SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014
YEAR 12 Mathematics
HSC Task #2

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answers must be given in simplest exact form unless otherwise stated.

Total marks - 72

Multiple Choice Section (5 marks)
- Answer Questions 1-5 on the Multiple Choice answer sheet provided.

Sections A, B, C and D (67 marks)

- Start a new answer booklet for each section.

Examiner: *D.McQuillan*

Multiple Choice Section (5 marks)

(1) A bag contains three white balls and seven yellow balls. Three balls are drawn one at a time from the bag without replacement. The probability that they are all yellow is

(A) $\frac{3}{500}$

(B) $\frac{27}{1000}$

(C) $\frac{21}{100}$

(D) $\frac{7}{24}$

(E) $\frac{243}{1000}$

(2) If $\int_1^4 f(x)dx = 2$, then $\int_1^4 (2f(x)+3)dx$ is equal to

(A) 2

(B) 4

(C) 7

(D) 10

(E) 13

(3) For the function $f(x) = (x+5)^2(x-1)$ the gradient of $f(x)$ is negative for

(A) $x < 1$

(B) $-5 < x < 1$

(C) $-5 < x < -1$

(D) $x < -5$

(E) $-5 < x < 0$

(4) For $y = \sqrt{1 - f(x)}$, $\frac{dy}{dx}$ is equal to

(A)
$$\frac{2f'(x)}{\sqrt{1-f(x)}}$$

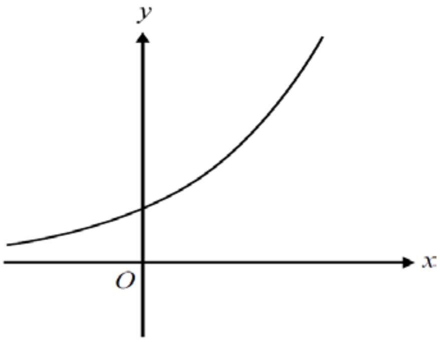
(B)
$$\frac{-1}{2\sqrt{1-f'(x)}}$$

(C)
$$\frac{1}{2}\sqrt{1-f'(x)}$$

(D)
$$\frac{3}{2\left(1-f'(x)\right)}$$

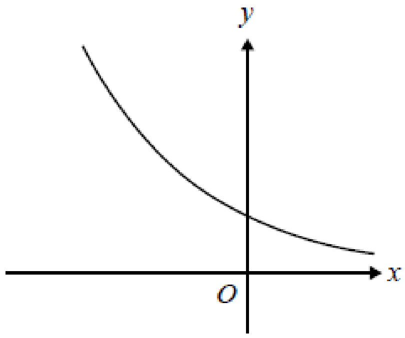
(E)
$$\frac{-f'(x)}{2\sqrt{1-f(x)}}$$

(5) The graph of $y = f(x)$ is shown below.

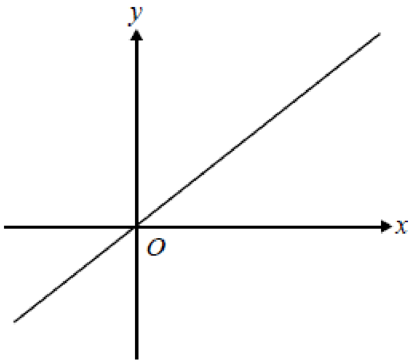


Which one of the following could be the graph of $y = f'(x)$?

