SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 66

- Attempt questions 1 – 6.
- All questions are not of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *P.R.Bigelow*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x,\ \ x > 0$

Section A (20 Marks)

START A NEW BOOKLET

Question 1. (5 marks)

Indicate which of the answers A, B, C, or D is the correct answer. Write the answer in your answer booklet. **Marks**

(a) $\frac{d}{dx}(\cos 3x)$ equals :- **1**

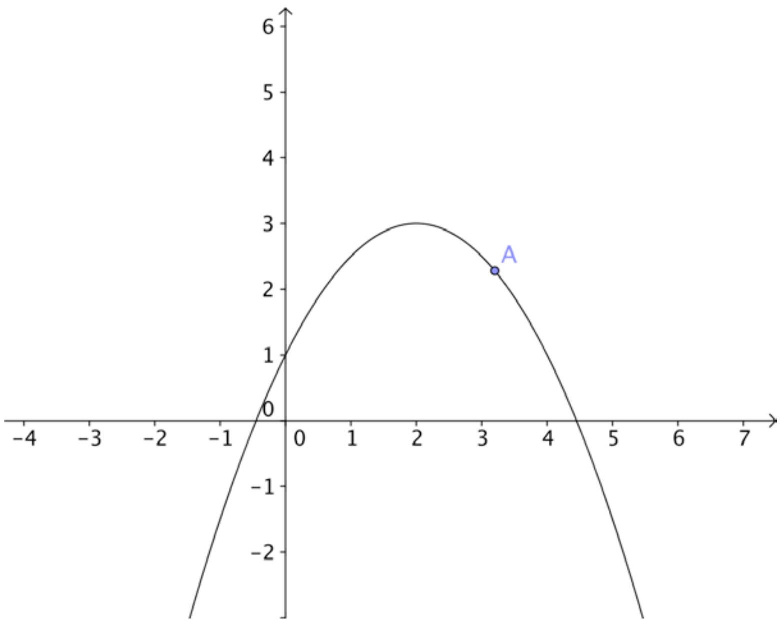
A: $\frac{1}{3}\sin 3x$

B: $3\sin 3x$

C: $-\frac{1}{3}\sin 3x$

D: $-3\sin 3x$

(b) **1**



The diagram shows $y = f(x)$. At point A:

A: $y' > 0,\ y'' > 0$

B: $y' < 0,\ y'' < 0$

C: $y' > 0,\ y'' < 0$

D: $y' < 0,\ y'' > 0$

(c) $4\ln\sqrt{e}$ equals:- **1**

A: 2

B: $2e$

C: 1

D: $\sqrt{2}$

(d) $\int_{-3}^{3} x^3\,dx$ equals:- **1**

A: $\left|\int_{-3}^{0} x^3 dx\right| + \int_0^3 x^3 dx$

B: $2\int_0^3 x^3 dx$

C: 0

D: $\frac{81}{2}$

(e) If $f(x) = \sin x$, then $f'\left(-\frac{\pi}{3}\right)$ is:- **1**

A: $\frac{\sqrt{3}}{2}$

B: $\frac{1}{2}$

C: $-\frac{1}{2}$

D: $-\frac{\sqrt{3}}{2}$

Question 2 (15 marks)

Marks

(a) Differentiate the following: **5**

(i) $y = \cos 4x$

(ii) $y = e^{-4x}$

(iii) $f(x) = x \tan x$

(iv) $f(x) = \dfrac{\sin x}{x}$

(v) $y = \cos^3 2x$

(b) Find **3**

(i) $\displaystyle\int \frac{dx}{x+3}$

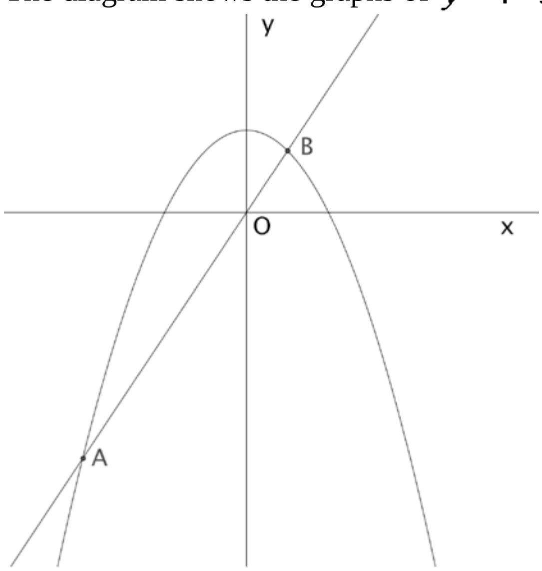
(ii) $\displaystyle\int_0^1 \frac{3}{e^{2x}}$

