Section I
10 marks
Use the multiple choice answer sheet for questions 1-10. **Marks**

1 What is the derivative of $f(x)=\sqrt{3-x}$? **1**

(A) $f'(x)=\dfrac{-1}{2\sqrt{3-x}}$

(B) $f'(x)=\dfrac{-2}{\sqrt{3-x}}$

(C) $f'(x)=\dfrac{2}{\sqrt{3-x}}$

(D) $f'(x)=\dfrac{1}{2\sqrt{3-x}}$

2 The gradient of the normal to the curve $f(x)=x^2-4x$ at $(1,-3)$ is **1**

(A) -10

(B) -2

(C) $-\dfrac{1}{2}$

(D) $\dfrac{1}{2}$

3 Which of the following is true for the equation $3x^2-x-2=0$? **1**

(A) No real roots

(B) Equal roots

(C) Two real distinct roots

(D) Irrational roots

4 The exact value of $\cos\dfrac{2\pi}{3}$ is ? **1**

(A) $-\dfrac{\sqrt{3}}{2}$

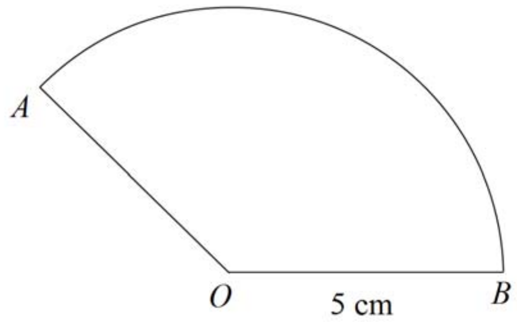
(B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{2}$

(D) $\dfrac{\sqrt{3}}{2}$

Marks

5 AOB is a sector of a circle, centre O and radius 5 cm. **1**
The sector has an area of 10π.



Not to scale

What is the arc length of the sector?

(A) 2π
(B) 4π
(C) 6π
(D) 10π

6 What is the derivative of $\log_2 x$? **1**

(A) $\dfrac{1}{x}$

(B) $\dfrac{1}{2x}$

(C) $\ln 2x$

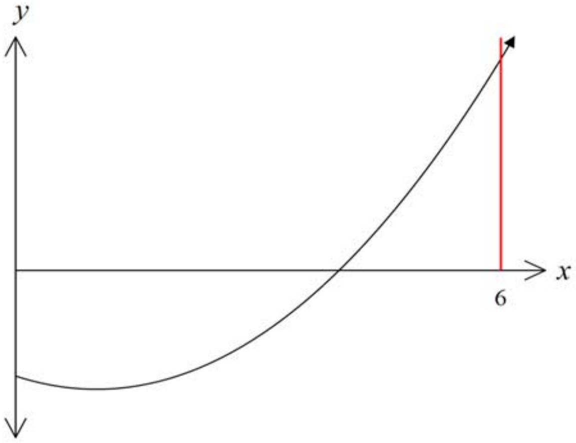
(D) $\dfrac{1}{x \ln 2}$

7 What is the value of $\int_{-1}^{2} x^2 + 1 dx$? **1**

(A) 4
(B) 5
(C) 6
(D) 7

Marks

8 The diagram below shows the graph of $y = x^2 - 2x - 8$. **1**



What is the correct expression for the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$?