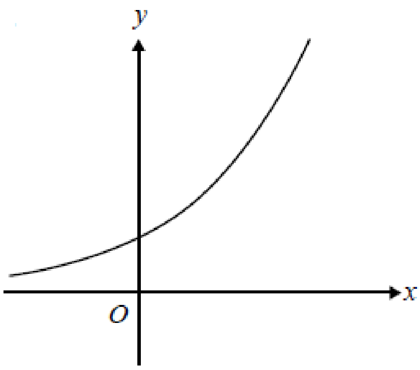
(A)



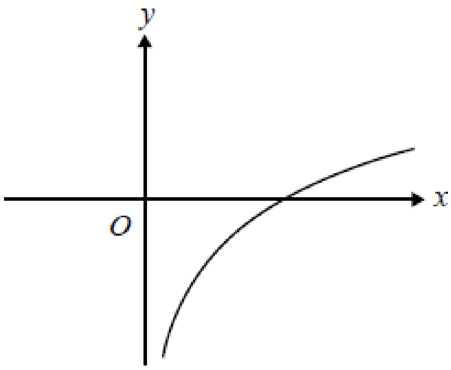
(B)



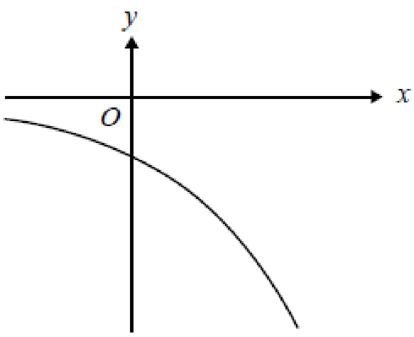
(C)



(D)



(E)

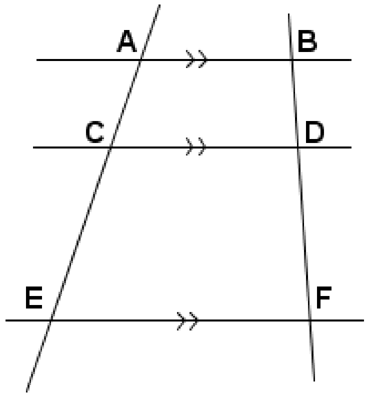


End of Multiple Choice Section

Section A (18 marks)

Start a NEW writing booklet for each section.

(1) Given AB//CD and CD//EF and AC:CE = 1:4 find the ratio of DF:FB. [1]



(2) To what sum will $1500 amount if invested for 3 years at 12% per annum compounded yearly? [2]

(3) Evaluate [4]

(a)

$$\lim_{x \to -2} \frac{x^2 + 7x + 10}{x + 2}$$

(b)

$$\lim_{x \to \infty} \frac{x^2 - 3x + 9}{2x^2 + 2}$$

(4) Find the derivative of [6]

(a) $2x^3 + 3x^{-2}$

(b) $(4x^3 + x)\left(\frac{1}{2}x^2 - 4\right)$

(c) $\frac{4x^2}{2x-1}$

(5) Find the equation of the tangent to the graph $y = e^{3x} + 3x$ at the point where $x = 0$. [3]

(6) Karthik has four pairs of identical purple socks and three pairs of identical green socks. His socks are randomly mixed in his drawer. He takes two individual socks at random from the drawer in the dark. Find the probability that he obtains a matching pair. [2]

End of Section A

SECTION B (16 marks)

Start a NEW writing booklet for each section.

(1) Find [6]

(a)

$$\int (4x^3 + 4x^2 + 1)dx$$

(b)

$$\int \frac{e^x}{e^x - 5}dx$$

(c)

$$\int \frac{e^x - 5}{e^x}dx$$

(2) Evaluate [2]

$$\int_1^3 \frac{x + 3}{x^2 + 6x - 6}dx$$

(3) Show that the points $A(3, -1)$, $B(7, 2)$ and $C(1, 10)$ are the vertices of a right angled triangle. Also find the area of the triangle ABC. [5]

(4) Using differentiation from first principles find the derivative of $f(x) = 2x + 5$. [3]

End of Section B

SECTION C (18 marks)

Start a NEW writing booklet for each section.

(1) The diagonal BD of the quadrilateral ABCD bisects each of the angles ABC and ADC. Prove that [4]

(a) $\Delta ABD \equiv \Delta CBD$

(b) $AC \perp BD$

(2) Find the greatest and least values of $f(x) = x^3 - 12x + 20$ on the interval $-3 \leq x \leq 5$. [4]

(3) [4]

(a) Find the approximate value of

$$\int_0^2 (x^3 + x)dx$$

using the Trapezoidal Rule with 2 subintervals.