(c) Use Simpson's Rule with three function values to find an approximation to **3**

$$\int_0^1 \sqrt{1+x^3}\, dx$$

(Answer correct to two decimal places.)

(d) The diagram shows the graphs of $y = 4 - x^2$ and $y = 3x$. **4**



(i) Find the x-values of the points of intersection *A* and *B*.

(ii) Calculate the area of the region between the two graphs.

Section B (20 Marks)

START A NEW BOOKLET

Question 3 (10 Marks)

Marks

(a) Given $f(x) = 3\ln(2+x)$ find $f'(4)$. **2**

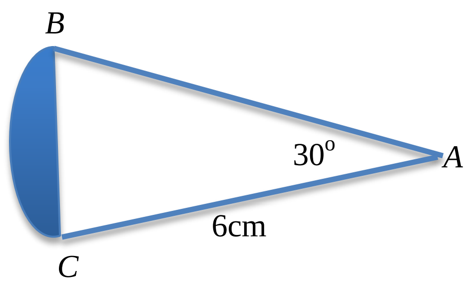
(b) Find $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$. **2**

(c) Find the value of $\sqrt{e^3}$ correct to three significant figures. **1**

(d) Use the identity $\sec^2 x = 1 + \tan^2 x$ to evaluate **2**

$$\int_0^{\frac{\pi}{4}} \tan^2 x \, dx.$$

(e) In the diagram $\angle BAC = 30^\circ$ and a circular arc of radius 6 cm, centre *A*, is constructed from *B* to *C*. **3**



(i) Find the area of ΔABC.

(ii) Calculate the exact area of the shaded segment.

Question 4 (10 Marks)

(a) Find $f'(x)$ and $f''(x)$ where $f(x) = \sin(x^2)$. **3**

(b) Given the function $y = x^3 - 9x + 3$ **7**

(i) Find the co-ordinates of the stationary points, and determine their nature.

(ii) Find the co-ordinates of any points of inflexion.

(iii) Sketch the curve in the domain $-4 \le x \le 4$.

(iv) What is the greatest value of $x^3 - 9x + 3$ in the domain $-4 \le x \le 4$?

Section C (26 Marks)

START A NEW BOOKLET

Question 5 (10 Marks)

(a) For the function $f(x) = 2\cos x + 1$ **5**

(i) State the range of $f(x)$.

(ii) Sketch $f(x) = 2\cos x + 1$ for $0 \le x \le 2\pi$.

(iii) Calculate the exact area of the region in the first quadrant bounded by the curve $f(x) = 2\cos x + 1$, the y-axis, and the line $y = 1$.

(b) **5**

(i) Sketch $y = \log_e x$. Mark on the curve the point P where $x = e$.

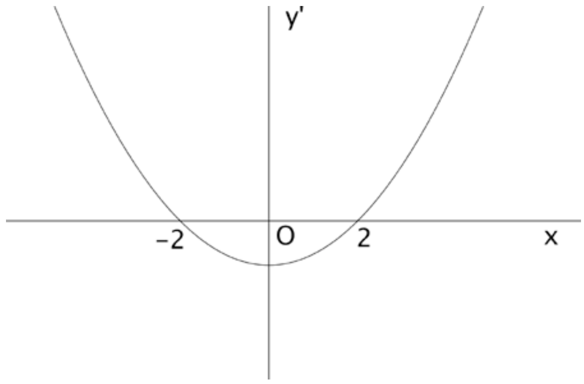
(ii) Find, in general form, the equation of the tangent to the curve $y = \log_e x$ at the point P.

