SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2002

FORM VI MATHEMATICS

Time allowed: 2 hours

Exam date: 15th May 2002

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new 4-page examination booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each booklet.

Checklist:

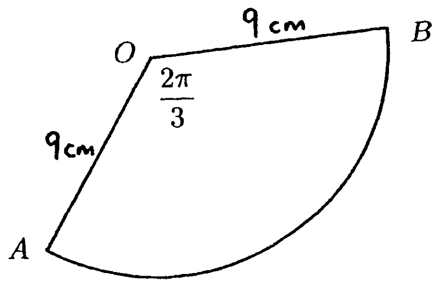
SGS Examination booklets required.

QUESTION ONE (Start a new examination booklet)

Marks

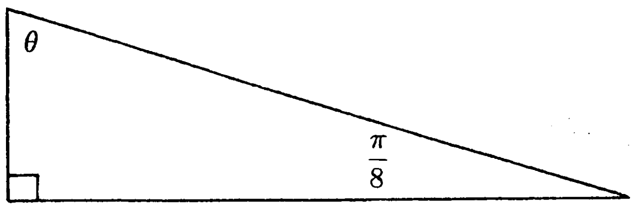
(a) Express $\frac{2\pi}{3}$ radians in degrees. **1**

(b) **1**



In the diagram above, AOB is a sector of the circle subtending an angle $\frac{2\pi}{3}$ at the centre O. Find the length of the arc AB.

(c) **1**



In the diagram above, find θ in radians.

(d) Find $\log_e 3$, correct to four significant figures. **1**

(e) Simplify:

(i) $\log_e \sqrt{e}$, **1**

(ii) $1 + \log_e \frac{10}{e}$. **1**

(f) Write down the range of $y = e^x$. **1**

(g) Complete the following sentence: **1**
"When $0 < x < 1$, the values of the function $f(x) = \log_e x$ are always ...."

(h) Write down the radius and the centre of the circle $(x-3)^2 + y^2 = 5$. **2**

(i) A variable point $P(x, y)$ moves so that it is equidistant from the point $(3, 0)$ and the line $x = -3$. Write down the equation of the locus of P. **1**

(j) Sketch the parabola $x^2 = 8y$, clearly showing the coordinates of the vertex and the focus, and the equation of the directrix. **3**

QUESTION TWO (Start a new examination booklet)

(a) Differentiate each of the following with respect to x:

Marks

(i) $y = \dfrac{2}{x^3}$, $\boxed{1}$

(ii) $y = \log_e 3x$, $\boxed{1}$

(iii) $y = \dfrac{1}{e^x}$, $\boxed{1}$

(iv) $y = (e^x - 2)^4$. $\boxed{1}$

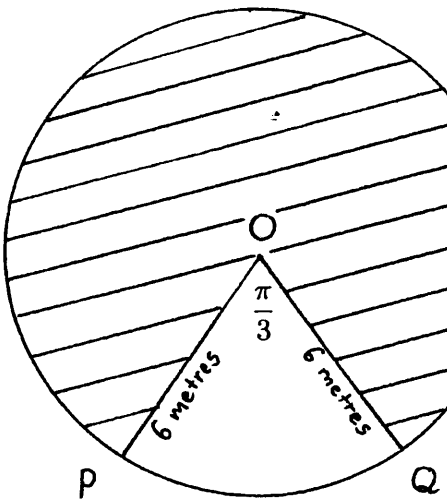
(b) Simplify $2\log 2 - \log 3 + \log 12$, expressing your answer in the form $a \log b$. $\boxed{1}$

(c) Solve $2\log_m 4 + \log_m 9 = 2$, where $m > 1$. $\boxed{2}$

(d) (i) Find $\tan \dfrac{3\pi}{4}$. $\boxed{1}$

(ii) Solve $\cos x = \dfrac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$. $\boxed{2}$

(e)



The circle above has centre O and radius 6 metres.