(A) $A = \int_0^5 x^2 - 2x - 8dx + \left|\int_5^6 x^2 - 2x - 8dx\right|$

(B) $A = \int_0^4 x^2 - 2x - 8dx + \left|\int_4^6 x^2 - 2x - 8dx\right|$

(C) $A = \left|\int_0^5 x^2 - 2x - 8dx\right| + \int_5^6 x^2 - 2x - 8dx$

(D) $A = \left|\int_0^4 x^2 - 2x - 8dx\right| + \int_4^6 x^2 - 2x - 8dx$

9 What are the solutions to the equation $e^{6x} - 7e^{3x} + 6 = 0$? **1**

(A) $x = 1$ and $x = 6$

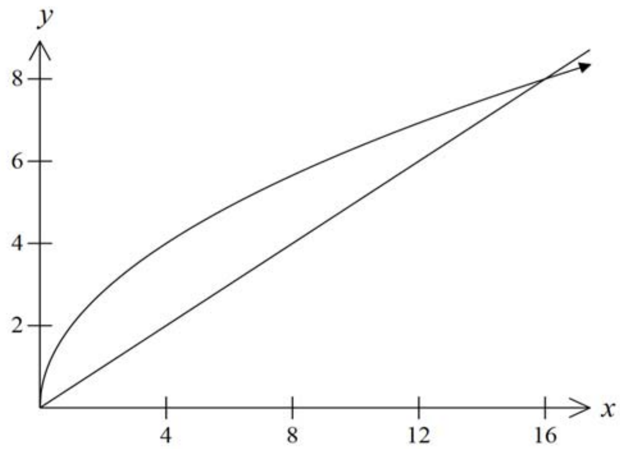
(B) $x = 0$ and $x = \dfrac{\ln 6}{2}$

(C) $x = 0$ and $x = \dfrac{\ln 6}{3}$

(D) $x = 1$ and $x = \dfrac{\ln 6}{2}$

Marks

10 The diagram below shows the graph of $y = 2\sqrt{x}$ and $y = \frac{x}{4}$. **1**



Which of the following is the correct expression for the volume of the solid of revolution when the area between the curve $y = 2\sqrt{x}$ and $y = \frac{x}{4}$ is rotated around the x-axis?

(A) $V = \int_0^8 (4y - \frac{y^2}{2})dy$

(B) $V = \int_0^{16} (2\sqrt{x} - \frac{x}{4})dx$

(C) $V = \pi\int_0^8 (16y^2 - \frac{y^4}{4})dy$

(D) $V = \pi\int_0^{16} (4x - \frac{x^2}{16})dx$

End of Section I

Section II

Answer questions in the writing booklet provided.
Start each question in a new booklet. Extra writing booklets are available.

Question 11 *(15 marks)* **Start a new booklet**. **Marks**

(a) Consider the parabola $x^2 - 6x + 4y + 17 = 0$

(i) Express the parabola in the form $(x-h)^2 = 4a(y-k)$ **1**

(ii) Write down the co-ordinates of the vertex. **1**

(iii) Write down the coordinates of the focus. **1**

(iv) Write down the equation of the directrix. **1**

(b) If α and β are the roots of the equation $3x^2 + 15x + 6 = 0$, find the value of

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ **2**

(c) If $A(x-1)^2 + B(x-1) + C = 3x^2 - 5x + 7$, find A, B and C. **3**

(d) Find the value of k for which the equation $x^2 + (k+1)x + (k+3) = 0$ has one root equal to 4. **1**

(e) Derive the equation of the locus of the point P (x,y) which moves so that AP:PB = 2:3, where A is (-2,1) and B is (3,–3). **3**

Question 12 *(16 marks)* **Start a new booklet.**

(a) The equation of a curve is given by $y = 6x^3 + 9x^2 - 3$

(i) Show that $6x^3 + 9x^2 - 3 = 3(x+1)^2(2x-1)$. **1**

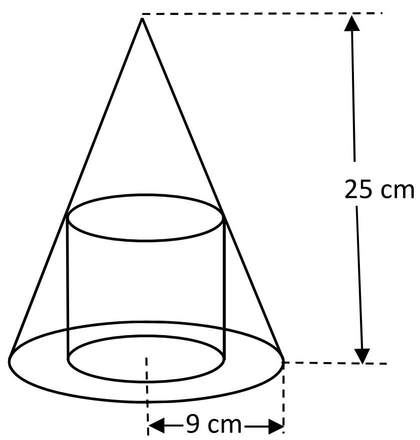
(ii) Hence write down the co-ordinates of the x intercepts. **1**

(iii) Find any stationary points and determine their nature. **4**

(iv) Find the co-ordinates of any points of inflexion. **2**

(v) Sketch the curve showing all essential details. **2**

(b) A cylinder is inscribed in a cone of radius 9 *cm* and height 25*cm*.



(i) Show that the height of the cylinder is $h = \dfrac{25(9-r)}{9}$, **2**

where r is the radius of the cylinder.

