by the curve $y = 4x - x^2$ and the x axis. (3)

(c) Below are the curve $y = (x-2)^2$ and the straight line $y = 4 - x$,



(i) Find the points of intersection A, B and C.

(ii) Hence, find the shaded area. (6)

(d) Without the use of a sketch, explain with reasoning, why $\int_{-a}^{a} x^5 - x^3\, dx = 0$

(2)

Q15…cont./page 8

Question 15 Start this question in a new booklet (15 marks) Marks

(a) Simplify $\log_3 27 - \log_9\left(\frac{1}{3}\right) + 7$ (2)

(b) (i) Find the first and second derivatives of $f(x) = \frac{x}{e^x}$.

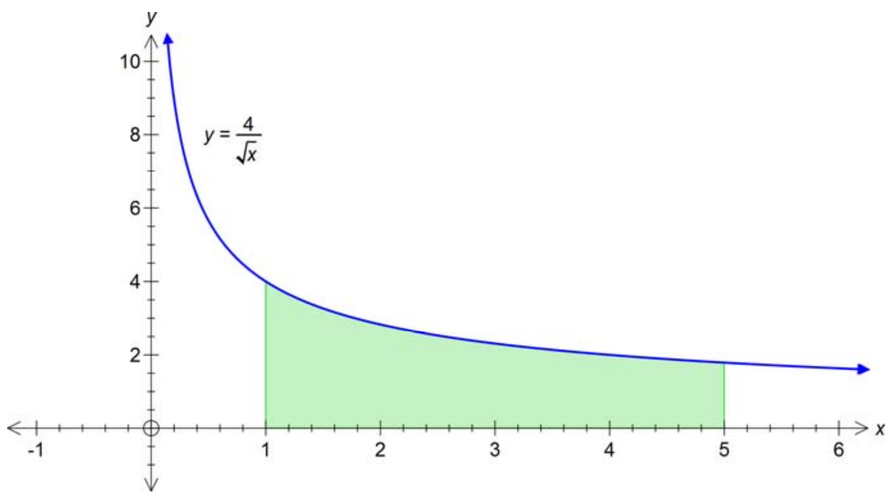
(ii) Find any stationary points for the curve determine their nature.

(iii) Find any points of inflexion.

(iv) Explain why $f(x) \to 0$ as $x \to \infty$.

(v) Sketch $f(x) = \frac{x}{e^x}$, using the information above. (8)

(c) The graph shows the curve $y = \frac{4}{\sqrt{x}}$, with the area under the curve from $1 \le x \le 5$ shaded.

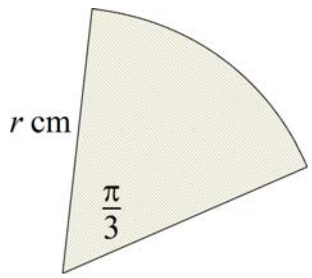


If this area is now revolved around the x axis find the exact value of the volume generated. (5)

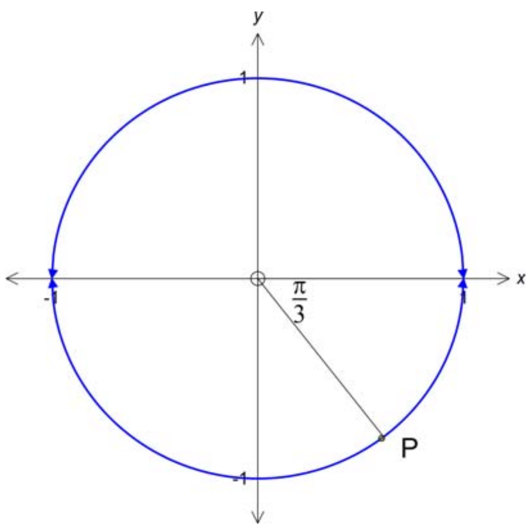
Q16…cont./page 9

Question 16 Start this question in a new booklet (15 marks) Marks

(a) In the diagram, the area of the sector is $\frac{3\pi}{2}$ cm^2. Find the radius of the sector. (2)



