Shore

Examination Number:
Set:

Year 12
Term II Examination
6 May 2013

Mathematics

General Instructions

- Reading time − 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Start each of Questions 11−16 in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Total marks − 100

Section I Pages 2−6

10 marks

- Attempt Questions 1−10
- Allow about 15 minutes for this section

Section II Pages 7−17

90 marks

- Attempt Questions 11−16
- Allow about 2 hours and 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 What is 5.098643 correct to three significant figures?

(A) 5.099

(B) 5.09

(C) 5.10

(D) 5.1

2 Which expression is a correct factorisation of $x^3 - 8$?

(A) $(x-2)(x^2+2x+4)$

(B) $(x+2)(x^2-2x+4)$

(C) $(x-2)(x^2+4x+4)$

(D) $(x+2)(x^2-4x+4)$

3 What is the solution of $x^2 > 4$?

(A) $x > 2$

(B) $-2 < x < 2$

(C) $x > -2$ or $x > 2$

(D) $x < -2$ or $x > 2$

4 The quadratic equation $x^2 + 6x - 1 = 0$ has roots α and β.

What is the value of $\alpha\beta - (\alpha + \beta)$?

(A) -7

(B) -5

(C) 5

(D) 7

5 What is the perpendicular distance of the point $(5, -1)$ from the line $4x - y - 3 = 0$?

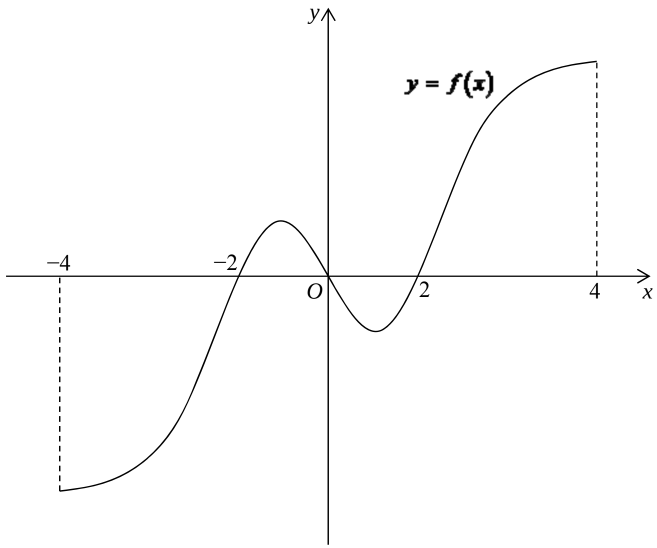
(A) $\frac{18}{\sqrt{15}}$ units

(B) $\frac{18}{\sqrt{17}}$ units

(C) $\frac{18}{\sqrt{24}}$ units

(D) $\frac{18}{\sqrt{26}}$ units

6 The graph of the odd function $y = f(x)$ has been drawn to scale for $-4 \le x \le 4$.



Which of the following integrals has the greatest value?

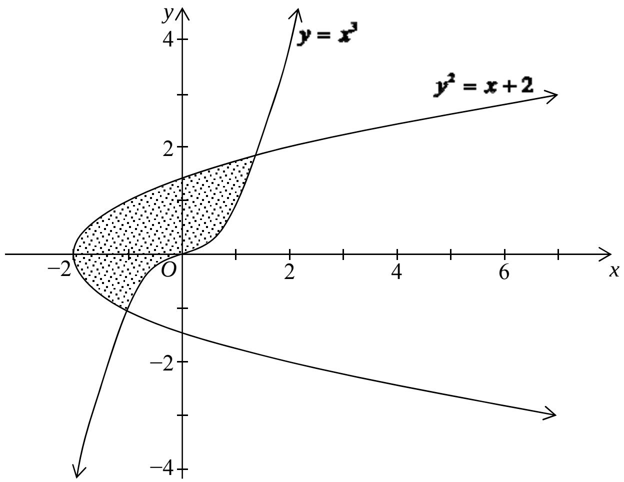
(A) $\int_{-4}^{4} f(x)\,dx$

(B) $\int_{-4}^{2} f(x)\,dx$

(C) $\int_{0}^{4} f(x)\,dx$

(D) $\int_{2}^{4} f(x)\,dx$

7 The diagram shows the region enclosed by $y^2 = x + 2$ and $y = x^3$.



Which of the following pairs of inequalities describes the shaded region in the diagram?