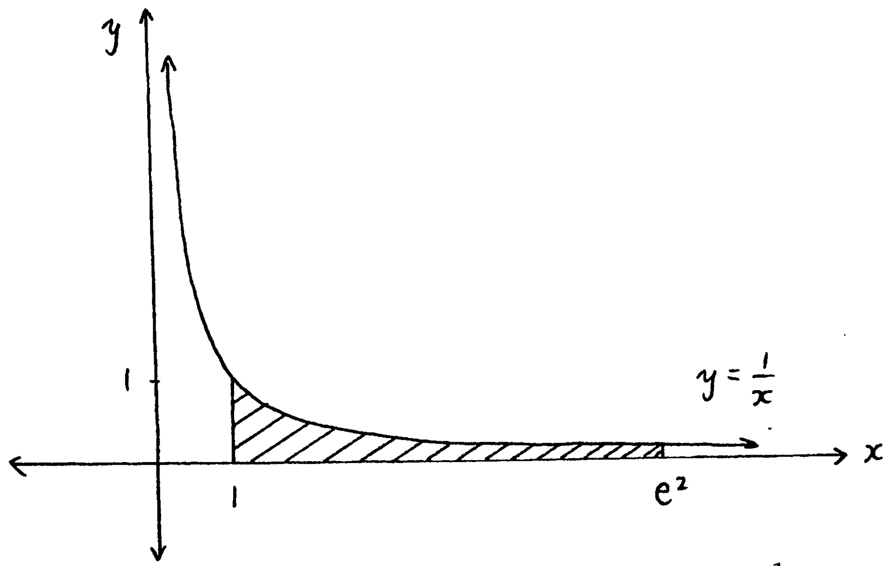
(i) Find the exact area of the major sector POQ. $\boxed{2}$

(ii) Find the exact area of the minor segment cut off by the chord PQ. $\boxed{2}$

QUESTION THREE (Start a new examination booklet)

Marks

(a) $\boxed{3}$



The diagram above shows the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = e^2$. Find the area of the shaded region.

(b) Consider the curve $y = -x^3 + 3x^2 + 9x - 11$.

(i) Show that $\dfrac{dy}{dx} = -3(x-3)(x+1)$. $\boxed{1}$

(ii) Find the coordinates of any stationary points and determine their nature. $\boxed{4}$

(iii) Find the coordinates of any points of inflexion. $\boxed{3}$

(iv) Sketch the curve, clearly showing the y-intercept and all stationary points and inflexions. $\boxed{2}$

(v) For what values of x is the curve increasing? $\boxed{1}$

QUESTION FOUR (Start a new examination booklet)

(a) Find the indefinite integrals:

Marks

(i) $\displaystyle\int e^{\frac{x}{2}}\,dx$, $\boxed{1}$

(ii) $\displaystyle\int \frac{2x^2 - 4x}{x}\,dx$, $\boxed{2}$

(iii) $\displaystyle\int (2x-1)^{10}\,dx$. $\boxed{1}$

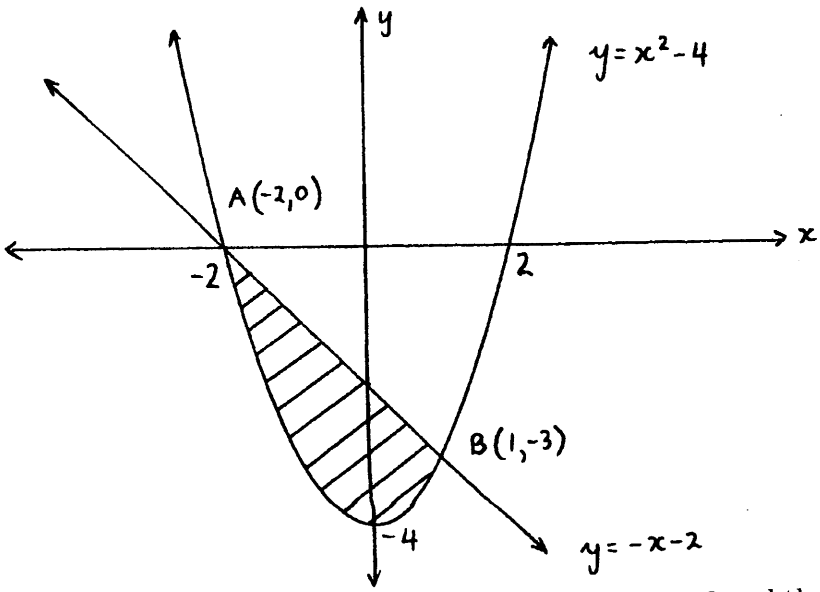
(b) Find the equation(s) of the locus of the point $P(x, y)$ such that the distance of P from the x-axis is always four times the distance of P from the y-axis. $\boxed{2}$

(c) Consider the parabola with equation $4y = -x^2 + 4x - 16$.