(ii) Show that the volume of the cylinder is $\dfrac{25\pi r^2(9-r)}{9}$. **1**

(iii) Hence find the maximum volume of the cylinder. **3**

Question 13 *(15 marks)* **Start a new booklet.**

(a) Find

(i) $\int (3x+4)^4 dx$ **1**

(ii) $\int \frac{4x^2+6}{x^2}\, dx$ **2**

(b) Evaluate $\int_0^5 \sqrt{25-x^2}\, dx$ **2**

(c) The derivative of the curve $y = f(x)$ is given by $f'(x) = x^2 - 3x - 4$. **2**

Find the equation of the curve, given that the curve passes through the point $(-2,1)$.

(d) (i) Sketch on the same axes the curves $y = 4 - x^2$ and $y = 2x^2 - 6x + 4$. **2**

(ii) Hence find the area bounded by the parabolas $y = 4 - x^2$ and $y = 2x^2 - 6x + 4$. **3**

(e) (i) Use the Trapezoidal Rule with 3 function values to find an approximation to $\int_1^3 \log_e x\, dx$. Answer correct to 2 decimal places. **2**

(ii) State whether the approximation found in part (i) is greater than or less the exact value of $\int_1^3 \log_e x\, dx$. Justify your answer. **1**

Question 14 *(15 marks)* **Start a new booklet.** **Marks**

(a) Differentiate

(i) $y = e^{4x+3}$ **1**

(ii) $y = x\ln x$ **1**

(iii) $y = \log_e(2x+7)$ **1**

(iv) $y = \ln\left(\dfrac{2x+3}{2x-3}\right)$ **2**

(b) Find the integral of

(i) $\displaystyle\int \frac{4x}{4x^2+7}\,dx$ **1**

(ii) $\displaystyle\int x^3 e^{x^4+2}\,dx$ **1**

(c) If $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$ find

(i) $\log_{10} 6$ **1**

(ii) $\log_{10} 5$ **1**

(d) (i) Sketch the graph of $y = \log_e(x-3)$. **1**

(ii) State its domain and range. **2**

(iii) The area between the curve $y = \log_e(x-3)$, the y axis and the lines $y = 1$ and $y = 4$ is rotated about the y axis. Find the exact volume of the solid formed. **3**

End of paper

Year 12 Mathematics H.S.C Mini Examination 2013

Section 1

1. $f(x) = \sqrt{3-x}$

$= (3-x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(3-x)^{-\frac{1}{2}} \times -1$

$= \frac{-1}{2\sqrt{3-x}}$ (A)

2. $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$ at $x = 1$

$= 2(1) - 4$

$= -2$

gradient of tangent $= -2$

gradient of normal $= \frac{1}{2}$

as $m_1 m_2 = -1$ for perpendicular lines (D)

3. $3x^2 - x - 2 = 0$

$\Delta = b^2 - 4ac$

$= (-1)^2 - 4(3)(2)$

$= 1 + 24$

$= 25$

As the discriminant is a perfect square, there are two distinct rational roots (C)

4. $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\times 180}{3}\right)$

$= \cos 120$

$= -\frac{1}{2}$ (B)

5. $A = \frac{1}{2}r^2\theta$

$10\pi = \frac{1}{2} \times 5^2 \theta$

$\theta = \frac{20\pi}{25}$

$\therefore \theta = \frac{4\pi}{5}$

$L = r\theta$

$= 5 \times \frac{4\pi}{5}$

$\therefore L = 4\pi$ (B)

$\therefore$ Length of the arc is 4π cm

6 $y = \log_2 x$

$= \frac{\log_e x}{\log_e 2}$

$\frac{dy}{dx} = \frac{\frac{1}{x}}{\log_e 2}$

$= \frac{1}{x \ln 2}$ (D)

7. $\int_{-1}^{2} x^2 + 1\, dx = \left[\frac{x^3}{3} + x\right]_{-1}^{2}$

$= \left[\frac{2^3}{3} + 2\right] - \left[\frac{(-1)^3}{3} - 1\right]$

$= 4\frac{2}{3} + 1\frac{1}{3}$

$= 6$

8. $y = x^2 - 2x - 8$

$= (x-4)(x+2)$

$\therefore A = \left| \int_0^4 x^2 - 2x - 8\, dx \right| + \int_4^6 x^2 - 2x - 8\, dx$ (D)