(A) $y^2 \le x+2$ and $y \ge x^3$

(B) $y^2 \ge x+2$ and $y \ge x^3$

(C) $y^2 \ge x+2$ and $y \le x^3$

(D) $y^2 \le x+2$ and $y \le x^3$

8 How many solutions does the equation $2\sin^2\theta = 1$ have in the domain $0° \le \theta \le 360°$?

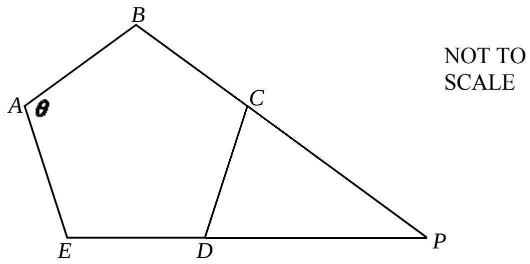
(A) 1

(B) 2

(C) 3

(D) 4

9 The diagram shows a regular pentagon *ABCDE*. Sides *ED* and *BC* are produced to meet at *P* and $\angle BAE = \theta$.



What is $\angle CPD$ in terms of θ?

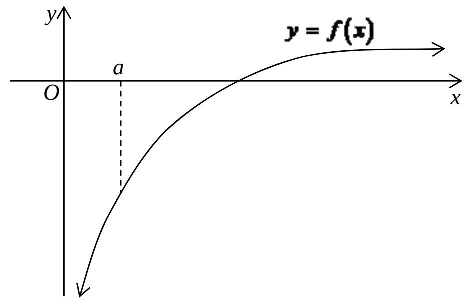
(A) θ

(B) $180° - 2\theta$

(C) $2\theta + 180°$

(D) $2\theta - 180°$

10 The diagram shows the graph of $y = f(x)$.



Which of the following statements is true?

(A) $f'(a) > 0$ and $f''(x) > 0$

(B) $f'(a) > 0$ and $f''(x) < 0$

(C) $f'(a) < 0$ and $f''(x) > 0$

(D) $f'(a) < 0$ and $f''(x) < 0$

End of Section I

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Differentiate $y=\left(x^3-1\right)^5$ with respect to x. **2**

(b) Evaluate $\log_4 6$, correct to 2 decimal places. **2**

(c) (i) Find $\int\left(x^2-\frac{x}{2}\right)dx$. **2**

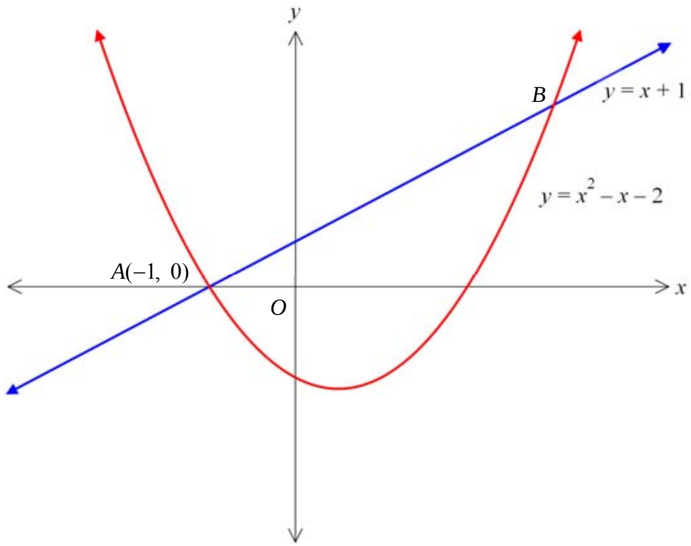
(ii) Find $\int\frac{1}{4x^2}\,dx$. **2**

(d) Evaluate $\int_1^4\sqrt{x}\,dx$. **2**

(e) Evaluate $\lim_{x\to(-2)}\frac{x+2}{x^2-4}$. **2**

Question 11 continues on page 9

(f) The graphs of $y=x+1$ and $y=x^2-x-2$ intersect at the points $A(-1,0)$ and B as shown in the diagram.