(b) What is the difference between the approximation from part (a) and the actual value of the integral?

(4) For the curve $y = (x+2)(5-x)^3$ [4]

(a) Find the stationary points and their natures.

(b) Hence sketch the curve $y = (x+2)(5-x)^3$.

(5) If two dice are rolled and one of the dice shows a 2 what is the probability that the sum of the upper most faces is less than 6? [2]

End of Section C

SECTION D (15 marks)

Start a NEW writing booklet for each section.

(1) Find the exact area of the region bounded by the x-axis, the y-axis, the line $y = 3$ and the curve $y = \log_e(x - 1)$. [3]

(2) A rectangular box is required to have a volume of 24 cm^3 and its length is to be double its width. It is to be strengthened by a strip of steel running along all its edges. If the total length of the steel strip is to be a minimum, find the dimensions of the box (length, width and depth). [4]

(3) On Brendan's 20th birthday his grandparents place \$500 000 in a trust account that earns 8% per annum. Brendan receives one payment a year from the trust account. He receives the first payment on his 21st birthday. [8]

(a) If the payments are \$M per year show that the amount, \$$A_2$, in the trust account after the second payment is

$$A_2 = 500\,000 \times 1.08^2 - M \times 1.08 - M$$

(b) How much would each payment be if the trust account was to last for 20 years?

(c) If the payments were \$75 000 per year, how many full payments would Brendan receive?

(d) After the last full payment of \$75 000 how much is still left in the trust.

End of Section D

End of Exam

2014 Year 12 Mathematics HSC Task #2 - Solutions

Multiple Choice

1.	D
2.	E
3.	C
4.	E
5.	C

1. 3 White, 7 Yellow.

$$P(YYY) = \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8}$$

$$= \frac{7}{24} \qquad \boxed{D}$$

2. $\int_1^4 f(x)\,dx = 2$, then $\int_1^4 (2f(x)+3)\,dx$ is

$$= \int_1^4 2f(x)\,dx + \int_1^4 3\,dx$$

$$= 2\int_1^4 f(x)\,dx + [3x]_1^4$$

$$= 2(2) + [3(4) - 3(1)]$$

$$= 4 + 9$$

$$= 13 \qquad \boxed{E}$$

3. negative gradient: $f'(x) < 0$

$$f(x) = (x+5)^2(x-1)$$

$u = (x+5)^2$, $v = (x-1)$

$u' = 2(x+5)$, $v' = 1$

$$\therefore f'(x) = uv' + vu'$$

$$= (x+5)^2 . 1 + (x-1) . 2(x+5)$$

$$= (x+5)[(x+5) + 2(x-1)]$$

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$$= (x+5)[3x+3]$$
$$= 3(x+5)(x+1)$$

$f'(x) < 0 \Rightarrow x > -5, \ x < -1$

ie. $-5 < x < -1$ $\boxed{C}$

4. $y = \sqrt{1 - f(x)}$

$= (1 - f(x))^{1/2}$

$\therefore \frac{dy}{dx} = \frac{1}{2}(1 - f(x))^{-1/2} \cdot -f'(x)$

$= \frac{-f'(x)}{2(1-f(x))^{1/2}}$

$= \frac{-f'(x)}{2\sqrt{1-f(x)}}$ $\boxed{E}$

5. $\boxed{C}$

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Section A

1. $AC : CE = 1 : 4$

$$DF = CE = 4$$

$$FB = BD + DF = AC + CE = 1 + 4 = 5$$

$$\therefore DF : FB = 4 : 5$$ [1]

2. $A = P(1+r)^n \Rightarrow A = 1500(1+0.12)^3$

$$= 2107.392$$

$$= \$2107.39$$ [2]