(i) By completing the square, write the equation of the parabola in the form $(x-h)^2 = -4a(y-k)$. $\boxed{1}$

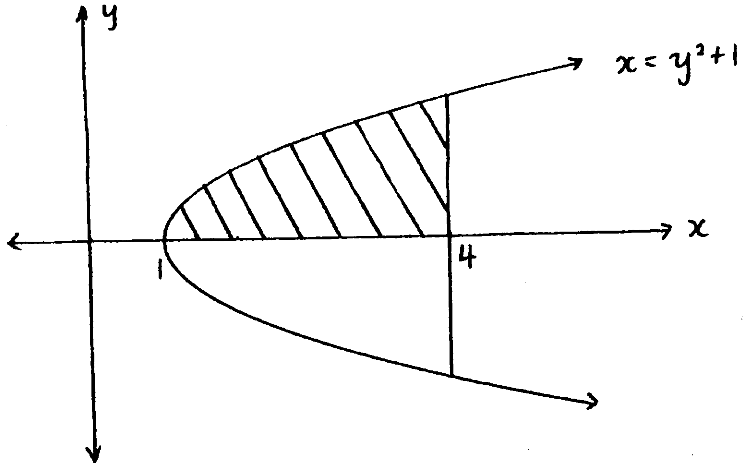
(ii) Find the coordinates of the focus. $\boxed{1}$

(d) $\boxed{3}$



The diagram above shows the region bounded by the line $y = -x - 2$ and the parabola $y = x^2 - 4$. The two functions intersect at $A(-2, 0)$ and $B(1, -3)$. Find the area of the shaded region.

(e) $\boxed{3}$

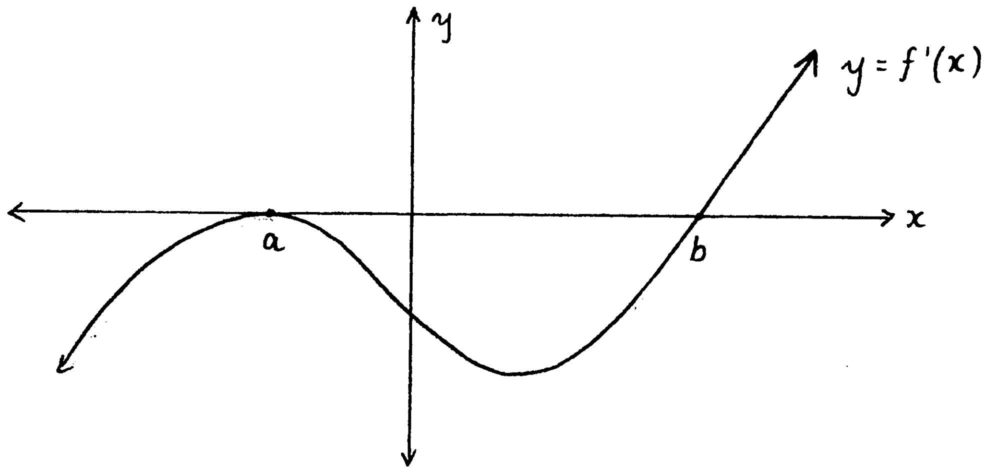


The diagram above shows the region above the x-axis bounded by the x-axis, the parabola $x = y^2 + 1$ and the line $x = 4$. Find the volume formed when the shaded region is rotated about the x-axis.

Exam continues overleaf ...

QUESTION FIVE (Start a new examination booklet)

(a)



The diagram above shows the graph of the derivative $y = f'(x)$ of a function $y = f(x)$. **Marks**

(i) Write down the nature of the two stationary points on the graph of $y = f(x)$ at $x = a$ and at $x = b$. **2**

(ii) Given that $f(0) = 0$, sketch a possible graph of $y = f(x)$. **2**

(b) (i) Show that the point $P(2, e)$ lies on the curve $y = e^{\frac{x}{2}}$. **1**

(ii) Show the gradient of the tangent at P is $\dfrac{e}{2}$. **1**

(iii) Show that the equation of the tangent to $y = e^{\frac{x}{2}}$ at P passes through the origin. **2**

(iv) Show that the equation of the normal at P has equation $2x + ey - e^2 - 4 = 0$. **1**

(v) Find the point Q where the normal crosses the y-axis. **1**

(vi) Find the area of $\triangle POQ$. **1**

(c) Consider the curve $y = \dfrac{x^2}{e^x}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x(2-x)}{e^x}$. **1**

(ii) Hence find the coordinates of the points on the curve where the tangent is horizontal. **2**

QUESTION SIX (Start a new examination booklet)

Marks

(a) The graph of $y = f(x)$ is known to have a maximum turning point at $(-1, 3)$. Given that $f''(x) = 12x + 4$, find the equation of the function $y = f(x)$. **4**

(b) (i) Show that $\frac{d}{dx}(x \log_e x) = \log_e x + 1$. **1**

(ii) Hence find the exact value of A, where **3**

$$A = \int_1^4 \log_e x \, dx.$$

(iii) Sketch the graph of $y = \log_e x$ and shade the region represented by A. **1**

(iv) Copy and complete the following table for the curve $y = \log_e x$, giving exact values for y: **1**

x	1	2	3	4
y				

(v) Use the trapezoidal rule with four function values to show that $A \doteqdot \log_e 12$. **2**

(vi) Find the second derivative of $y = \log_e x$ and hence explain why the approximation of A in part (v) is less than the exact area found in part (ii). **2**

QUESTION SEVEN (Start a new examination booklet)

Marks

(a) (i) Find $\lim_{x \to \infty} \left(1 - e^{2-x}\right)$. **1**

(ii) Sketch $y = 1 - e^{2-x}$, showing the horizontal asymptote and any intercepts with the coordinate axes. **2**

(b) Consider the function $f(x) = e^x + e^{-x}$.