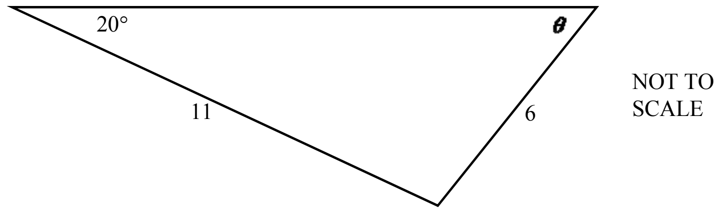
(i) Show that the coordinates of B are $(3,4)$. **1**

(ii) Hence, find the area of the region enclosed by $y=x+1$ and $y=x^2-x-2$. **2**

End of Question 11

Question 12 (15 marks) **Marks**

(a) Find the value of θ in the diagram. Give your answer to the nearest degree. **2**



(b) Solve $|x-2|<3$. **2**

(c) Consider the geometric series

$$1+3x+9x^2+27x^3+\ldots \quad \text{where } x\neq 0.$$

(i) State the value of the common ratio. **1**

(ii) For what values of x does the limiting sum of the series exist? **1**

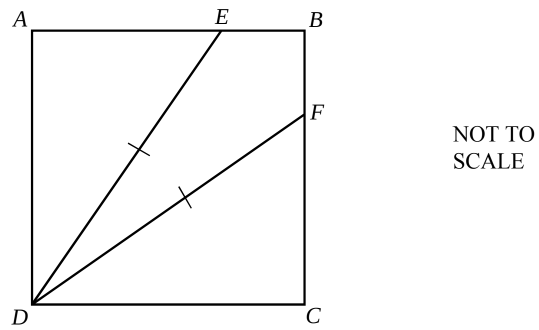
(iii) Calculate the value of the limiting sum for $x=0.25$. **1**

(d) Solve $2\ln x - \ln 6 = \ln 3$. **3**

Question 12 continues on page 11

(e) In the diagram, *ABCD* is a square.

E lies on *AB* and *F* lies on *BC* so that *DE* = *DF*



Copy or trace the diagram into your writing booklet.

(i) Prove that $\Delta AED \equiv \Delta CFD$. **3**

(ii) Hence, prove that *BE* = *BF*. **2**

End of Question 12

Question 13 (15 marks) **Marks**

(a) Find the value(s) of k for which the quadratic equation $x^2 - kx + 1 = 0$ has equal roots. **2**

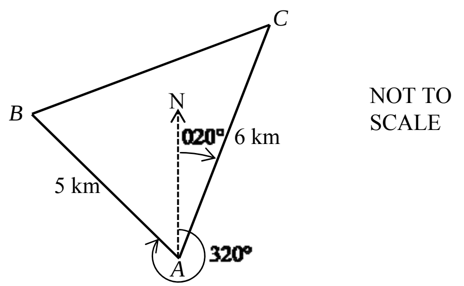
(b) Differentiate $y = x^2 \log_e x$ with respect to x. **2**

(c) Differentiate $y = \dfrac{e^{3x}}{x^3}$ with respect to x. Leave your answer in simplest form. **3**

(d) Find $\int e^{4x+2}\, dx$. **1**

(e) Find $\int \dfrac{6x}{9 - x^2}\, dx$. **2**

(f) The diagram shows a triangular paddock ABC. The bearing of B from A is $320°$ and the bearing of C from A is $020°$. The distance from A to B is 5 km and the distance from A to C is 6 km.