(iii) Using the sketch of $y = \log_e x$ find the values of k for which $kx = \log_e x$ has at least one real root.

Question 6 (16 Marks)

(a) Simplify e^{2ln5}. **1**

(b) **4**



The diagram shows the graph of the gradient function of the curve $y = f(x)$.

(i) What type of point occurs on $y = f(x)$ at $x = -2$?
Justify your answer.

(ii) If $f(-2) = 5$ and $f(2) > 0$ sketch $y = f(x)$.

(c) On the same (new) diagram: **4**

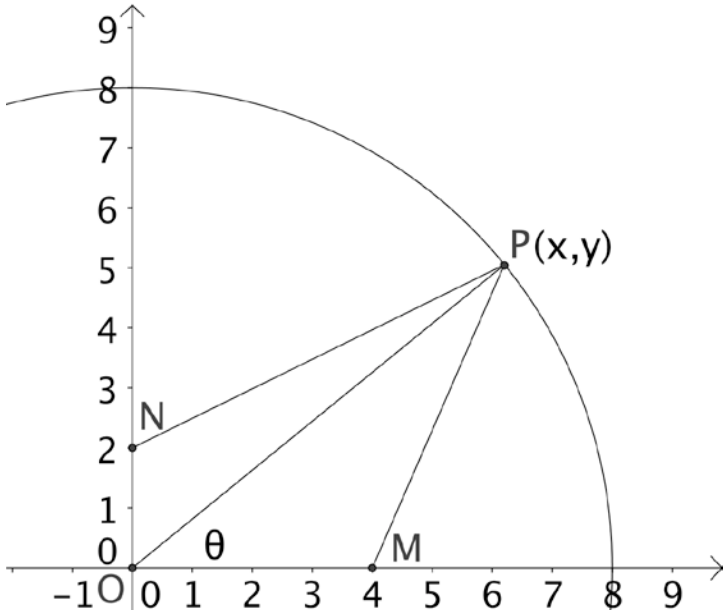
(i) Sketch $y = x$ and $y = x^2$ in the first quadrant.

(ii) Mark the points of intersection and indicate, by shading, the area between the two graphs.

(iii) Find the volume generated when this area is rotated about the x-axis.

(Question continued overleaf)

(d) **7**



The diagram shows the part of the circle $x^2 + y^2 = 64$ in the first quadrant. The point $P(x, y)$ lies on the circle, centre *O*.

M is on the x-axis at $x = 4$, *N* is on the y-axis at $y = 2$, and $\angle MOP = \theta$ in radians.

(i) Show that the area, *A*, of the quadrilateral *OMPN* is given by

$$A = 16\sin\theta + 8\cos\theta$$

(ii) Find the value of $\tan\theta$ for which *A* is a maximum.

(iii) Hence find in surd form the co-ordinates of *P* for which *A* is a maximum.

This is the end of the paper.

2012 2U TASK 2 . YR12

SECTION A

1. a) D
b) B
c) A
d) C
e) B. ✓

2. a) i) $-4\sin 4x$ ✓

ii) $-4e^{-4x}$ ✓

iii) $\tan x + x\sec^2 x$ ✓

iv) $\dfrac{x\cos x - \sin x}{x^2}$ ✓

v) $3(\cos 2x)^2(-2\sin 2x)$
$= -6\cos^2 2x \sin 2x$ ✓

b) i) $\ln(x+3) + C$ ✓

ii) $3\int_0^1 e^{-2x}\,dx = 3\left[-\frac{1}{2}e^{-2x}\right]_0^1$
$= -\frac{3}{2}e^{-2} - \left(-\frac{3}{2}e^0\right)$
$= \frac{3}{2}\left(1 - \frac{1}{e^2}\right)$

c) $h = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}$

x	0	0.5	1
	1	$\sqrt{1.125}$	$\sqrt{2}$

$\int_0^1 \sqrt{1+x^3} = \int_a^b \doteq \frac{h}{3}\left(y_0 + y_n + 4(y_{odd}) + 2(y_{even})\right)$
$\doteq \frac{1}{6}\left(1 + \sqrt{2} + 4(\sqrt{1.125})\right)$
$= 1.110947 \doteq 1.11$ (2dp)

