SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

HSC Mathematics Task 2

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Start each **NEW SECTION** in a separate answer booklet.
- Hand in three separate sections labelled A, B and C.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.

Total Marks – 72

- Attempt all questions 1-6.
- All answers are to be given in simplest exact form unless specified.

Examiner: *P.Bigelow*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A

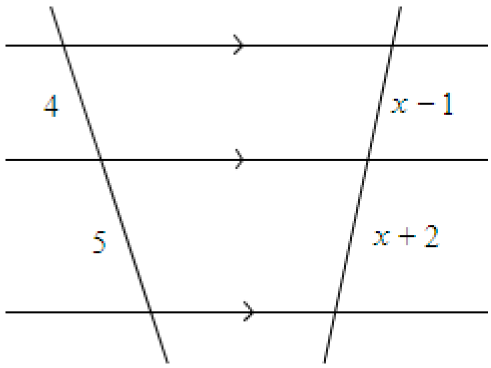
Question One [12 marks]

(a) Find a primitive of $3+x^2$. **1**

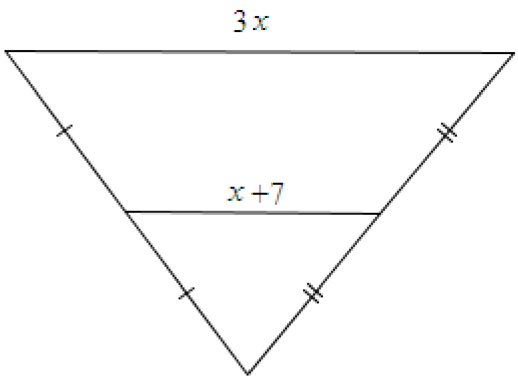
(b) Find the value of $\log_e 6$ correct to two decimal places. **1**

(c) Find the value of *x* in the following: **2**

(i)



(ii)



(d) Find the exact value of: **2**

$$\cos\frac{\pi}{4}+\sin\frac{2\pi}{3}$$

(e) Differentiate **6**

(i) e^{-3x}

(ii) $\ln(x+4)$

(iii) $x\sin(x+2)$

(iv) $\dfrac{x}{\cos x}$

Question Two [12 marks]

(a) Find the equation of the tangent to the curve $y = 3\log_e x$ at the point (1, 0). **2**

(b) Find **8**

(i) $\int \sec^2 3x\,dx$

(ii) $\int \dfrac{4}{3+x}\,dx$

(iii) $\int (1-x)^2\,dx$

(iv) $\int_0^{\ln 4} e^{2x}\,dx$

(c) The graph of $y = F(x)$ passes through (4, 1) and $F'(x) = \dfrac{2}{\sqrt{x}}$, find $F(x)$. **2**

End of Section A

Section B

(Start a NEW answer booklet)

Question Three [12 marks]

(a) **4**

(i) Without using calculus draw a neat sketch of $y = 3\cos 2x$ for $-\pi \le x \le \pi$.

(ii) On the same set of axes draw the line $y = x$.

(iii) How many solutions has the equation $3\cos 2x = x$?

(b) The following table lists the values of a function for five values of *x*. **4**

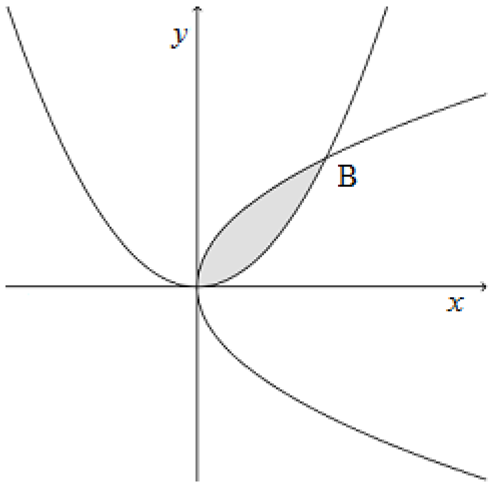
x	2	3	4	5	6
$f(x)$	1.6	2.9	7.6	3.2	0.8

Use these five function values to estimate $\int_2^6 f(x)dx$ by

(i) the trapezoidal rule.

(ii) Simpson's rule.