(i) Show that $f(x)$ is an even function. **1**

(ii) Find $f(1)$, correct to three significant figures. **1**

(iii) Show that $f''(x) = f(x)$. **1**

(iv) Find any stationary points on $y = f(x)$ and determine each point's nature. **2**

(v) Explain why $y = f(x)$ is concave up for all real x. **1**

(vi) Sketch $y = f(x)$, clearly showing any intercepts with the coordinate axes and one other point on the curve. **1**

(c) (i) Write down the domain of $y = \ln(x - 2)$. **1**

(ii) The region bounded by $y = \ln(x - 2)$, the coordinate axes and the line $y = \ln 3$ is rotated about the y-axis. Show that the volume of the solid formed is $4\pi(3 + \ln 3)$ cubic units. **3**

TCW

QUESTION 1

(a) 120°

(b) $l = r\theta$

arc $AB = 9 \times \frac{2\pi}{3}$

$= 6\pi$ cm

(c) $\theta = \frac{\pi}{2} - \frac{\pi}{8}$

$= \frac{3\pi}{8}$

(d) $\log_e 3 = 1.099$ (4 sig figs)

(e) (i) $\log_e \sqrt{e} = \frac{1}{2}$

(ii) $1 + \log_e \frac{10}{e} = 1 + \log_e 10 - \log_e e$

$= \log_e 10$

(f) $y > 0$

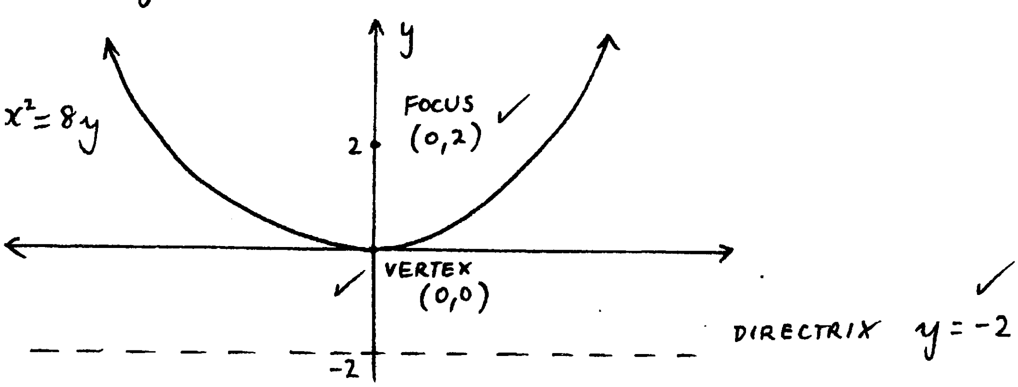
(g) negative

(h) radius $= \sqrt{5}$ units

centre $(3, 0)$

(i) $y^2 = 12x$

(j)



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QUESTION 2

(a) (i) $y = 2x^{-3}$

$y' = -6x^{-4}$

$= -\dfrac{6}{x^4}$

(ii) $y = \log_e 3x$

$y' = \dfrac{3}{3x}$

$= \dfrac{1}{x}$

(iii) $y = e^{-x}$

$y' = -e^{-x}$

(iv) $y = (e^x - 2)^4$

$y' = 4e^x(e^x - 2)^3$

(b) $2\log 2 - \log 3 + \log 12 = \log 4 - \log 3 + \log 4 + \log 3$

$= 2\log 4$

(c) $2\log_m 4 + \log_m 9 = 2$, $m > 1$

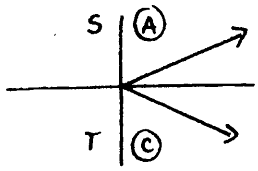
$\log_m 144 = 2$

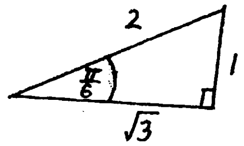
$m^2 = 144$

$m = 12$

(d) (i) $\tan \dfrac{3\pi}{4} = -1$

(ii) $\cos x = \dfrac{\sqrt{3}}{2}$





$x = \dfrac{\pi}{6}$ or $\dfrac{11\pi}{6}$

(e) (i) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 36 \times \dfrac{5\pi}{3}$