(b) In the unit circle shown, find the exact value of the co-ordinates of the point P. (3)



(c) Solve the following equation, for $0 \le x \le 2\pi$, (3)

$$2\sin^2 x - 1 = 0$$

(d) (i) Sketch the graph, $y = 2\sin\frac{x}{2}$, from $0 \le x \le 2\pi$

(ii) Use Simpson's Rule and 5 function values to find the approximate area bounded by the curve and the x-axis. (7)

END OF EXAMINATION

Question 1

C

Question 2

B.

Question 3

A

Question 4

D

Question 5

D

Question 6

D

Question 7

C

Question 8

A

Question 9

A

Question 10

B

Question 11 Start this question in a new booklet (15 Marks) Marks

(a) (i) $\alpha+\beta=-\frac{-5}{1}=5$ [1]

(ii) $\alpha\beta=\frac{6}{1}=6$ [1]

(iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=5^2-2\times 6$

$=13$

[2]

(iv) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$ [2]

$=\frac{13}{36}$

(b) $y=-\left(x^2-3x+6\right)$ is **negative definite** if for the general equation:

$a<0$ and $\Delta<0$

In this equation $a=-1<0$ so true, and, $\Delta=3^2-4(-1)(-6)=9-24=-13<0$ which is also true so this equation is negative definite. [2]

(c)

$y=x^2-4x$

$y+4=x^2-4x+4$ [3]

$y+4=(x-2)^2$

$(x-2)^2=4\left(\frac{1}{4}\right)(y-(-4))$

(i) the vertex at $(2,-4)$ [1]

(ii) the focal length = $\frac{1}{4}$ [1]

(iii) the focus at $\left(2,-3\frac{3}{4}\right)$ [1]

(iv) the equation of the directrix is given by $y=-4\frac{1}{4}$ [1]

Question 12 Start this question in a new booklet (15 marks)

(a) (i)

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$$
$$=\lim_{x\to 3} x+3$$
$$=6$$

[1]

(ii)

$$\lim_{x\to\infty}\frac{1-x}{x}=\lim_{x\to\infty}\frac{1}{x}-1$$
$$=0-1$$
$$=-1$$

[2]

(b) (i) [2]

$$\frac{d\left(x^3-5\right)^7}{dx}=3x^2(7)\left(x^3-5\right)^6$$
$$=21x^2\left(x^3-5\right)^7$$

(ii) [3]

$$\frac{d\left[x^3(2-x)^4\right]}{dx}=3x^2(2-x)^4+x^3(-4)(2-x)^3$$
$$=x^2(2-x)^3\left[3(2-x)-4x\right]$$
$$=x^2(2-x)^3(6-7x)$$

(iii)

$$\frac{d\left(\frac{x}{x^2-4}\right)}{dx}=\frac{\left(x^2-4\right)-2x(x)}{\left(x^2-4\right)^2}=-\frac{x^2+4}{\left(x^2-4\right)^2}$$

[3]

(c) (i) . [2]

$$y=x^2-3x+2$$
$$x^2-3x+2=0$$
$$(x-2)(x-1)=0$$
$$x=1 \text{ or } 2$$

(ii) If $x=2$, then [2]

$$\frac{dy}{dx}=2x-3$$

If $x=2$ then

$$\frac{dy}{dx}=2(2)-3=1$$

At $x=2$, $y=2^2-3(2)+2=0$

$$y - 0 = 1(x-2)$$
$$y = x - 2$$

Question 13 Start this question in a new booklet (15 marks)

(a) (i) $y = x^3 + x^2 - x + 5$ If $\frac{dy}{dx} = 0$, then $x = \frac{1}{3}$ or -1

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$
$$= (3x-1)(x+1)$$

Hence, stationary points at $\left(\frac{1}{3}, 4\frac{22}{27}\right)$ and $(-1, 6)$