9. $e^{6x} - 7e^{3x} + 6 = 0$

Let $y = e^{3x}$

$y^2 = (e^{3x})^2$

$= e^{6x}$

$\therefore y^2 - 7y + 6 = 0$

$(y-6)(y-1) = 0$

$y = 6, 1$

But $y = e^{3x}$

$e^{3x} = 1$ $\quad$ $e^{3x} = 6$

$\therefore x = 0$ $\quad$ $\ln e^{3x} = \ln 6$

$3x \ln e = \ln 6$

$x = \frac{\ln 6}{3}$

$\therefore x = 0, x = \frac{\ln 6}{3}$ $\quad$ C

10 $\quad y = 2\sqrt{x}$ $\quad\quad y = \frac{x}{4}$

$y^2 = 4x$ $\quad\quad y^2 = \frac{x^2}{16}$

$$V = \pi \int_a^b y^2 \, dx$$

$$= \pi \int_0^{16} \left(4x - \frac{x^2}{16}\right) dx$$

D

Section II

a) $x^2 - 6x + 4y + 17 = 0$

$x^2 - 6x = -4y - 17$

$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = -4y - 17 + \left(\frac{-6}{2}\right)^2$

$(x-3)^2 = -4y - 8$

$(x-3)^2 = -4(y+2)$

$(x-h)^2 = -4a(y-k)$

$a = 1$

ii) $\therefore$ vertex is $(3, -2)$

iii) $\therefore$ focus is $(3, -3)$

iv) directrix is $y = -1$

b) $3x^2 + 15x + 6 = 0$

$a = 3 \quad b = 15 \quad c = 6$

i) $\alpha + \beta = \frac{-b}{a}$

$= \frac{-15}{3}$

$= -5$

ii) $\alpha\beta = \frac{c}{a}$

$= \frac{6}{2}$

$= 3$

iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(-5)^2 - 2 \times 3}{3}$

$= \frac{25 - 6}{3}$

$= \frac{19}{3}$

$= 6\frac{1}{3}$

c) $A(x-1)^2 + B(x-1) + C = 3x^2 - 5x + 7$

$A(x^2 - 2x + 1) + Bx - B + C = 3x^2 - 5x + 7$

$Ax^2 - 2Ax + A + Bx - B + C = 3x^2 - 5x + 7$

$Ax^2 + (B - 2Ax) + (A - B + C) = 3x^2 - 5x + 7$

$\therefore A = 3$ $\quad$ $B - 2A = -5$ $\quad$ $A - B + C = 7$

$B - 2(3) = -5$ $\quad$ $3 - 1 + C = 7$

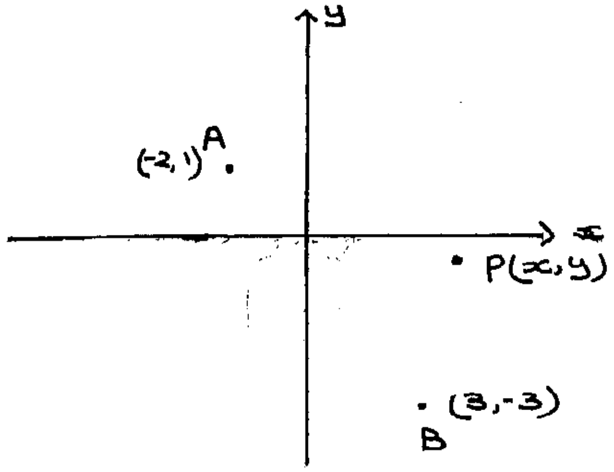
$B = 1$ $\quad$ $C = 5$

$\therefore A = 3 \quad B = 1 \quad C = 5$

d) $x^2+(k+1)x+(k+3)=0$

$$4^2+4(k+1)+(k+3)=0$$
$$16+4k+4+k+3=0$$
$$5k=-23$$
$$k=\frac{-23}{5}$$

c)



$$\frac{AP}{PB}=\frac{2}{3}$$
$$3AP=2PB$$
$$(3AP)^2=(2PB)^2$$
$$9AP^2=4PB^2$$

Let $P(x,y)$ be a point on the locus

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$AP=\sqrt{(x+2)^2+(y-1)^2}$$
$$AP^2=(x+2)^2+(y-1)^2$$