d) i) $y = 4 - x^2 = 3x$

$x^2 + 3x - 4 = 0$
$(x-1)(x+4) = 0$

P.O.I $B(1, 3), A(-4, -12)$

ii) $\int_{-4}^{1} 4 - x^2 - 3x\,.\,dx$
$= \left[4x - x^3/3 - \frac{3}{2}x^2\right]_{-4}^{1}$
$= \left[4 - 1/3 - 3/2\right] - \left[-16 + \frac{64}{3} - \frac{48}{2}\right]$
$= 2\frac{1}{6} - {}^-18\frac{2}{3}$
$= 20\,5/6$ units^2

3 (a) $f(x) = 3\ln(2+x)$

$f'(x) = \frac{3}{2+x}$

$f'(4) = \frac{3}{2+4}$

$= \frac{1}{2}$

(b) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= \frac{1}{2}$

(c) $\sqrt{e^3} = 4.48$ (3 SIG. FIG.)

(d) $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$

$= \left[\tan x - x\right]_0^{\frac{\pi}{4}}$

$= \left(1 - \frac{\pi}{4}\right) - (0 - 0)$

$= 1 - \frac{\pi}{4}$

(e) (i) Area $\triangle ABC = \frac{1}{2} \cdot 6^2 \cdot \sin 30$

$= 9 \text{ cm}^2$

(ii) Area of shaded $= \frac{1}{2} r^2 \theta - 9$

$= \frac{1}{2} \cdot 6^2 \cdot \frac{\pi}{6} - 9$

$= 3\pi - 9$

4 (a) $f(x) = \sin(x^2)$ (Chain Rule)

$f'(x) = 2x \cdot \cos(x^2)$

$f''(x) = 2x \cdot -2x\sin(x^2) + \cos(x^2) \cdot 2$ (Prod. Rule)

$= 2\cos(x^2) - 4x^2 \cdot \sin(x^2)$

(b) $y = x^3 - 9x + 3$

$y' = 3x^2 - 9$

For Stat Pts $y' = 0$

$\therefore 3(x^2 - 3) = 0$

$x = \pm\sqrt{3}$

Test in $y'' = 6x$

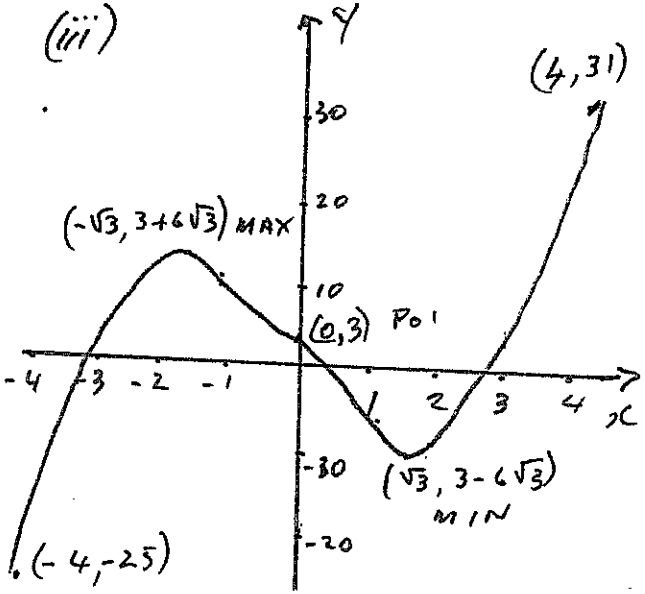
at $x = \sqrt{3}$; $y'' > 0 \therefore$ min

$\therefore (\sqrt{3}, 3 - 6\sqrt{3})$ MIN

Similarly $(-\sqrt{3}, 3 + 6\sqrt{3})$ MAX.