Since, $y = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \frac{1}{3} + 5 = 4\frac{22}{27}$, and

$y = (-1)^3 + (=1)^2 - (-1) + 5 = 6$

x	-2	-1	0	1/3	1
$\frac{dy}{dx}$	+ve	0	-ve	0	+ve
slope	/	_	\	_	/
		Max tp		Min tp	

[3]

(ii) $y = x^3 + x^2 - x + 5$ If $\frac{d^2y}{dx^2} = 0$, then $x = -\frac{1}{3}$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$
$$\frac{d^2y}{dx^2} = 6x + 2$$

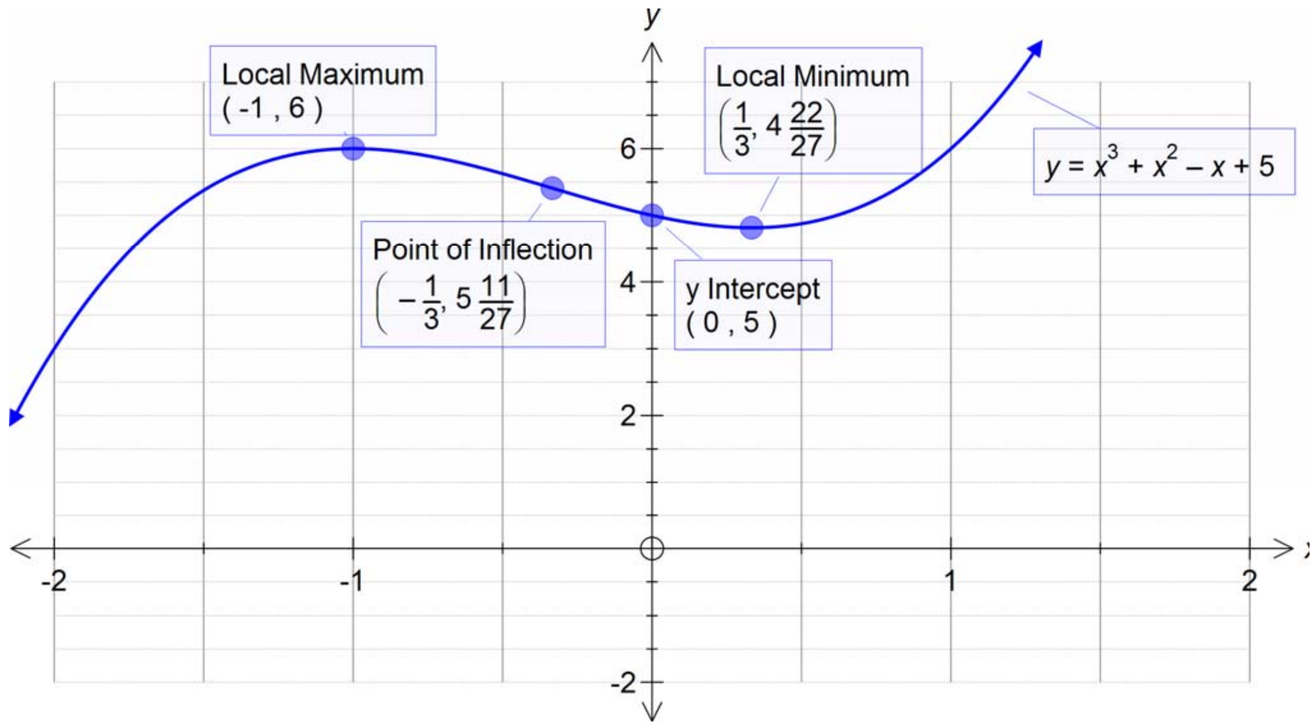
$$y = \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 5 = 5\frac{11}{27}$$