$$PB=\sqrt{(x-3)^2+(y+3)^2}$$
$$PB^2=(x-3)^2+(y+3)^2$$

But

$$9AP^2=4PB^2$$
$$9\left[(x+2)^2+(y-1)^2\right]=4\left[(x-3)^2+(y+3)^2\right]$$
$$9\left[x^2+4x+4+y^2-2y+1\right]=4\left[x^2-6x+9+y^2+6y+9\right]$$

$$9x^2+36x+9y^2-18y+45=4x^2-24x+4y^2+24y+72$$
$$5x^2+60x+5y^2-42y-27=0$$

Question 12

a) i) $y=6x^3+9x^2-3$

$$6x^3+9x^2-3=3(x+1)^2(2x-1)$$
$$\begin{aligned}R.HS&=3(x+1)^2(2x-1)\\&=3(x^2+2x+1)(2x-1)\\&=(3x^2+6x+3)(2x-1)\\&=6x^3+12x^2+6x-3x^2-6x-3\\&=6x^2+9x^2-3\\&=LHS\end{aligned}$$
$$\therefore 6x^3+9x^2-3=3(x+1)^2(2x-1)$$

(ii) x - intercepts $\Rightarrow y=0$

$$6x^3+9x^2-3=0$$
$$3(x+1)^2(2x-1)=0$$
$$x=-1,\tfrac{1}{2}$$

$\therefore$ x intercepts are $(-1,0)$ $(\tfrac{1}{2},0)$

iii) $y=6x^3+9x^2-3$

$$\frac{dy}{dx}=18x^2+18x$$
$$\frac{d^2y}{dx^2}=36x+18$$

For stationary points $\frac{dy}{dx}=0$

$$18x^2+18x=0$$
$$18x(x+1)=0$$
$$x=0,-1$$

when $x=0$

$$y=6x^3+9x^2-3$$
$$=-3$$

$\therefore$ $(0,-3)$ is a stationary point

Test the nature

$$\frac{d^2y}{dx^2} = 36x+18 \text{ at } x=0$$
$$= 18$$

As $\frac{d^2y}{dx^2} > 0$ the curve is concave up ↺

$\therefore$ $(0,-3)$ is a local minimum

when $x = -1$

$$y = 6x^3 + 9x^2 - 3$$
$$= 6(-1)^3 + 9(-1)^2 - 3$$
$$= -6 + 9 - 3$$
$$= 0$$

$\therefore$ $(-1,0)$ is a stationary point

Test the nature

$$\frac{d^2y}{dx^2} = 36x+18 \text{ at } x=-1$$
$$= 36(-1)+18$$
$$= -18$$

As $\frac{d^2y}{dx^2} < 0$ the curve is concave down ↷

$\therefore$ $(-1,0)$ is a local maximum.

iv) For points of inflexion $\frac{d^2y}{dx^2} = 0$ and there is a change in concavity