(i) Find the size of $\angle BAC$. **1**

(ii) Calculate the exact length of the fence BC. **2**

(iii) Calculate the exact area of the paddock ABC. **2**

Question 14 (15 marks) **Marks**

(a) Consider the parabola given by $4y = x^2 - 2x + 13$.

(i) Write the equation in the form $(x-h)^2 = 4a(y-k)$. **2**

(ii) Write down the coordinates of the vertex. **1**

(iii) Find the coordinates of the focus. **1**

(iv) Write down the equation of the directrix. **1**

(v) Draw a sketch of the parabola showing all important features. **1**

(b) Two points $A\left(\sqrt{2},\ 1\right)$ and $B\left(\sqrt{3},\ k\right)$ lie on a line that makes an angle of $60°$ with the positive direction of the x-axis.

(i) Show that the gradient of this line is $\sqrt{3}$. **1**

(ii) Calculate the exact value of k. **2**

(c) On the first day of the harvest, an apple orchard produces 780 kg of fruit. On the next day, the orchard produces 767 kg, and the amount produced continues to decrease by the same amount each day for a total of 60 days.

(i) How much fruit is produced on the 9th day of the harvest? **2**

(ii) What is the total amount of fruit that is produced in the first 9 days of the harvest? **1**

(iii) The owner of the apple orchard has an order for 18 512 kg of apples. After how many days of the harvest would the total mass of apples harvested be enough to fill the order? **3**

Question 15 (15 marks) **Marks**

(a) A function is defined by $f(x) = 4x^3 - 3x$.

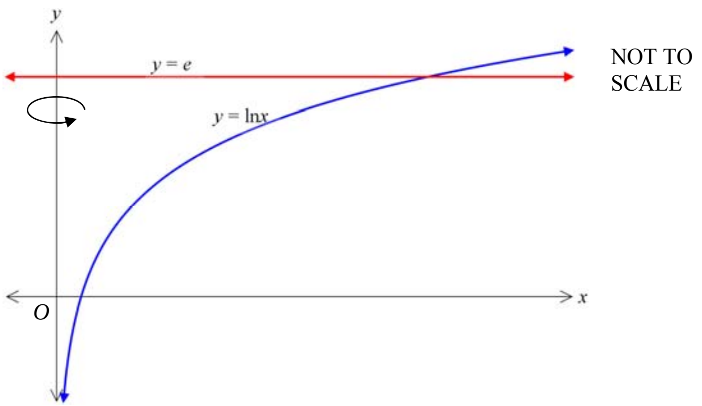
(i) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature. **3**

(ii) Hence, sketch the graph of $y = f(x)$, showing the stationary points and the points where the curve meets the x-axis. **2**

(iii) For what values of x is the curve concave down? **1**

(b) Find the equation of the tangent to the curve $y = \ln x^3$ at the point where $x = e$ on the curve. **3**

(c) The diagram below shows the region enclosed by the curve $y = \ln x$, the line $y = e$, the y-axis and the x-axis. **3**

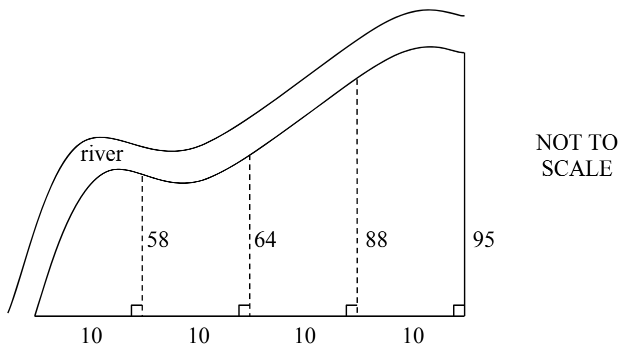


The region is rotated about the y-axis to form a solid.

Find the exact volume of the solid so formed.

Question 15 continues on page 15

(d) The diagram shows a field bordered by two straight fences and a river. The widths of the field are shown in metres, at 10 metre intervals. **3**



Use Simpson's rule with 5 function values to find an approximate value for the area of the field.