$= 30\pi \quad m^2$

(ii) $A = \frac{1}{2} r^2(\theta - \sin\theta)$

$= 18\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= 6\pi - 9\sqrt{3} \quad m^2$

[NO PENALTY FOR UNITS]

/14

QUESTION 3

(a) $A = \int_1^{e^2} \frac{1}{x}\, dx$ ✓

$= \left[\log_e x\right]_1^{e^2}$ ✓

$= \log_e e^2 - \log_e 1$

$= 2$ units^2 ✓

(b) (i) $y = -x^3 + 3x^2 + 9x - 11$

$\frac{dy}{dx} = -3x^2 + 6x + 9$

$= -3(x^2 - 2x - 3)$ ✓

$= -3(x-3)(x+1)$

(ii) stationary points when $\frac{dy}{dx} = 0$

ie $x = 3,\ y = 16$ ✓

$x = -1,\ y = -16$. ✓

x	-2	-1	0	3	4
y'	-15	0	9	0	-15
slope:	\	—	/	—	\

✓

so $(3,16)$ is a maximum turning point and $(-1,-16)$ is a minimum turning point. ✓

(iii) $\frac{d^2y}{dx^2} = -6x + 6$ ✓

when $\frac{d^2y}{dx^2} = 0$

$6x = 6$

$x = 1,\ y = 0$ ✓

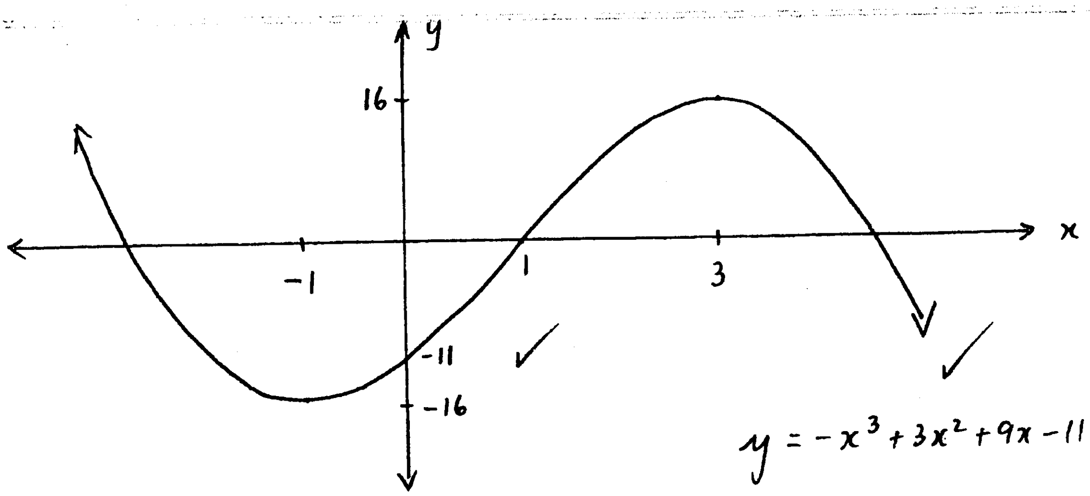
Test for a change in concavity:

x	0	1	2
y''	6	0	-6
	∪		∩

✓

so $(1,0)$ is a point of inflexion

(iv)



$y = -x^3 + 3x^2 + 9x - 11$

when $x = 0$, $y = -11$

(v) increasing when $\frac{dy}{dx} > 0$

$-3(x-3)(x+1) > 0$

$-1 < x < 3$

14

QUESTION 4

(a) (i) $\int e^{\frac{x}{2}}\,dx = 2e^{\frac{x}{2}} + c$ ✓

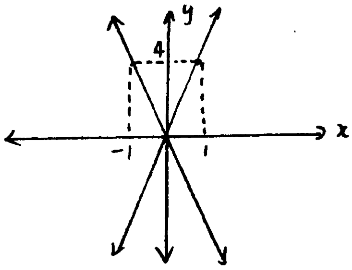
(ii) $\int \frac{2x^2 - 4x}{x}\,dx = \int 2x - 4\,dx$ ✓

$= x^2 - 4x + c$ ✓

(iii) $\int (2x-1)^{10}\,dx = \frac{1}{22}(2x-1)^{11} + c$ ✓

(b)

$y = 4x$ and $y = -4x$



✓✓.