3. (a) $$\lim_{x \to -2} \frac{x^2 + 7x + 10}{x + 2} = \lim_{x \to -2} \frac{(x+5)\cancel{(x+2)}}{\cancel{(x+2)}}$$

$$= \lim_{x \to -2} (x+5)$$

$$= -2 + 5$$

$$= 3$$ [2]

(b) $$\lim_{x \to \infty} \frac{x^2 - 3x + 9}{2x^2 + 2} = \lim_{x \to \infty} \frac{x^2/x^2 - 3x/x^2 + 9/x^2}{2(x^2/x^2 + 1/x^2)}$$

$$= \lim_{x \to \infty} \frac{1 - 3/x + 9/x^2}{2(1 + 1/x^2)}$$

$$= \frac{1 - 3/\infty + 9/\infty^2}{2(1 + 1/\infty^2)}$$

$$= \frac{1 - 0 + 0}{2(1 + 0)}$$

$$= \frac{1}{2}$$ [2]

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4. (a) $\dfrac{d(2x^3 + 3x^{-2})}{dx} = 3.2x^2 + (-2).3x^{-3}$

$= 6x^2 - 6x^{-3}$

$= 6x^2 - \dfrac{6}{x^3}$ [1]

(b) $\dfrac{d(4x^3 + x)(\frac{1}{2}x^2 - 4)}{dx}$

Let $u = 4x^3 + x$

$u' = 12x^2 + 1$

$v = \frac{1}{2}x^2 - 4$

$v' = x$

$= uv' + vu'$

$= (4x^3 + x).x + (\frac{1}{2}x^2 - 4)(12x^2 + 1)$

$= 4x^4 + x^2 + 6x^4 + \frac{1}{2}x^2 - 48x^2 - 4$

$= 10x^4 - \dfrac{93x^2}{2} - 4$ [2]

(c) $\dfrac{d\left(\dfrac{4x^2}{2x-1}\right)}{dx}$

Let $u = 4x^2$

$u' = 8x$

$v = 2x - 1$

$v' = 2$

$= \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(2x-1).8x - 4x^2.2}{(2x-1)^2}$

$= \dfrac{16x^2 - 8x - 8x^2}{(2x-1)^2}$

$= \dfrac{8x^2 - 8x}{(2x-1)^2}$

$= \dfrac{8x(x-1)}{(2x-1)^2}$ [3]

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5. $y = e^{3x} + 3x$ at $x = 0$.

when $x = 0$, $y = e^{3(0)} + 3(0)$
$= 1$ $\therefore (0, 1)$

$y' = 3e^{3x} + 3$
$= 3(e^{3x} + 1)$

when $x = 0$, $y' = 3(e^{3(0)} + 1)$
$= 3(2)$
$= 6$

$\therefore$ $y - y_1 = m(x - x_1)$
$y - 1 = 6(x - 0)$
$\therefore$ $6x - y + 1 = 0$ [3]

6. $P(\text{a pair}) = P(PP) + P(GG)$

$= \frac{8}{14} \times \frac{7}{13} + \frac{6}{14} \times \frac{5}{13}$
$= \frac{43}{91}$ [2]

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Section B

1.(a) $\int 4x^3 + 4x^2 + 1 \quad dx$

$= \frac{4x^4}{4} + \frac{4x^3}{3} + x + C$

$= x^4 + \frac{4x^3}{3} + x + C$ [2]

(b) $\int \frac{e^x}{e^x - 5} \, dx \Rightarrow \left[\int \frac{f'(x)}{f(x)} \, dx \right]$

$= \ln(e^x - 5) + C$ [2]

(c) $\int \frac{e^x - 5}{e^x} \, dx = \int \frac{e^x}{e^x} - \frac{5}{e^x} \, dx$

$= \int 1 - 5e^{-x} \, dx$

$= x + 5e^{-x} + C$

$= x + \frac{5}{e^x} + C$ [2]