(c) The curves $y = x^2$ and $y^2 = 8x$ intersect at the origin and the point B. **4**



(i) Find the coordinates of B.

(ii) Find the area between the curves $y = x^2$ and $y^2 = 8x$.

Question Four [12 marks]

(a) Solve $\cos\theta = -0.3$ for $0 \le \theta \le 2\pi$. **2**
[Give your answer in radians correct to two decimal places]

(b) Let $f(x) = 2x^3 - 3x^2 - 12x - 15$. **6**

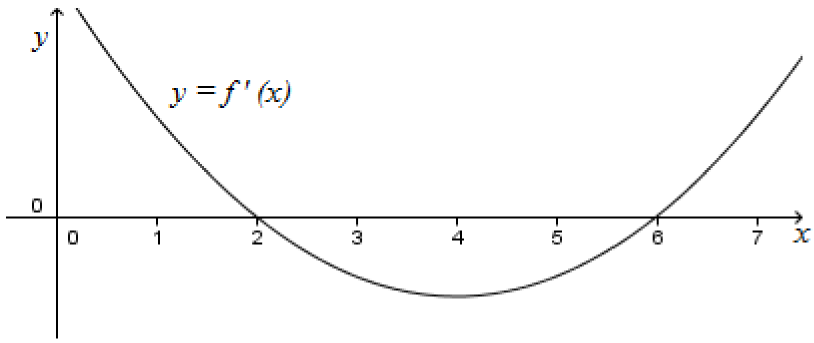
(i) Find the coordinates of the stationary points and determine their nature.

(ii) Find the coordinates of the point of inflexion.

(iii) Sketch the graph of $y = f(x)$ clearly indicating the stationary points and the point of inflexion.

(c) Show that $f(x) = \dfrac{1}{1 + \tan x}$ is decreasing for all values of x for which it is defined. **2**

(d) **2**



The diagram shows the graph of the gradient function of the curve $y = f(x)$. For what values of x does *f(x)* have a local minimum? Justify your answer.

End of Section B

Section C

(Start a NEW answer booklet)

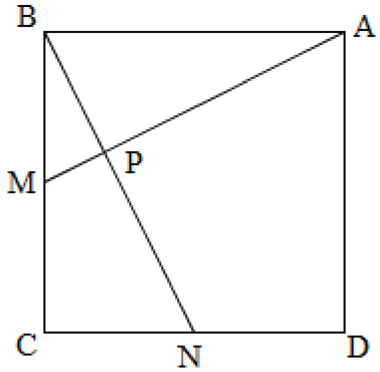
Question Five [12 marks]

(a) Given $f(x)=\frac{1}{e^{x^2}}$, find **3**

(i) $f'(x)$

(ii) $f''(x)$

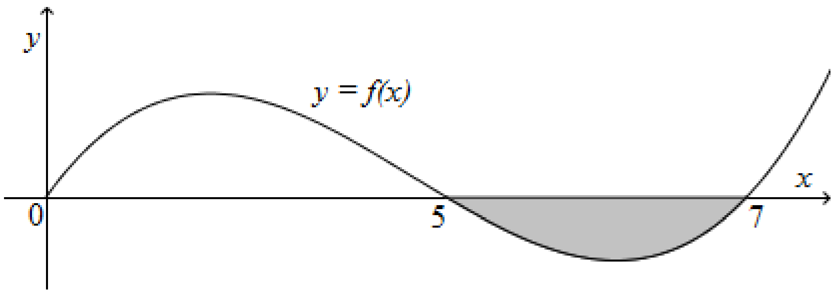
(b) ABCD is a square, M and N are the mid-points of BC and CD respectively. **4**



(i) Prove that triangles ABM and BCN are congruent.

(ii) Prove that AM and BN are perpendicular.

(c) **2**

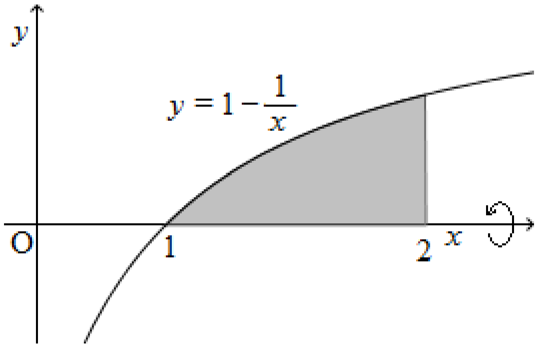


Given that $\int_0^5 f(x)dx=20$ and $\int_0^7 f(x)dx=14$.