$$\frac{d^2y}{dx^2} = 36x+18$$

$$36x+18 = 0$$
$$36x = -18$$
$$x = -\tfrac{1}{2}$$

when $x = -\frac{1}{2}$

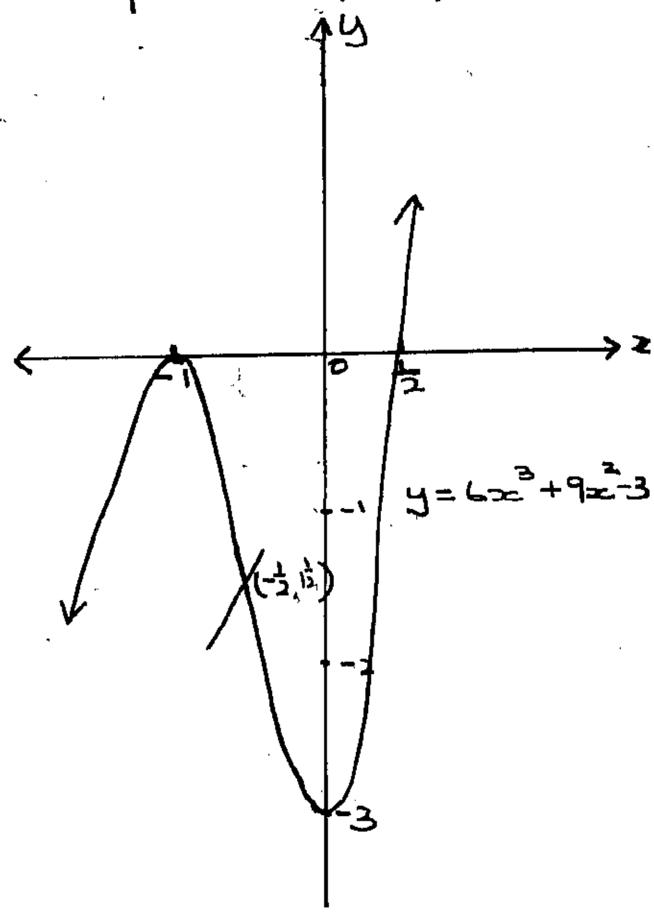
$$y = 6x^3 + 9x^2 - 3$$
$$= 6(-\tfrac{1}{2})^3 + 9(-\tfrac{1}{2})^2 - 3$$
$$= -1\tfrac{1}{2}$$

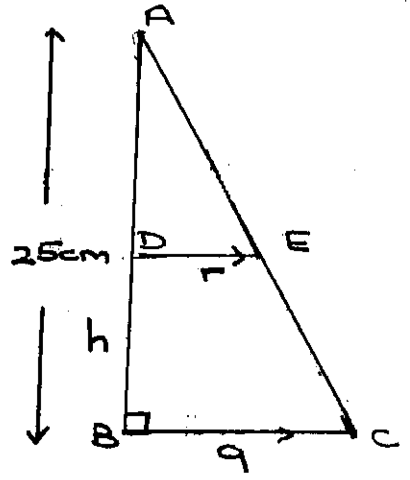
$\therefore$ $(-\frac{1}{2}, -1\frac{1}{2})$ may be a point of inflexion.

check concavity

x	$-\frac{1}{2}^-$	$-\frac{1}{2}$	$-\frac{1}{2}^+$
$\frac{d^2y}{dx^2} = 36x+18$	$-$	0	$+$

As there is a change in concavity $(-\frac{1}{2}, -1\frac{1}{2})$ is a point of inflexion





In $\triangle ADE$, $\triangle ABC$

$A\hat{D}E = A\hat{B}C$

$= 90^\circ$

$A\hat{E}D = A\hat{C}B$ corresponding angles are equal $DE \parallel BC$

$\therefore \triangle ADE \parallel\!\mid \triangle ABC$ equiangular

$\frac{BC}{DE} = \frac{AB}{AD}$ corresponding sides of similar triangles are in proportion