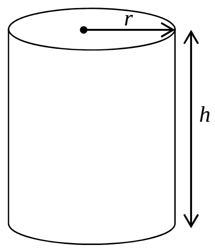
End of Question 15

Question 16 (15 marks) **Marks**

(a) (i) Show that $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=\dfrac{2}{\sin\theta}$. **3**

(ii) Hence, or otherwise, solve $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=4$, for $0°\le\theta\le 360°$. **2**

(b) A small business creates a new soft drink. To minimise production costs, the business sells the soft drink in a 375 mL cylindrical can with the smallest surface area. A sketch of the can with radius r and height h is shown below.



NOT TO SCALE

(i) Show that the surface area of the soft drink can is given by **2**

$$S=2\pi r^2+\frac{750}{r} \quad \text{where } r \text{ is the radius of the can.}$$

(ii) Calculate the radius of the can with the smallest surface area. Justify your answer. **3**

Question 16 continues on page 17

(c) A small business borrowed \$30 000 at the beginning of 2013. The annual interest rate is 6%, compounded monthly. Each month, interest is calculated on the balance at the beginning of the month and added to the balance owing.

The debt is to be repaid in equal monthly repayments of \$580, with the first repayment being made at the end of January 2013.

Let A_n be the balance owing after the nth repayment.

(i) Show that $A_2=30\,000(1.005)^2-580(1+1.005)$. **1**

(ii) Show that $A_n=116\,000-86\,000(1.005)^n$. **2**

(iii) In what month and year will the business make the final repayment? **2**

End of paper

Year 12 Mathematics Mid-year exam solutions

1. C
2. A
3. D
4. C
5. B
6. D
7. A
8. D
9. D
10. B

11.

a.
$$y=(x^3-1)^5$$
$$\frac{dy}{dx}=15x^2(x^3-1)^4$$

b.
$$\log_4 6=\frac{\log_{10}6}{\log_{10}4}$$
$$=1.29248...$$
$$\approx 1.29$$

c.

i. $$\int\left(x^2-\frac{x}{2}\right)dx=\frac{x^3}{3}-\frac{x^2}{4}+C$$

ii. $$\int\frac{1}{4x^2}\,dx=-\frac{x^{-1}}{4}+C$$
$$=\frac{-1}{4x}+C$$

d.
$$\int_1^4\sqrt{x}\,dx=\left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^4$$
$$=\left(\frac{2\times 4^{\frac{3}{2}}}{3}-\frac{2\times 1^{\frac{3}{2}}}{3}\right)$$
$$=\frac{14}{3}$$

e.
$$\lim_{x\to(-2)}\frac{x+2}{x^2-4}=\lim_{x\to(-2)}\frac{x+2}{(x+2)(x-2)}$$
$$=\lim_{x\to(-2)}\frac{1}{(x-2)}$$
$$=\frac{1}{(-2-2)}$$
$$=-\frac{1}{4}$$

f.

i.
$y=x^2-x-2$ equation 1

$y=x+1$ equation 2

For $B(3,4)$

Sub into equation 1

$LHS=4$

$RHS=3^2-3-2$

$=9-3-2$

$=4$

$=LHS$

Sub into equation 2

$LHS=4$

$RHS=3+1$

$=4$

$=LHS$

ii.
$$A=\int_{-1}^{3}(x+1)-(x^2-x-2)dx$$
$$=\int_{-1}^{3}-x^2+2x+3dx$$
$$=\left[\frac{-x^3}{3}+x^2+3x\right]_{-1}^{3}$$
$$=\left(\frac{-3^3}{3}+3^2+3(3)\right)-\left(\frac{-(-1)^3}{3}+(-1)^2+3(-1)\right)$$
$$=10\frac{2}{3}\text{ units}^2$$

12.