(i) What is the area of the shaded region?

(ii) What is the value of $\int_5^7 f(x)dx$?

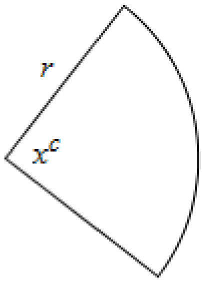
(d) **3**



The area under the curve $y = 1 - \frac{1}{x}$ between $x = 1$ and $x = 2$ is rotated about the x-axis. Find in exact form the volume generated.

Question Six [12 marks]

(a) A sector of a circle has a perimeter of 21 cm and an area of 27 cm^2. **5**



(i) Use the information to form a pair of simultaneous equations.

(ii) Solve these equations to find all values of r and x, which satisfy the information.

(b)

(i) Find the minimum value of **3**

$$x+\frac{900}{x} \text{ for } x>0.$$

[show all working]

(ii) A transport company operates a shipping service between port A and port B, at a constant speed of v km/h. The distance between the ports is S km. **4**

For a given v, the cost *per hour* of running the ship is $\$\left(9000+10v^2\right)$.

Find the value of v which minimizes the cost of the trip.

End of Section C

End of Exam

2010 HSC Mathematics Task 2 – Section A:

Question One:

a) $\int 3 + x^2 \, .dx = 3x + \frac{x^3}{3} + C$

b) $\log_e 6 = 1.79$

c)

(i)
$$\frac{x-1}{x+2} = \frac{4}{5}$$
$$5x - 5 = 4x + 8$$
$$x = 13$$

(ii)
$$x + 7 = \frac{1}{2}(3x)$$
$$2x + 14 = 3x$$
$$x = 14$$

d) $\cos\frac{\pi}{4} + \sin\frac{2\pi}{3} = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}$

e)

(i) $\frac{d}{dx}(e^{-3x}) = -3e^{-3x}$

(ii) $\frac{d}{dx}\ln(x+4) = \frac{1}{x+4}$

(iii) $\frac{d}{dx}[x\sin(x+2)] = \sin(x+2) + x\cos(x+2)$

(iv) $\frac{d}{dx}\left[\frac{x}{\cos x}\right] = \frac{\cos x + x\sin x}{\cos^2 x}$

Question Two:

a) $y = 3\log_e x$

$y' = \frac{3}{x}$

At $x = 1$,

$y' = \frac{3}{1} = 3$

The equation of the tangent is

$y - 0 = 3(x - 1)$

$\therefore y = 3x - 3$

b)

(i) $\int \sec^2 3x\,.dx = \frac{1}{3}\tan 3x + C$

(ii) $\int \frac{4}{3+x}.dx = 4\ln(3 + x) + C$

(iii) $\int (1 - x)^2\,.dx = -\frac{(1-x)^3}{3} + C$

(iv) $\int_0^{\ln 4} e^{2x}\,.dx = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 4}$

$= \frac{1}{2}e^{2\ln 4} - \frac{1}{2}e^0$

$= 8 - \frac{1}{2}$

$= 7\frac{1}{2}$

c) $F(x) = \int \frac{2}{\sqrt{x}}.dx$

$= 2\int x^{-1/2}\,.dx$

$= 2 \times 2x^{1/2} + C$

$F(x) = 4\sqrt{x} + C$

At $(4, 1)$,

$1 = 4\sqrt{4} + C$

$1 = 4 \times 2 + C$

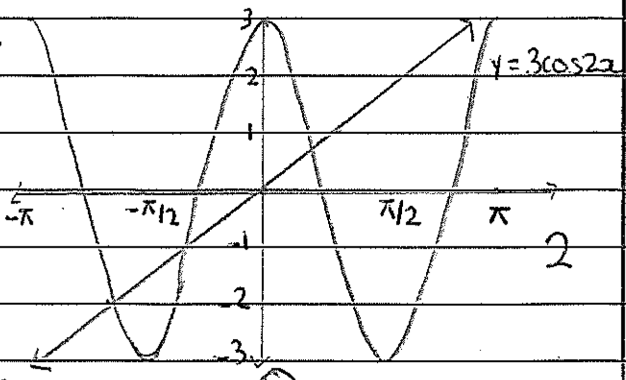
$\therefore C = -7$

$\therefore F(x) = 4\sqrt{x} - 7$

Section B

Question 3

a) i



2

ii) $y=x$ (1)