$\frac{9}{r} = \frac{25}{25-h}$

$9(25-h) = 25r$

$225 - 9h = 25r$

$9h = 225 - 25r$

$= 25(9-r)$

$h = \frac{25(9-r)}{9}$

$V = \pi r^2 h$

$= \pi r^2 \times \frac{25(9-r)}{9}$

$= \frac{25\pi r^2 (9-r)}{9}$

$V = \frac{225\pi r^2 - 25\pi r^3}{9}$

$\frac{dv}{dr} = \frac{450\pi r - 75\pi r^2}{9}$

$= \frac{\cancel{75}^{25}\pi r(6-r)}{\cancel{9}\,3}$

For stationary points $\frac{dv}{dr} = 0$

$\frac{75\pi r(6-r)}{9} = 0$

$75\pi r(6-r) = 0$

$r = 0, 6 \quad r > 0$

$\therefore r = 6$

when $r = 6$

$V = \frac{25\pi r^2 (9-r)}{9}$

$= \frac{25\pi (6)^2 (9-6)}{9}$

$= 300\pi$

Test the nature

$\frac{dv}{dr} = \frac{150\pi r - 25\pi r^2}{3}$

$\frac{d^2v}{dr^2} = 50\pi - \frac{50\pi r}{3}$

$= \frac{50\pi - 50\pi(6)}{3}$

$= 50\pi - 100\pi$

$= -50\pi$

As $\frac{d^2v}{dr^2} < 0$ a maximum value occurs

$\therefore$ The maximum volume is 300π

Question 13

a) (i) $\int (3x+4)^4 \, dx = \frac{1}{15}(3x+4)^5 + c$

ii) $\int \frac{4x^2+6}{x^2} \, dx = \int 4 + 6x^{-2} \, dx$

$= 4x - 6x^{-1} + c$

$= 4x - \frac{6}{x} + c$

b) $\int_0^5 \sqrt{25-x^2} \, dx = \frac{1}{4}\pi r^2$

$= \frac{1}{4}\pi (5)^2$

$= \frac{25\pi}{4}$

c) $f'(x) = x^2 - 3x - 4$

$f(x) = \int x^2 - 3x - 4 \, dx$

$= \frac{x^3}{3} - \frac{3x^2}{2} - 4x + c$

when $x = -2$, $f(x) = 1$

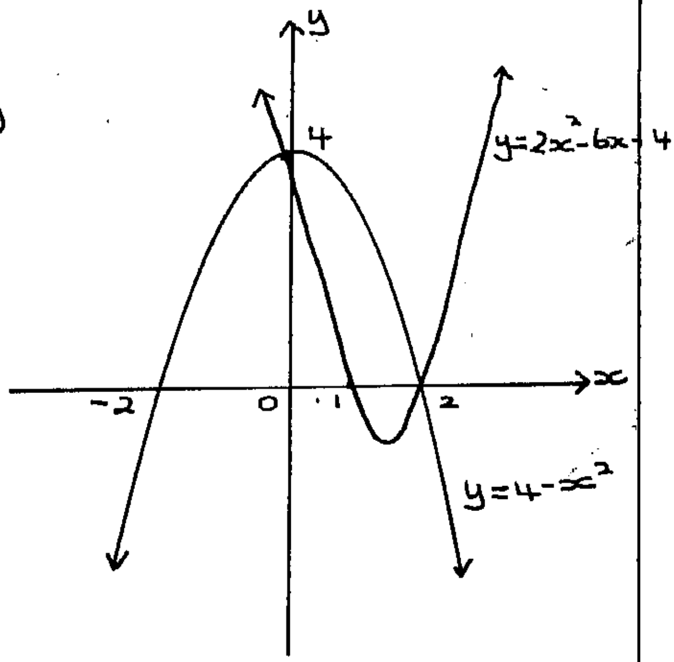
$1 = \frac{(-2)^3}{3} - \frac{3(-2)^2}{2} - 4(-2) + c$

$1 = -\frac{8}{3} - 6 + 8 + c$

$c = \frac{5}{3}$

$\therefore f(x) = \frac{x^3}{3} - \frac{3x^2}{2} - 4x + \frac{5}{3}$

d)



$A = \int_0^2 4 - x^2 - (2x^2 - 6x + 4) \, dx$

$= \int_0^2 4 - x^2 - 2x^2 + 6x - 4 \, dx$

$= \int_0^2 -3x^2 + 6x \, dx$

$= \left[-x^3 + 3x^2 \right]_0^2$

$= \left[-2^3 + 3(2)^2\right] - [0]$

$= 4$

$\therefore$ Area is 4 units^2

e)

x	1	2	3
$f(x)=\ln x$	0	$\ln 2$	$\ln 3$

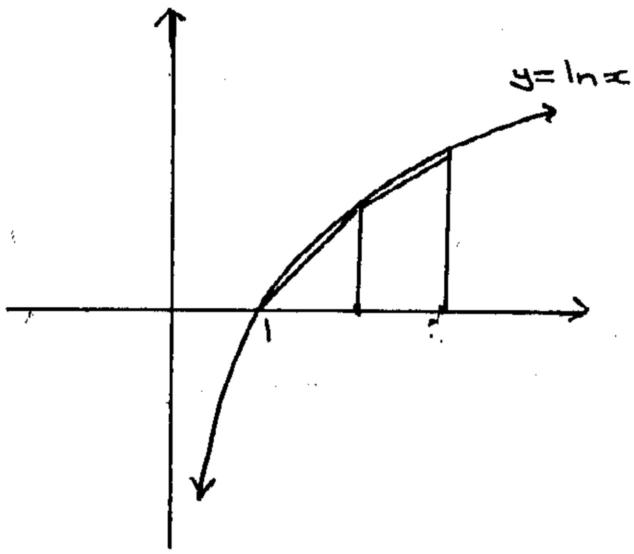
$\int_a^b f(x) \, dx \doteq \frac{b-a}{2}\left[f(a)+f(b)\right]$