P.O.I at $x = 0$ and concavity changes. $(0, 3)$

(iii)



(iv) Clearly max value of function in given domain is 31 at $x = 4$

SECTION C

Question 5

(a) i) $-1 \le \cos x \le 1$

$-2 \le 2\cos x \le 2$

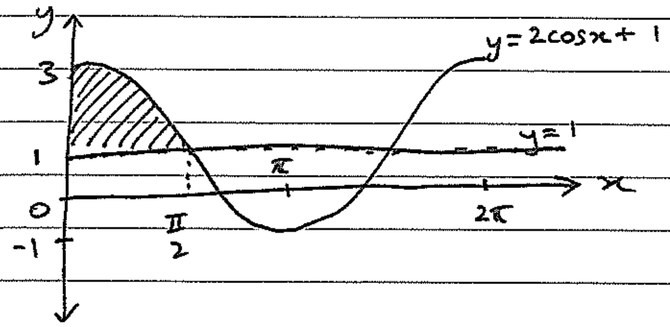
$-1 \le 2\cos x + 1 \le 3$

R: $-1 \le y \le 3$

ii) $f(x) = 2\cos x + 1$ for $0 \le x \le 2\pi$

$a = 2$

$p = \frac{2\pi}{1}$



iii) $A = \int_0^{\frac{\pi}{2}} (2\cos x + 1 - 1)\, dx$

$= \int_0^{\frac{\pi}{2}} 2\cos x \, dx$

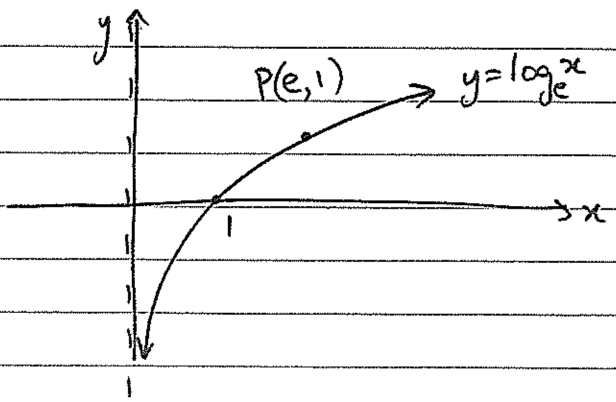
$= \left[2\sin x \right]_0^{\frac{\pi}{2}}$

$= 2\sin\frac{\pi}{2} - 2\sin 0$

$= 2(1)$

$= 2$ units^2

(b) i)



ii) $y = \log_e x$

$y' = \frac{1}{x}$

at $P(e, 1)$

$m_T = \frac{1}{e}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{e}(x - e)$

$ey - e = x - e$

$x - ey = 0$

iii) The tangent at P found in (ii) passes through the origin since it's in the form $y = kx$ where $k = \frac{1}{e}$.

Answer is $-\infty < k \leq \frac{1}{e}$

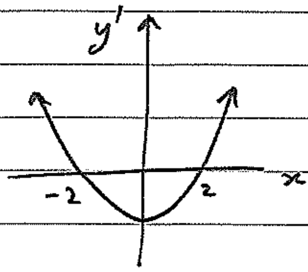
Question 6

(a) $e^{2\ln 5} = e^{\ln 5^2}$

$= 5^2$

$= 25$

(b) i)



when $x = -2$

$y' = 0$ (y' cuts the x-axis)

$y'' < 0$ (gradient of y' is negative)

↳ concave down

$\therefore$ Maximum Turning Point when $x = -2$

OR consider y' on either side of $x = -2$

x	-3	-2	-1
y'	$+$	0	$-$

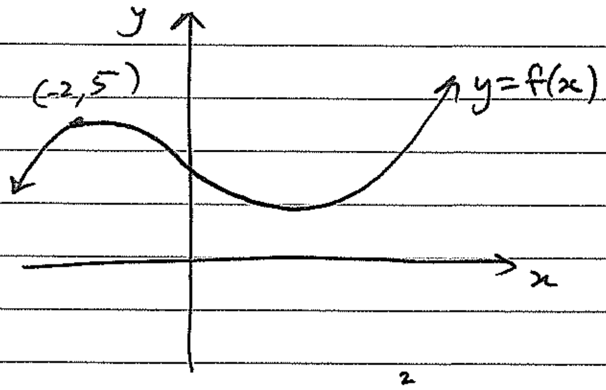
← since y' is below x-axis (for $x=-1$)