x	-1	$-\frac{1}{3}$	0
$\frac{dy}{dx}$	-ve	0	+ve
cconcavity	down		up
		Pt of I	

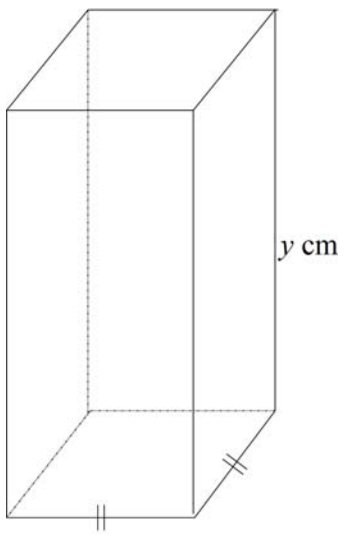
Hence, point of inflexion at $\left(-\frac{1}{3}, 5\frac{11}{27}\right)$

[2]

(iii)



(b) (i)



[1]

(ii) Surface Area = 96 cm^2. Hence,

$$2x^2 + 4xy = 96$$
$$4xy = 96 - 2x^2$$
$$y = \frac{2\left(48 - x^2\right)}{4x}$$
$$y = \frac{48 - x^2}{2x}$$

[2]

(iii)

$$\text{Volume} = x^2 y$$

$$V = \frac{x^2\left(48 - x^2\right)}{2x} = \frac{1}{2}x\left(48 - x^2\right)$$ [1]

$$V = 24x - \frac{x^3}{2}$$

(iv) Now, maximum volume when $\frac{dV}{dx} = 0$,

[3]

$$V = 24x - \frac{x^3}{2}$$

$$\frac{dV}{dx} = 24 - \frac{3x^2}{2}$$

$$3x^2 = 48$$

$$x^2 = 16$$

$$x = 4\,\text{cm}, \quad x > 0$$

$$y = \frac{48 - 16}{8} = 4 \text{ cm}$$

$$\text{Max Volume} = 64\,\text{cm}^3$$

Question 14

(a) (i)

$$\int 3x^2 + \frac{2}{x^2}\ dx = \int 3x^2 + 2x^{-2}\ dx$$

$$= \frac{3x^3}{3} + \frac{2x^{-1}}{(-1)} + c$$ [1]

$$= x^3 - \frac{2}{x} + c$$

(ii)

$$\int_0^1 (4 - x)^5\ dx = \left[\frac{(4 - x)^6}{(-1)(6)}\right]_0^1$$ [2]

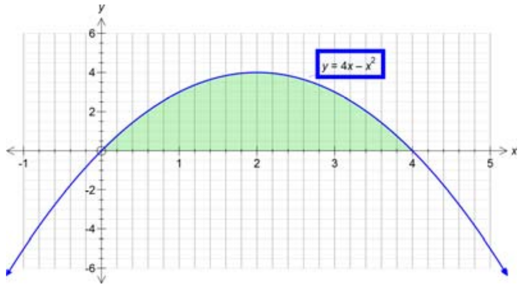
$$= -\frac{1}{6}\left[(4 - x)^6\right]_0^1$$

$$= -\frac{1}{6}\left(3^6 - 4^6\right)$$

$$= \frac{3367}{6}$$

(b) Find the area bounded by the curve $y = 4x - x^2$ and the x axis.

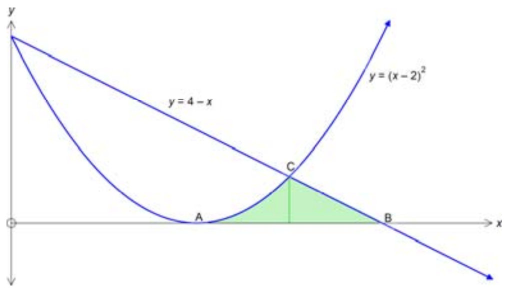
Area wholly above the x-axis, hence



$$\text{Area} = \int_0^4 4x - x^2 \ dx$$
$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$
$$= \left(32 - \tfrac{64}{3}\right)$$
$$= \frac{32}{3} \text{ units}^2$$

[3]

(c) (i) At A, for the curve, $y = (x-2)^2$, $y = 0$.