(c) (i)

$$4y = -x^2 + 4x - 16$$
$$x^2 - 4x = -4y - 16$$
$$x^2 - 4x + 4 = -4y - 12$$
$$(x-2)^2 = -4(y+3)$$ ✓

(ii) Focus $= (2,-4)$ ✓

(d)

$$\text{Area} = \int_{-2}^{1} -x - 2 - x^2 + 4\,dx$$
$$= \int_{-2}^{1} -x^2 - x + 2\,dx$$ ✓
$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$ ✓
$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4$$
$$= 4\tfrac{1}{2} \text{ units}^2$$ ✓

(e)

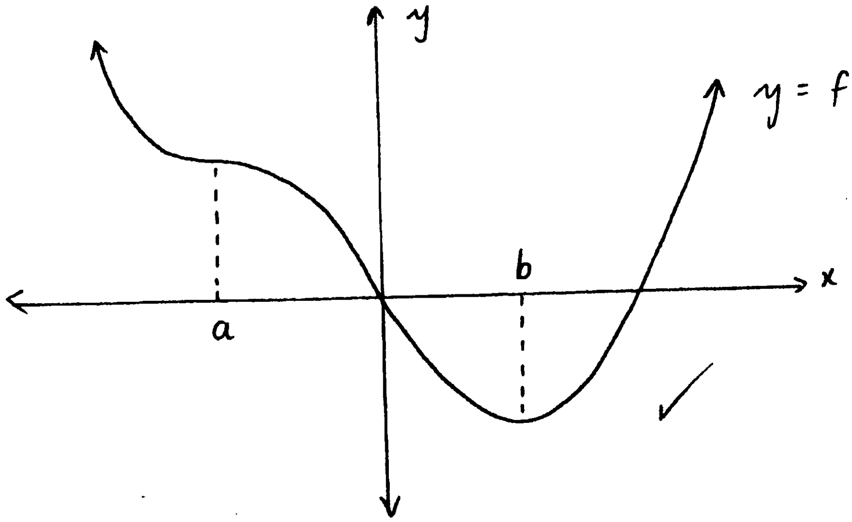
$$V = \pi \int_{1}^{4} (x-1)\,dx$$ ✓
$$= \pi\left[\frac{x^2}{2} - x\right]_{1}^{4}$$ ✓
$$= \pi\left(8 - 4 - \frac{1}{2} + 1\right)$$
$$= \frac{9\pi}{2} \text{ units}^3$$ ✓

$\frac{}{14}$

QUESTION 5

(a) (i) Stationary point of inflexion at $x = a$. ✓
Minimum turning point at $x = b$. ✓

(ii)



✓

$f(0) = 0$ ✓

(b) (i) $y = e^{\frac{x}{2}}$

when $x = 2$, $y = e^{\frac{2}{2}}$

$= e$ ✓

so $(2, e)$ lies on $y = e^{\frac{x}{2}}$

(ii) $y' = \frac{1}{2} e^{\frac{x}{2}}$

when $x = 2$, $y' = \frac{1}{2} e^{\frac{2}{2}}$

$= \frac{e}{2}$ ✓

so $\frac{e}{2}$ is the gradient of the tangent at $P(2, e)$

(iii) tangent at $P(2, e)$: $y - e = \frac{e}{2}(x - 2)$

$2y - 2e = ex - 2e$

$y = \frac{e}{2} x$ ✓

when $x = 0$, $y = \frac{e}{2} \times 0$

$= 0$ ✓

so the tangent passes through the origin.

(iv) gradient of normal at $P = -\frac{2}{e}$

normal at P: $y - e = -\frac{2}{e}(x-2)$

$ey - e^2 = -2x + 4$ ✓

$2x + ey - e^2 - 4 = 0$ as required

(v) when $x = 0$, $ey = 4 + e^2$

$y = e + \frac{4}{e}$

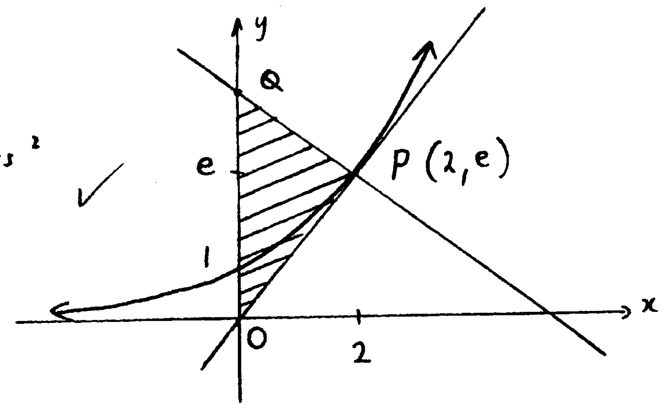
so Q is $\left(0, e + \frac{4}{e}\right)$ ✓

(vi) Area $= \frac{1}{2} \times 2 \times \left(e + \frac{4}{e}\right)$

$= e + \frac{4}{e}$ units^2 ✓

OR

$\frac{4+e^2}{e}$ units^2



(c) (i) $y = \frac{x^2}{e^x}$

$\frac{dy}{dx} = \frac{2x \times e^x - e^x \times x^2}{(e^x)^2}$ by the quotient rule

$= \frac{x\cancel{e^x}(2-x)}{e^{\cancel{2x}}}$

$= \frac{x(2-x)}{e^x}$ ✓

(ii) For a horizontal tangent, $\frac{dy}{dx} = 0$

$x(2-x) = 0$

$x = 0$ or 2 ✓

$y = 0$ or $\frac{4}{e^2}$ ✓

so the tangent is horizontal at $(0,0)$ and $\left(2, \frac{4}{e^2}\right)$.