iii) 3 (1)

b) i $\int_2^6 f(x)\,dx$ $h = \frac{6-2}{4} = 1$

$\doteq \frac{1}{2}\left[(1.6+0.8)+2(2.9+7.6+3.2)\right]$

$= 14.9$ 2

ii $\int_2^6 f(x)\,dx$

$\doteq \frac{1}{3}\left[(1.6+0.8)+4(2.9+3.2)\right.$

$\left.+2(7.6)\right]$

$= 14$ 2

c) i $y = x^2$

$y^2 = 8x$

$\therefore (x^2)^2 = 8x$

$x^4 = 8x$

$x^4 - 8x = 0$

$x(x^3-8) = 0$

$\therefore x=0, \quad x^3-8=0$

$x^3 = 8$

$x = 2$

when $x=2$ $y = 2^2$

$= 4$

$\therefore B = (2,4)$ 2

ii $A = \int_0^2 (\sqrt{8x} - x^2)\,dx$

$= \int_0^2 (\sqrt{8}x^{1/2} - x^2)\,dx$

$= \left[\frac{\sqrt{8}x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^2$

$= \left[\frac{4\sqrt{2}x^{3/2}}{3} - \frac{x^3}{3}\right]_0^2$

$= \left[\frac{4\sqrt{2}\times 2^{3/2}}{3} - \frac{2^3}{3}\right] - \left[0-0\right]$

$= \left[\frac{4\sqrt{2}\times\sqrt{8}}{3} - \frac{8}{3}\right]$

$= \left[\frac{16}{3} - \frac{8}{3}\right]$

$= \frac{8}{3}$ units^2 2

Question 4

a) $\cos\theta = -0.3$
for $0 \leq \theta \leq 2\pi$

S ✓	A
T ✓	C

$\theta = 1.88, 4.41$ (2dp)

2

b) $f(x) = 2x^3 - 3x^2 - 12x - 15$

i) $f'(x) = 6x^2 - 6x - 12$
st pts. $f'(x) = 0$

$0 = 6x^2 - 6x - 12$
$0 = x^2 - x - 2$
$0 = (x-2)(x+1)$

$\therefore x = 2, \quad x = -1$
$\therefore f(x) = -35 \quad f(x) = -8$

x	1	2	3
$f'(x)$	-12	0	24
	\	—	/

$\therefore$ local minimum

x	-2	-1	0
$f'(x)$	24	0	-12
	/	—	\

$\therefore$ local maximum

2

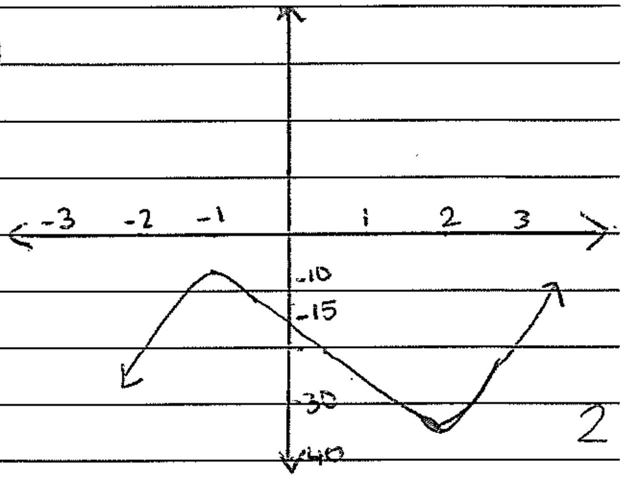
ii) $f''(x) = 12x - 6$
$f''(x) = 0$ for p.o.i