a. $\frac{Sin\theta}{11} = \frac{Sin20°}{6}$

$\theta = Sin^{-1}\left(\frac{11 \times Sin20°}{6}\right)$

$= 38°49'54.66''$

$\approx 39°$

b. $|x-2| < 3$

$x - 2 < 3$ or $-x + 2 < 3$

$x < 5$ $\quad$ $x > -1$

$\therefore -1 < x < 5$

c. $1 + 3x + 9x^2 + 27x^3 + ...$

i. $r = 3x$

ii. $-\frac{1}{3} < x < \frac{1}{3};\ x \neq 0$

iii. $S_\infty = \frac{1}{1 - \frac{3}{4}}$

$= 4$

d. $2\ln x - \ln 6 = \ln 3$

$2\ln x = \ln 3 + \ln 6$

$2\ln x = \ln 18$

$\therefore x^2 = 18$

$x = \pm\sqrt{18}$

$x = 3\sqrt{2}$ (only positive solution as domain of function is $x > 0$)

e.

i. In triangle AED and triangle CFD

$\angle EAD = \angle FCD$ (angles in a square)

$= 90°$

$DE = DF$ (given)

$AD = DC$ (adjacent sides of a square)

$\therefore \Delta AED \equiv \Delta CFD$ (RHS)

ii. In Quadrilateral ABCD

$AB = BC$ (adjacent sides of a square)

$AE + EB = BF + FC$

But $AE = FC$ (Corresponding sides in congruent triangles)

$\therefore BE = BF$

13.

a. $x^2 - kx + 1 = 0$

For equal roots

$\Delta = 0$

$0 = (-k)^2 - 4 \times 1 \times 1$

$0 = k^2 - 4$

$k = \pm 2$

b. $y = x^2 \log_e x$

$\frac{dy}{dx} = \left(x^2 \times \frac{1}{x}\right) + \left(\log_e x \times 2x\right)$

$= x + 2x\log_e x$

$= x(1 + 2\log_e x)$

c.

$y = \frac{e^{3x}}{x^3}$

$\frac{dy}{dx} = \frac{\left(x^3 \times 3e^{3x}\right) - \left(e^{3x} \times 3x^2\right)}{x^6}$

$= \frac{3x^2e^{3x}\left(x-1\right)}{x^6}$

$= \frac{3e^{3x}\left(x-1\right)}{x^4}$

d. $\int e^{4x+2}dx = \frac{1}{4}e^{4x+2} + C$

e. $\int \frac{6x}{9 - x^2}\, dx = -3\int \frac{-2x}{9 - x^2}\, dx$

$= -3\ln(9 - x^2) + C$

f.

i. $\angle BAC = (360° - 320°) + 20°$

$= 60°$

ii. $BC^2 = 5^2 + 6^2 - 2\times5\times6\times\cos 60°$
$= 31$
$BC = \sqrt{31}$

iii. $Area = \frac{1}{2}\times5\times6\times\sin 60°$
$= \frac{15\sqrt{3}}{2}$ units ^{2}

14.

a. $4y = x^2 - 2x + 13$

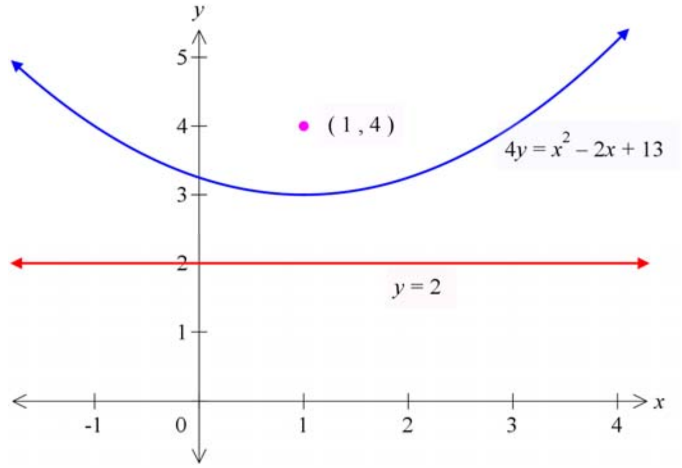
i. $4(y-3) = (x-1)^2$

ii. Vertex at $(1,3)$

iii. Focus at $(1,4)$

iv. Equation of directrix $y = 2$

v.