$\int_1^3 \ln x \, dx \doteq \frac{2-1}{2}\left[0 + 2\ln 2 + \ln 3\right]$

$\doteq 1.242453325$

$= 1.24$ to 2dp

b)



The area using the trapezoidal rule is less than the area under the curve. The curve is concave down

Question 14

a) (i) $y = e^{4x+3}$

$$\frac{dy}{dx} = 4e^{4x+3}$$

(ii) $y = x \ln x \qquad u = x \quad v = \ln x$

$= u \quad v \qquad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= \ln x + x \times \frac{1}{x}$$

$$= \ln x + 1$$

(iii) $y = \log_e(2x+7)$

$$\frac{dy}{dx} = \frac{2}{2x+7}$$

iv) $y = \ln\left(\frac{2x+3}{2x-3}\right)$

$= \ln(2x+3) - \ln(2x-3)$

$$\frac{dy}{dx} = \frac{2}{2x+3} - \frac{2}{2x-3}$$

$$= \frac{2(2x-3) - 2(2x+3)}{(2x+3)(2x-3)}$$

$$= \frac{-12}{4x^2-9}$$

b) i) $\int \frac{4x}{4x^2+7}\,dx = \frac{1}{2}\int \frac{8x}{4x^2+7}\,dx$

$= \frac{1}{2}\ln(4x^2+7) + c$

ii) $\int x^3 e^{x^4+2}\,dx = \frac{1}{4}e^{x^4+2} + C$

c) $\log_{10} 6 = \log_{10}(2 \times 3)$

$= \log_{10} 2 + \log_{10} 3$

$= 0.301 + 0.477$

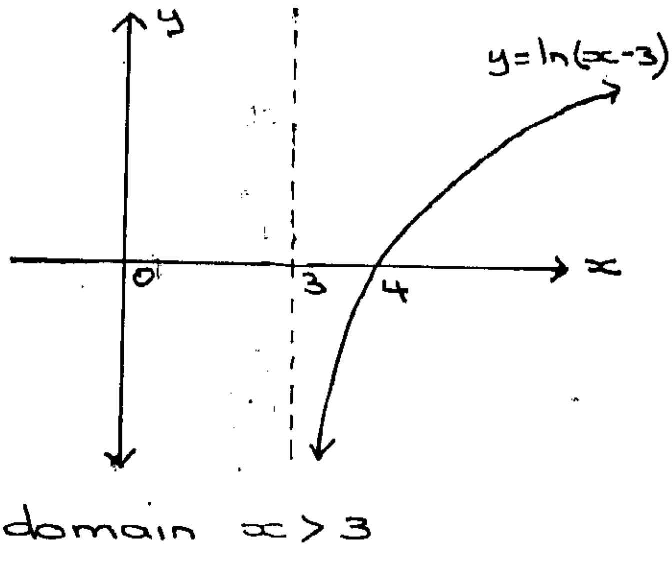
$= 0.778$

ii) $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right)$

$= \log_{10} 10 - \log_{10} 2$

$= 1 - 0.301$

$= 0.699$



domain $x > 3$

range $\mathbb{R}$

$y = \log_e(x-3)$

$e^y = x-3$

$x = e^y + 3$

$x^2 = (e^y+3)^2$

$= e^{2y} + 6e^y + 9$

$V = \pi\int_a^b x^2\,dy$

$= \pi\int_1^4 e^{2y} + 6e^y + 9\,dy$

$= \pi\left[\frac{1}{2}e^{2y} + 6e^y + 9y\right]_1^4$

$= \pi\left[\frac{1}{2}e^8 + 6e^4 + 36\right] - \pi\left[\frac{1}{2}e^2 + 6e + 9\right]$

$= \pi\left[\frac{1}{2}e^8 + 6e^4 - \frac{1}{2}e^2 - 6e + 27\right]$

$\therefore$ Volume is $\pi\left[\frac{1}{2}e^8 + 6e^4 - \frac{1}{2}e^2 - 6e + 27\right]$ units^3