↑ since y' is above x-axis (for $x=-3$)

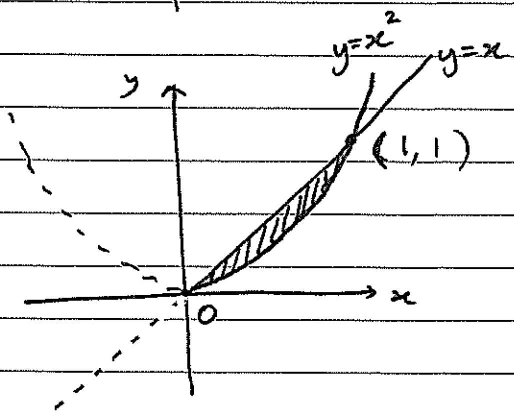
/ — \

$\therefore$ Maximum Turning Point when $x=-2$

ii)



(c) i)
ii)



iii) $V = \pi \int_a^b y^2 \, dx$

$= \pi \int_0^1 \left((x)^2 - (x^2)^2 \right) dx$

$= \pi \int_0^1 \left(x^2 - x^4 \right) dx$

$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$

$= \pi \left[\frac{(1)^3}{3} - \frac{(1)^5}{5} - (0) \right]$

$= \frac{2\pi}{15}$ units^3

(d) assuming P lies in the first quadrant
θ is acute.

i) Area $\triangle OPM = \frac{1}{2}(8)(4)\sin\theta$

$= 16\sin\theta$

Area $\triangle OPN = \frac{1}{2}(2)(8)\sin(\frac{\pi}{2}-\theta)$

$= 8\sin(\frac{\pi}{2}-\theta)$

$= 8\cos\theta$ (complimentary angles)

Area of quadrilateral OMPN $= 16\sin\theta + 8\cos\theta$

ii) $A = 16\sin\theta + 8\cos\theta$

$\frac{dA}{d\theta} = 16\cos\theta - 8\sin\theta$

$\frac{d^2A}{d\theta^2} = -16\sin\theta - 8\cos\theta$

let $\frac{dA}{d\theta} = 0$ for stat. points

$16\cos\theta - 8\sin\theta = 0$

$16\cos\theta = 8\sin\theta$ $(\div\cos\theta)$

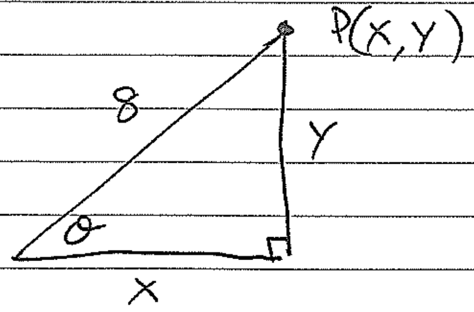
$16 = 8\tan\theta$

$\tan\theta = 2$

when θ is acute $\frac{d^2A}{d\theta^2} < 0$ concave down

$\therefore$ A is a maximum when $\tan\theta = 2$

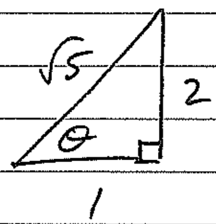
iii)



$\cos\theta = \frac{X}{8}$ $\sin\theta = \frac{Y}{8}$

$X = 8\cos\theta$ $Y = 8\sin\theta$

$\tan\theta = 2$



$\cos\theta = \frac{1}{\sqrt{5}}$ $\sin\theta = \frac{2}{\sqrt{5}}$

$X = 8\left(\frac{1}{\sqrt{5}}\right)$ $Y = 8\left(\frac{2}{\sqrt{5}}\right)$

$= \frac{8}{\sqrt{5}}$ $= \frac{16}{\sqrt{5}}$

$= \frac{8\sqrt{5}}{5}$ $= \frac{16\sqrt{5}}{5}$

$\therefore$ P has coordinates $\left(\frac{8\sqrt{5}}{5}, \frac{16\sqrt{5}}{5}\right)$