b.

i. $\tan 60° = \frac{\sqrt{3}}{1}$
$\therefore m = \sqrt{3}$

ii. $\frac{k-1}{\sqrt{3}-\sqrt{2}} = \sqrt{3}$
$k = 4 - \sqrt{6}$

c.

i. $T_9 = 780 + (9-1)\times -13$
$= 676\ kg$

ii. $S_9 = \frac{9}{2}(780 + 676)$
$= 6552\ kg$

iii. $18512 = \frac{n}{2}\left(2\times780 + (n-1)\times(-13)\right)$
$37024 = 1573n - 13n^2$
$n^2 - 121 + 2848 = 0$
$(n-32)(n-89) = 0$
$n = 32\ or\ n = 89$
But:The harvest lasts for 60 days only.
$\therefore$ The order can be filled on the 32nd day.

15.

a. $f(x) = 4x^3 - 3x$

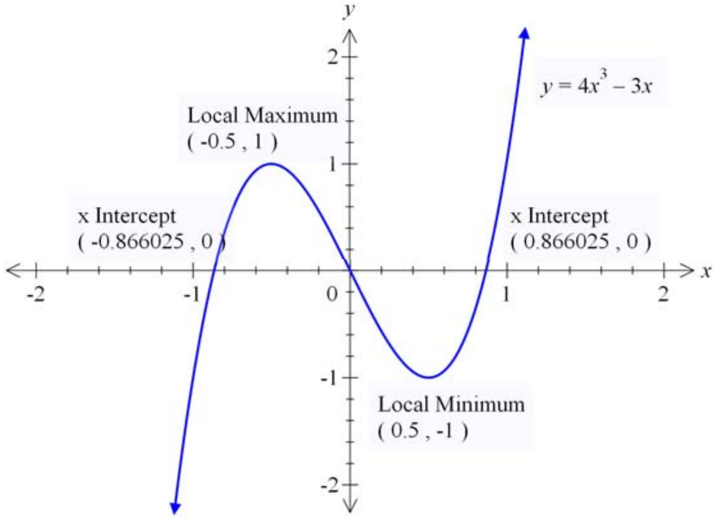
i.
$f(x) = 4x^3 - 3x$
$f'(x) = 12x^2 - 3$
Stationary points when $f'(x) = 0$
$0 = 3(4x^2 - 1)$
$0 = 3(2x+1)2x-1)$
$\therefore x = \frac{1}{2}\ or\ x = \frac{-1}{2}$

$f''(x) = 24x$	$f''(x) = 24x$
$f''\left(\frac{1}{2}\right) = 24\times\frac{1}{2}$	$f''\left(\frac{-1}{2}\right) = 24\times\frac{-1}{2}$
$= 12$	$= -12$
> 0	< 0
$\therefore$ *concave up*	$\therefore$ *concave down*
$\therefore$ *local* min	$\therefore$ *local* max

$$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3-3\left(\frac{1}{2}\right)$$
$$=\frac{1}{2}-\frac{3}{2}$$
$$=-1$$

$$f\left(\frac{-1}{2}\right)=4\left(\frac{-1}{2}\right)^3-3\left(\frac{-1}{2}\right)$$
$$=\frac{-1}{2}+\frac{3}{2}$$
$$=1$$

ii.



iii. $f(x)$ *is concave down*$: x<0$

b.

$$y=\ln x^3$$
$$y'=\frac{3x^2}{x^3}$$
$$=\frac{3}{x}$$
$$y'=\frac{3}{e} \text{ when } x=e.$$

the equation of the tangent is given by;

$$y-y_1=m(x-x_1)$$
$$y-3=\frac{3}{e}(x-e)$$
$$e(y-3)=3x-3e$$
$$0=3x-ey$$
or
$$y=\frac{3}{e}x$$

c.