$\frac{}{14}$

QUESTION 6

(a)

$$f''(x) = 12x + 4$$
$$f'(x) = 6x^2 + 4x + c_1$$
$$0 = 6 - 4 + c_1 \quad \text{since } f'(-1) = 0$$
$$c_1 = -2$$
$$f'(x) = 6x^2 + 4x - 2$$
$$f(x) = 2x^3 + 2x^2 - 2x + c_2$$
$$3 = -2 + 2 + 2 + c_2 \quad \text{since } f(-1) = 3$$
$$c_2 = 1$$
so
$$f(x) = 2x^3 + 2x^2 - 2x + 1$$

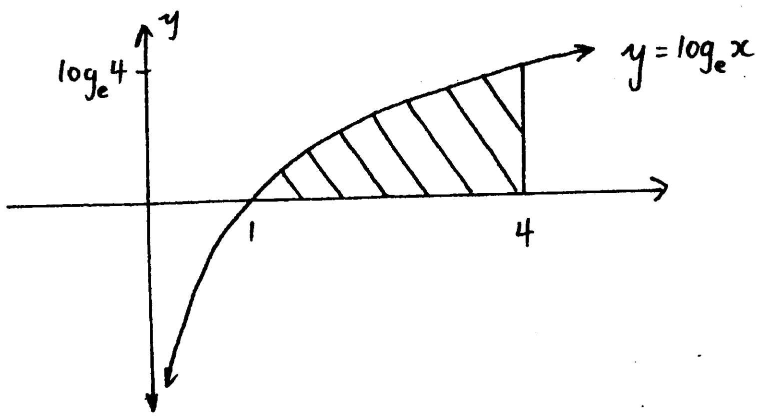
(b) (i)

$$\frac{d}{dx}(x\log_e x) = 1 \times \log_e x + \frac{1}{x} \times x \quad \text{by the product rule}$$
$$= \log_e x + 1$$

(ii)

$$\int_1^4 \log_e x + 1 \, dx = \left[x\log_e x\right]_1^4$$
$$\int_1^4 \log_e x \, dx + \int_1^4 1 \, dx = \left[x\log_e x\right]_1^4$$
$$\int_1^4 \log_e x \, dx = 4\log_e 4 - 0 - \left[x\right]_1^4$$
$$= 4\log_e 4 - 3$$
$$\therefore \quad A = 8\log_e 2 - 3$$

(iii)



(iv)

x	1	2	3	4
y	0	$\log 2$	$\log 3$	$\log 4$

✓

(v)

$$A \doteqdot \tfrac{1}{2}(0 + \log 2) + \tfrac{1}{2}(\log 2 + \log 3) + \tfrac{1}{2}(\log 3 + \log 4)$$ ✓

$$\doteqdot \tfrac{1}{2}(2\log 2 + 2\log 3 + \log 4)$$

$$\doteqdot \log 4 + \log 3$$

$$\doteqdot \log 12$$ ✓

(vi)

$$y = \log_e x \quad , \quad x > 0$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$ ✓

$$\frac{d^2y}{dx^2} < 0 \quad \text{for all } x$$

So $y = \log_e x$ is always concave down. Then the three trapezia applied in the trapezoidal rule all lie under the curve. Hence the approximation for A will be less than the exact value for A.