So, $x = 2$, i.e. A (2, 0)

At B, for the line, $y = 4 - x$, $y = 0$

So, $x = 4$, i.e. B (4, 0)

At C, the curves $y = (x-2)^2$ and $y = 4 - x$ intersect, so,

$$(x-2)^2 = 4 - x$$
$$x^2 - 4x + 4 = 4 - x$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0 \text{ or } 3, \ x > 0$$
$$x = 3$$

If $x = 3$ then $y = 1$, so C (3, 1) [3]

(ii)

$$\text{Area} = \int_2^3 (x-2)^2 \ dx + \int_3^4 4 - x \ dx$$
$$= \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[4x - \frac{x^2}{2}\right]_3^4$$
$$= \left(\tfrac{1}{3} - 0\right) + \left[(16-8) - \left(12 - \tfrac{9}{2}\right)\right]$$
$$= \tfrac{1}{3} + 8 - \tfrac{15}{2}$$
$$= \frac{5}{6} \text{ square unitss}$$

[3}

(d) $\int_{-a}^{a} x^5 - x^3 \; dx = 0$. The graph of $y = x^5 - x^3$ is an odd function, i.e.

$f(a) = -f(-a)$, and as such has point symmetry about the origin.

Thus the areas above and below the *x*-axis on either sides of the *y*-axis must be equal, hence the integral:

$$\int_{-a}^{a} x^5 - x^3 \; dx = 0$$ [2]

Question 15 **(15 marks)** **Marks**

(a)

$$\begin{aligned}\log_3 27 - \log_9\left(\frac{1}{3}\right) + 7 &= \log_3 (3)^3 - \log_9\left(9^{-\frac{1}{2}}\right) + 7 \\ &= 3\log_3 3 + \tfrac{1}{2}\log_9 9 + 7 \\ &= 10\tfrac{1}{2}\end{aligned}$$ [2]

(b) (i)

$$\begin{aligned}f(x) &= \frac{x}{e^x} \\ f'(x) &= \frac{e^x(1) - x(e^x)}{e^{2x}} \\ &= \frac{e^x(1-x)}{e^{2x}} \\ &= \frac{1-x}{e^x} \\ f''(x) &= \frac{e^x(-1) - (1-x)(e^x)}{e^{2x}} \\ &= \frac{e^x(x-2)}{e^{2x}} \\ &= \frac{x-2}{e^x}\end{aligned}$$ [2]

(ii) If $f'(x)=0$

$1-x=0$

$x=1$

If $x=1$ then $y=e^{-x}$ and $f''(x)=\dfrac{-1}{e^x}<0$. [2]

Hence, $(1,e^{-x})$ is a max turning point.

(iii) If $f''(x)=0$

$x-2=0$

$x=2$

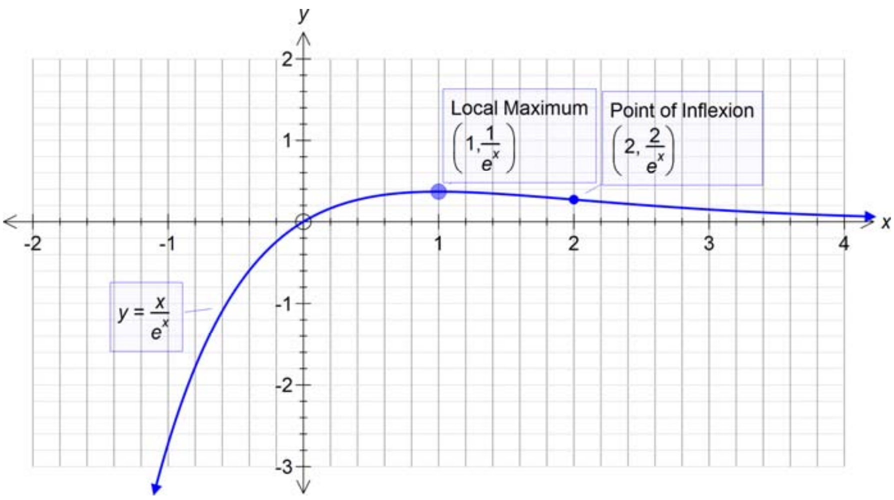
If $x=2$ and $y=2e^{-x}$

x	1	2	3
$f''(x)$	-ve	0	+ve
	Concave down		Concave up

Hence point of inflexion at $(2,2e^{-x})$

(iv) Let $y=xe^{-x}$, so $e^{-x}\to 0$, as $x\to\infty$ [1]