2. $\int_1^3 \frac{x+3}{x^2+6x-6} \, dx$

If $f(x) = x^2 + 6x - 6$
$f'(x) = 2x + 6$
$= 2(x+3)$

$= \frac{1}{2} \int_1^3 \frac{2(x+3)}{x^2+6x-6} \, dx$

$= \frac{1}{2} \left[\ln(x^2 + 6x - 6) \right]_1^3$

$= \frac{1}{2} \left[\left(\ln(3^2 + 6(3) - 6) \right) - \left(\ln(1^2 + 6(1) - 6) \right) \right]$

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$= \frac{1}{2} \ln(21) \quad (\approx 1.5226)$

3. $m_{AB} = \frac{2-(-1)}{7-3}$

$= \frac{3}{4}$

$m_{AC} = \frac{10-(-1)}{1-3}$

$= -\frac{11}{2}$

$m_{BC} = \frac{10-2}{1-7}$

$= -\frac{8}{6}$

$= -\frac{4}{3}$

$\therefore m_{AB} \times m_{BC} = \frac{3}{4} \times -\frac{4}{3}$

$= -1$

$\therefore AB \perp BC$

$d_{AB} = \sqrt{(7-3)^2 + (2-(-1))^2}$

$= 5$

$d_{BC} = \sqrt{(1-7)^2 + (10-2)^2}$

$= 10$

$\therefore \text{Area} = \frac{1}{2} \times 5 \times 10$

$= 25 \quad \text{units}^2$

[5]

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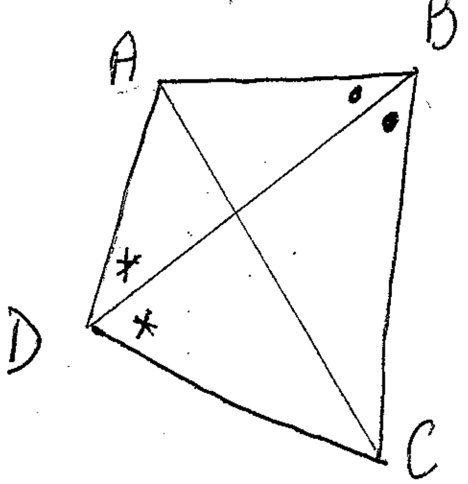
2U 2014 Assess Task 2

Section C

/18

(1)



a) data: as per diagram

aim: to prove $\triangle ABD \equiv \triangle CBD$

Proof: DB bisects $A\hat{B}C$ given

So $A\hat{B}D = C\hat{B}D$

DB bisects $A\hat{D}C$ given

So $A\hat{D}B = C\hat{D}B$

Line DB common side

$\therefore \triangle ABD \equiv \triangle CBD$ AAS rule.

(3)

(b) Because $\triangle ABD \equiv \triangle CBD$, quadrilateral is a square, rhombus, kite where a diagonal bisects a pair of vertex angles

$\therefore AB = AD$, $\triangle ABC$ is isosceles.

Line from vertex to base DB bisects base and is $\perp$ to it.

(1)

(2) $f(x) = x^3 - 12x + 20$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

Stat pts $f'(x) = 0$ $\quad 3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