$$y=\log_e x$$
$$x=e^y$$
$$x^2=e^{2y}$$
$$V=\pi\int_0^e e^{2y}dy$$
$$=\pi\left[\frac{e^{2y}}{2}\right]_0^e$$
$$=\pi\left[\frac{e^{2e}}{2}-\frac{e^0}{2}\right]$$
$$=\frac{\pi}{2}\left(e^{2e}-1\right)\ units^2$$

d.

$$A\approx\frac{10}{3}(0+95+4(58+88)+2(64))$$
$$=2690\ \text{m}^2$$

16.

a.

i. Prove $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=\frac{2}{\sin\theta}$

$$\begin{aligned} LHS &= \frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \\ &= \frac{2}{\sin\theta} \\ &= RHS \end{aligned}$$

ii.

$$\frac{2}{\sin\theta}=4$$

$$\sin\theta=\frac{1}{2}$$

$$\theta=30°,150°$$

b.

i.

$$S=2\pi r^2+2\pi rh$$

$$V=\pi r^2h$$

$$h=\frac{375}{\pi r^2}$$

$$S=2\pi r^2+2\pi r\times\frac{375}{\pi r^2}$$

$$=2\pi r^2+\frac{750}{r}$$

ii.

$$S=2\pi r^2+\frac{750}{r}$$

$$\frac{dS}{dr}=4\pi r-\frac{750}{r^2}$$

Stationary points when $\frac{dS}{dr}=0$

$$0=4\pi r-\frac{750}{r^2}$$

$$\begin{aligned} r &= \sqrt[3]{\frac{750}{4\pi}} \\ &= \sqrt[3]{\frac{375}{2\pi}} \\ &= 3.907... \\ &\approx 3.9 \text{ cm} \end{aligned}$$

$$\frac{d^2S}{dr^2}=4\pi+\frac{1500}{r^3}$$

when $r=\sqrt[3]{\frac{375}{2\pi}}$

$$\begin{aligned} \frac{d^2S}{dr^2} &= 4\pi+\frac{1500}{\left(\sqrt[3]{\frac{750}{4\pi}}\right)^3} \\ &= 4\pi+8\pi \\ &= 12\pi \\ &> 0 \end{aligned}$$

$\therefore$ concave up

$\therefore$ local min at $r=\sqrt[3]{\frac{375}{2\pi}}$

c.

i.

$$A_1=30000(1.005)-580$$

$$\begin{aligned} A_2 &= A_1(1.005)-580 \\ &= \big(30000(1.005)-580\big)(1.005)-580 \\ &= 30000(1.005)^2-580(1.005)-580 \\ &= 30000\left(1.005\right)^2-580\left(1+1.005\right) \end{aligned}$$

$$A_2 = 30000(1.005)^2 - 580(1+1.005)$$

$$A_3 = 30000(1.005)^3 - 580(1+1.005+1.005^2)$$

$$A_n = 30000(1.005)^n - 580(1+1.005+...+1.005^{n-1})$$

$$S_n = \frac{1(1.005^n - 1)}{0.005}$$

ii.

$$A_n = 30000(1.005)^n - 580\frac{1(1.005^n - 1)}{0.005}$$

$$A_n = 30000(1.005)^n - 580\frac{1(1.005^n - 1)}{0.005}$$

$$A_n = 30000(1.005)^n - 116000(1.005^n - 1)$$

$$A_n = 30000(1.005)^n - 116000(1.005)^n + 116000$$

$$A_n = 116000 - 86000(1.005)^n$$

$$A_n = 116000 - 86000(1.005)^n$$

For repaid loan $A_n = 0$

iii.

$$0 = 116000 - 86000(1.005)^n$$

$$1.005^n = \frac{116000}{86000}$$

$$= \frac{58}{43}$$

$$n = \frac{\log_e\left(\frac{58}{43}\right)}{\log_e 1.005}$$

$= 59.998...$ months

$\therefore$ last payment made in the 60th month.

ie: December 2017