(v)



[3]

(c)

$$y = \frac{4}{\sqrt{x}}$$

$$y^2 = \frac{16}{x}$$

[5]

$$\text{Volume} = \pi \int_1^5 \frac{16}{x}\, dx$$

$$= \pi \left[16 \log_e x\right]_1^5$$

$$= 16\pi \left(\log_e 5 - \log_e 1\right)$$

$$= 16\pi \log_e 5$$

Question 16

(a)

$$\text{Area} = \frac{r^2\theta}{2}$$

[2]

$$\frac{3\pi}{2} = \frac{r^2 \frac{\pi}{3}}{2}$$

$$r^2 = 9$$

$$r = 3$$

(b) At P,

$$\left(\cos\left(-\frac{\pi}{3}\right), \sin\left(-\frac{\pi}{3}\right)\right) = \left(\cos\frac{\pi}{3}, -\sin\frac{\pi}{3}\right)$$

[3]

$$= \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

(c)

$$2\sin^2 x - 1 = 0$$

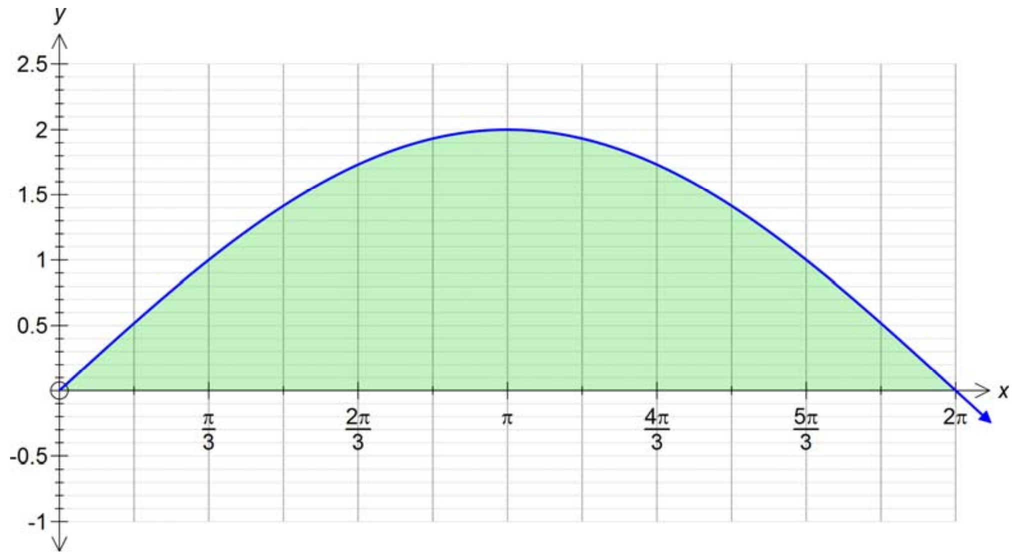
[3]

$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

(d) (i)



[3]

(ii) $h = \frac{2\pi - 0}{4} = \frac{\pi}{2}$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) = 2sin\frac{x}{2}$	0	$\sqrt{2}$	2	$\sqrt{2}$	0
Factor	x1	x4	x2	x4	x1
Value	0	$4\sqrt{2}$	4	$4\sqrt{2}$	0

Area $= \frac{\pi}{6}\left(4 + 8\sqrt{2}\right) \approx 8$ [5]