at $x = 2$, $f(x) = 4$ $\quad (2, 4)$

$x = -2$, $f(x) = 36$ $\quad (-2, 36)$

at $(2, 4)$ $\quad f''(x) = 12 > 0$ min stat pt

at $(-2, 36)$ $\quad f''(x) = -12 < 0$ max stat pt

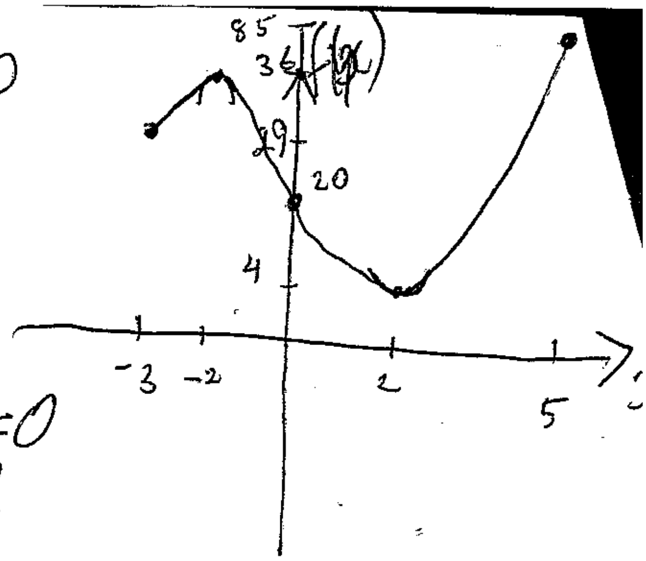
At the end points!

at $x = -3$, $f(x) = 29$

$x = 5$, $f(x) = 85$

Greatest (y) value in domain is 85

Least (y) value in domain is 4



(4)

(3) (a) $\int_0^2 (x^3+x)\,dx$ Trapezoidal rule. 2 sub intervals

x	0	1	2
$f(x)$	0	2	10
	y_0	y_1	y_n

$$\int_a^b f(x)\,dx \doteq \frac{h}{2}\left[(y_0+y_n)+2(y_1)\right]$$
$$= \frac{1}{2}\left[0+10+2\times 2\right]$$
$$= \frac{1}{2}\times 14 = 7u^2 \quad (2)$$

(b) $\int_0^2 (x^3+x)\,dx = \left.\frac{x^4}{4}+\frac{x^2}{2}\right]_0^2$

$$= (4+2)-(0) = 6.$$

difference $7-6 = 1$ (2)

④ (a) $y = (x+2)(5-x)^3$

$$y' = (x+2) \times 3(5-x)^2 \times -1 + (5-x)^3 \times 1$$
$$= -3(x+2)(5-x)^2 + (5-x)^3$$
$$= (5-x)^2\left[-3(x+2) + (5-x)\right]$$
$$= (5-x)^2\left[-3x - 6 + 5 - x\right]$$
$$= (5-x)^2\left[-4x - 1\right] \quad \checkmark$$

$$y'' = (5-x)^2 \times -4 + (-4x-1) \times 2(5-x)^1 \times -1$$
$$= -4(5-x)^2 - 2(-4x-1)(5-x)$$
$$= -4(5-x)^2 + 2(1+4x)(5-x)$$
$$= 2(5-x)\left[-2(5-x) + (1+4x)\right]$$
$$= 2(5-x)\left[-10 + 2x + 1 + 4x\right]$$
$$= 2(5-x)\left[-9 + 6x\right]$$
$$= 6(5-x)\left[2x - 3\right]$$

Stat points when $y' = 0$ ie $-(5-x)^2(1+4x) = 0$

$x = 5$ and $x = -\frac{1}{4}$

When $x = 5$, $y = 0$ $(5, 0)$

When $x = -\frac{1}{4}$ $y = 253\frac{59}{256}$ $\left(-\frac{1}{4}, 253\frac{59}{256}\right)$