$0 = 12x - 6$
$\therefore 0 = 2x = 1$
$\therefore 2x = 1$
$x = 1/2$
when $x = 1/2$
$f(x) = -21\frac{1}{2}$
$\therefore (1/2, -21\frac{1}{2})$ is a possible point of inflexion

x	0	1/2	1
$f''(x)$	-6	0	+6

Since concavity changes $(1/2, -21\frac{1}{2})$ is a horizontal point of inflexion. 2

iii)



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c) $f(x) = (1 + \tan x)^{-1}$
$f'(x) = -1(1 + \tan x)^{-2} \times \sec^2 x$
$= \dfrac{-\sec^2 x}{(1 + \tan x)^2}$

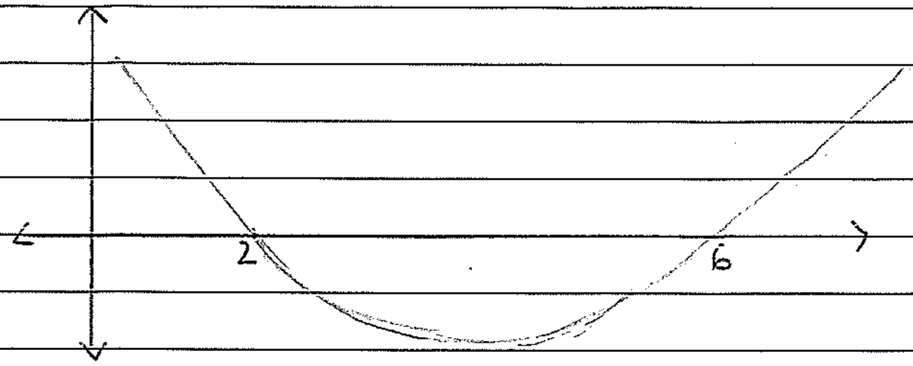
as anything squared is always positive you have

$f'(x) = - \times \dfrac{+}{+}$

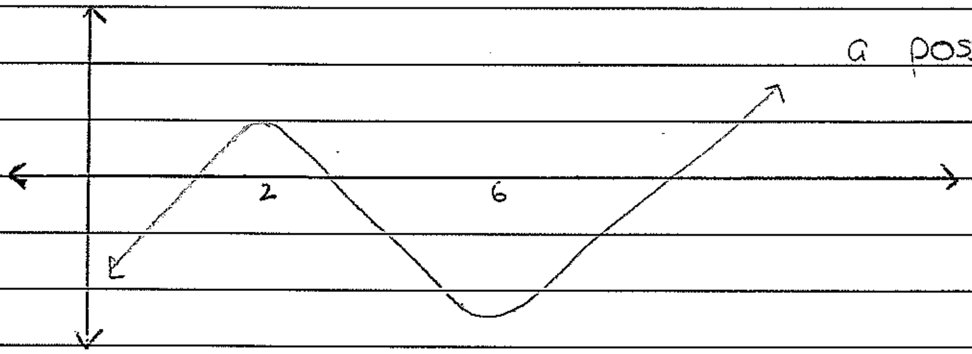
which will always be negative
$\therefore f(x)$ is decreasing for all values of x for which it is defined.

2

d)



$x < 2$ gradient is positive
$x = 2$, $x = 6$ zero gradient ∴ stationary point
$2 < x < 6$ gradient is negative
$x > 6$ gradient is positive



2

∴ $f(x)$ is a local minimum when $x = 6$

OR for local minimum gradient needs to go from negative to positive ie this happens @ $x = 6$.

SECTION C

Question 5

(a) $f(x) = \frac{1}{e^{x^2}}$

$= e^{-x^2}$

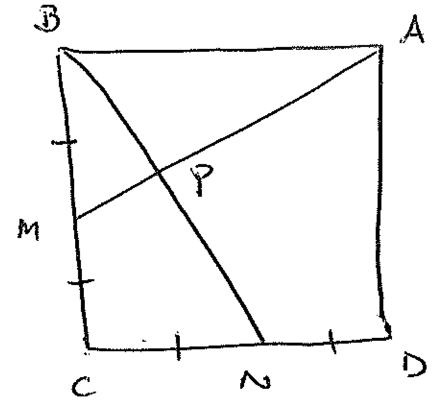
(i) $f'(x) = e^{-x^2} \cdot -2x = -2xe^{-x^2}$ 1