✓

14

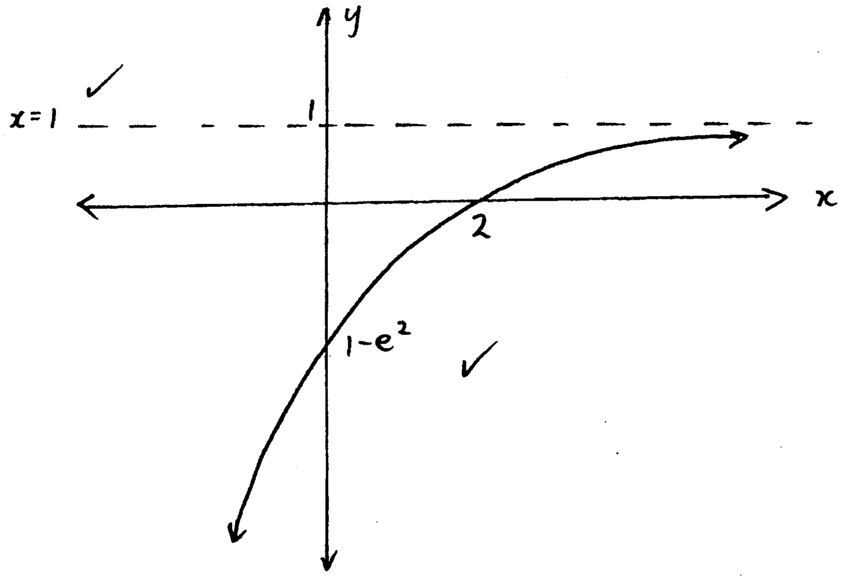
QUESTION 7

(a) (i) $\lim_{x \to \infty} (1 - e^{2-x}) = 1$ ✓

(ii) when $x = 0$, $y = 1 - e^2 \Rightarrow (0, 1-e^2)$

when $y = 0$, $e^{2-x} = 1$

$x = 2 \Rightarrow (2, 0)$



(b) (i) $f(x) = e^x + e^{-x}$

$f(-x) = e^{-x} + e^{-(-x)}$

$= e^{-x} + e^x$ ✓

$= f(x)$

so $f(x)$ is an even function.

(ii) $f(1) = e + \frac{1}{e}$

$\doteqdot 3.09$ (3 sig figs) ✓

(iii) $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

$f''(x) = e^x + e^{-x}$

$= f(x)$ ✓

(iv)

$$f'(x) = 0$$
$$e^x - e^{-x} = 0$$
$$e^x = \frac{1}{e^x}$$
$$e^{2x} = 1$$
$$2x = 0$$
$$x = 0, \quad y = 2$$

✓

$$f''(0) = 1 + 1 = 2 > 0$$

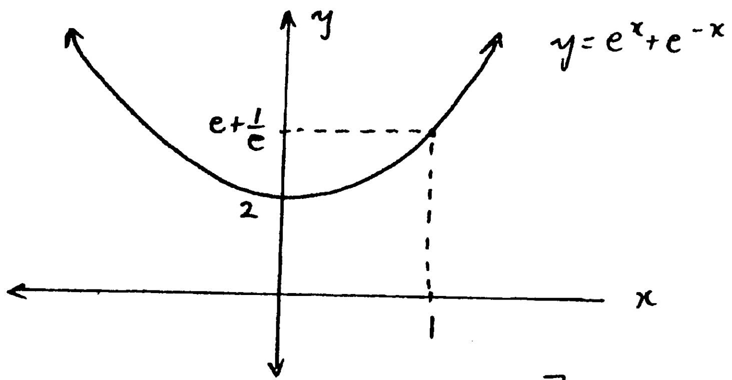
so $(0,2)$ is a minimum turning point. ✓

(v) Both e^x and e^{-x} are positive for all real x.

$$f''(x) = e^x + e^{-x} > 0 \quad \text{for all real } x$$

so $y = f(x)$ is concave up for all real x. ✓

(vi)



✓

[NOTE: MUST SHOW ONE OTHER POINT]

(c) (i) Domain: $x > 2$ ✓

(ii)
$$V = \pi \int_0^{\ln 3} (e^y + 2)^2 \, dy$$ ✓

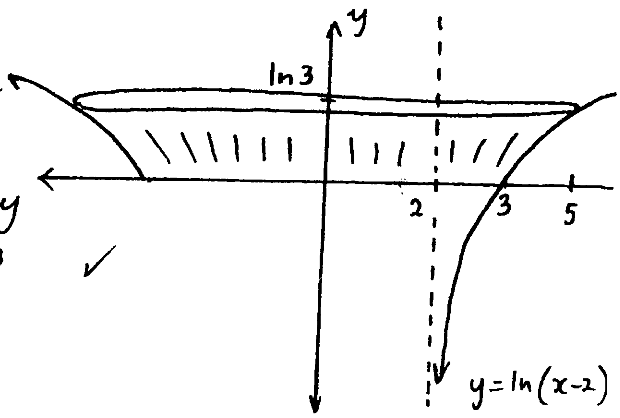
$$= \pi \int_0^{\ln 3} e^{2y} + 4e^y + 4 \quad dy$$

$$= \pi \left[\tfrac{1}{2} e^{2y} + 4e^y + 4y \right]_0^{\ln 3}$$ ✓

$$= \pi \left(\tfrac{9}{2} + 12 + 4\ln 3 - \tfrac{1}{2} - 4 - 0 \right)$$

$$= \pi \left(12 + 4 \ln 3 \right)$$

$$= 4\pi \left(3 + \ln 3 \right) \quad \text{units}^3$$ ✓



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