at $(5,0)$ $y'' = 0$ so need $x = 5 - \varepsilon$ (4.9) $y' = -0.206$

$x = 5$ $y' = 0$

$x = 5 + \varepsilon$ (5.1) $y' = -0.214$

but probably a horizontal point of inflexion

not max nor min

at $\left(-\frac{1}{4}, 253\frac{59}{256}\right)$ $y'' < 0$ MAX stat point

4 ⓐ continued (5,0) is not a max nor min stat point

An inflexion? $x = 5 - \varepsilon$ (4.9) $y'' > 0$

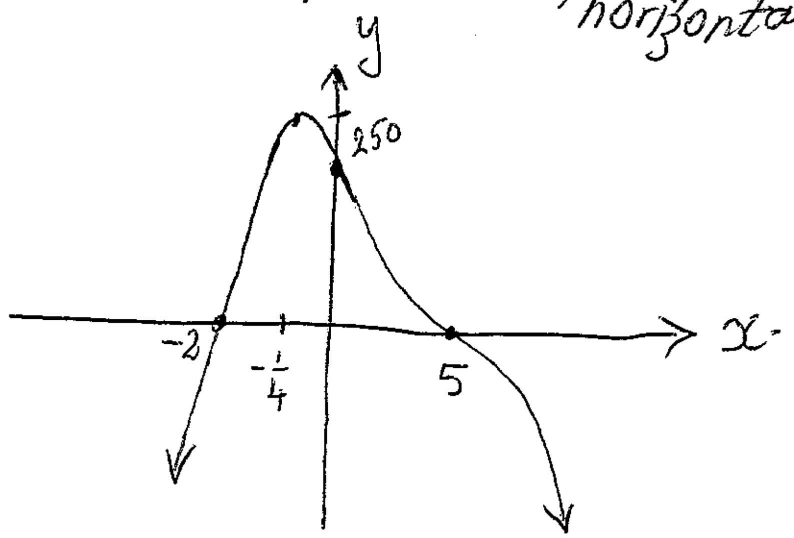
$x = 5$ $y'' = 0$

$x = 5 + \varepsilon$ (5.1) $y'' < 0$

(5,0) is a point of inflexion horizontal

3

ⓑ



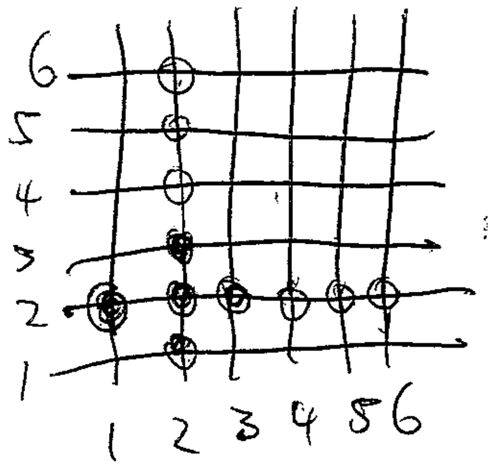
1

5

2,1
2,2
2,3
1,2
2,3

2

$\frac{5}{11}$



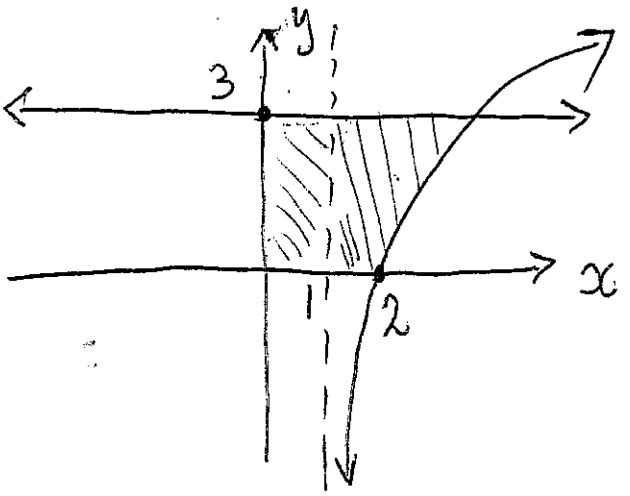
Section D

/15

(1) $y = \ln(x-1)$

so $x - 1 > 0$

$x > 1$



$\log_e(x-1) = y$

$e^y = x - 1$

$e^y + 1 = x$

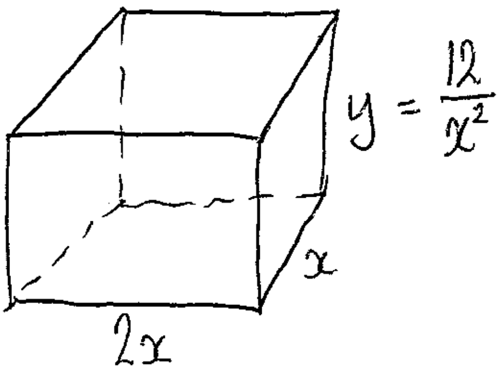
$$A = \int_0^3 (e^y + 1)\, dy$$

$$= \left(e^y + y\right)\Big]_0^3$$

$$= (e^3 + 3) - (e^0 + 0)$$

$$= e^3 + 3 - 1 = (e^3 + 2) \quad u^2$$

(2)