(ii) $f''(x) = e^{-x^2} \cdot -2 + (-2x) \cdot (-2x) e^{-x^2}$

$= -2e^{-x^2} + 4x^2 e^{-x^2}$

$= (4x^2 - 2) e^{-x^2}$ 2

(b)



(i) $AB = BC$ (sides of square)

$BM = \frac{1}{2} BC = \frac{1}{2} CD = CN$ (BC = CD sides of square) 2

$\angle ABM = \angle BCN = 90°$ (angles of square)

$\therefore \triangle ABM \equiv \triangle BCN$ (SAS)

(ii) $\angle PBM = \angle NBC$ (common) 2

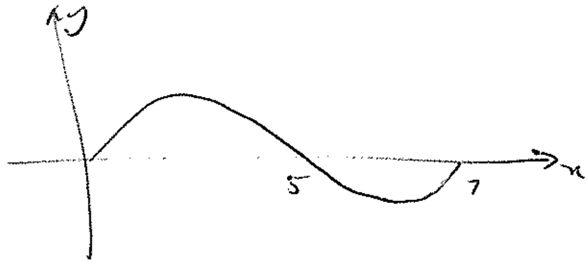
$\angle PMB = \angle AMB$ (common)

$= \angle BNC$ (corresponding ∠s in ≡ Δs)

$\therefore \triangle BPM \parallel\parallel \triangle BCN$ (equiangular)

$\therefore \angle BPM = \angle BCN = 90°$ (corresponding ∠s in ||| Δs) $\therefore AM \perp BN$

(c)



$\int_0^5 f(x)\,dx = 20 \quad \int_0^7 f(x)\,dx = 14$

(i) Area $= 6$ units^2 1

(ii) $\int_5^7 f(x)\,dx = -6$ 1

(d) $V = \pi \int_1^2 \left(1 - \frac{1}{x}\right)^2 dx$

$= \pi \int_1^2 \left(1 - \frac{2}{x} + \frac{1}{x^2}\right) dx$

$= \pi \left[x - 2\ln x - \frac{1}{x} \right]_1^2$ 3

$= \pi \left\{ \left[2 - 2\ln 2 - \frac{1}{2}\right] - \left[1 - 0 - 1\right] \right\}$

$= \pi \left[\frac{3}{2} - 2\ln 2\right]$ units^3

12

Question 6

(a) (i) $P = 2r + xr = 21$ | 1

$A = \frac{1}{2}r^2 x = 27$ | 1

$\therefore x = \frac{54}{r^2}$

$\therefore 2r + \frac{54}{r^2} \cdot r = 21$

$\therefore 2r + \frac{54}{r} = 21$

$\therefore 2r^2 - 21r + 54 = 0$

$\therefore 2r^2 - 9r - 12r + 54 = 0$

$\therefore r(2r-9) - 6(2r-9) = 0$

$\therefore (2r-9)(r-6) = 0$

$\therefore r = 4\frac{1}{2}$ or $r = 6$. 3

$\therefore x = 2\frac{2}{3}$ $x = 1\frac{1}{2}$

×	+
108	-27
2, 54	
3, 36	
4, 27	
6, 18	
9, 12	

(b) (i) $V = x + \frac{900}{x}$

$V' = 1 - \frac{900}{x^2}$ | 1

$V'' = \frac{1800}{x^3}$ | 1

For min value $V' = 0$, $V'' > 0$

If $V' = 0$, $1 - \frac{900}{x^2} = 0$ 1.

$\therefore x^2 = 900$

$x = 30$ (as $x > 0$)

When $x = 30$, $V'' > 0$

$\therefore$ min value when $x = 30$

$$\text{min value} = 30 + \frac{900}{30}$$

$$= 60$$

(ii) $$T = \frac{\text{Distance}}{\text{speed}}$$

$$= \frac{5}{v}$$

$$\text{Cost} = T \times (9000 + 10v^2)$$

$$= \frac{5}{v}(9000 + 10v^2)$$

$$= 105\left(v + \frac{900}{v}\right)$$

min cost occurs when $v = 30$ (using (i))