(3)

$V = 2x^2 y = 24$

$x^2 y = 12$

$y = \frac{12}{x^2}$

All edges $\cancel{2x} + \cancel{x} + \cancel{2x} + \cancel{x} + \cancel{2x} + \cancel{x} + \cancel{2x} + \cancel{x} + 4 \times \frac{12}{x^2}$

$E = 12x + \frac{48}{x^2}$

1) 2 continued

$E = 12x + 48x^{-2}$

$E' = 12 - 96x^{-3}$

$E'' = 288x^{-4}$

let $E' = 0$, $12 - \frac{96}{x^3} = 0$

$12 = \frac{96}{x^3}$

$12x^3 = 96$

$x^3 = 8$

$x = 2$

when $x = 2$, $E'' = \frac{288}{x^4} > 0$ minimum

dimensions are 4cm, 2cm, 3 cm

4

③ (a) $A_1 = 500000 + 500000 \times \frac{8}{100} - m$

$= 500000\left(1 + \frac{8}{100}\right) - m$

$= \left[500000(1.08) - m\right]$

$A_2 = \left[500000(1.08) - m\right] + \left[500000(1.08) - m\right] \times \frac{8}{100} - m$

$= \left[500000(1.08) - m\right]\left[1 + \frac{8}{100}\right] - m$

$= \left[500000(1.08) - m\right]\left[1.08\right] - m$

$= 500000(1.08)^2 - m \times 1.08 - m$ ②

(b) above $500000(1.08)^2 - m(1 + 1.08^1)$

account last 20 years.

$A_{20} = 500000(1.08)^{20} - m(1 + 1.08 + \cdots + 1.08^{19})$

$$m = \frac{500000(1.08)^{20}}{1 + 1.08 + \cdots + 1.08^{19}}$$

so denominator $1 + 1.08 + \cdots + 1.08^{19}$

Use $S_{20} = \frac{rL - a}{r - 1} = \frac{1.08 \times 1.08^{19} - 1}{1.08 - 1}$

$= \frac{1.08^{20} - 1}{.08} = 45.7619643$

$m = \$50926.10$ ②

D 3

c $A_n = 500000(1.08)^n - 75000(1+1.08+\dots+1.08^{n-1})$

So $(1+1.08+\dots+1.08^{n-1})$

$$S_n = \frac{rl-a}{r-1} = \frac{1.08\times1.08^{n-1}-1}{1.08-1} = \frac{1.08^n-1}{0.08}$$

$$A_n = 500000(1.08)^n - 75000\frac{(1.08^n-1)}{0.08}$$

$$= 500000(1.08)^n - 937500(1.08)^n + 937500$$

$$= -437500(1.08)^n + 937500$$

Let $A_n = 0$

$$437\cancel{5}\cancel{00}(1.08)^n = 937\cancel{5}\cancel{00}$$

$$1.08^n = 2.142857143\dots$$

$$n\log_{10}1.08 = \log_{10}2.142857143$$

$$n = \frac{\log_{10}2.142857143}{\log_{10}1.08}$$

$$n = 9.9029\dots$$

Brendan receives 9 full payments.

2

1) 3)

d)

$$A_9 = 500000(1.08)^9 - 75000\frac{(1.08^9-1)}{0.08}$$

$= \$999502.31 - \$936566.84$

$= \$62,935.47$

2)