FORM VI MATHEMATICS

Time allowed: 2 hours

Exam date: 12th May 2003

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

The writing booklets will be collected in one bundle.
Start each question in a new writing booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each booklet.

Checklist:

SGS Writing Booklets required — 8 per boy.
Candidature 123 boys.

QUESTION ONE (Start a new writing booklet)

Marks

(a) Use your calculator to evaluate each of the following, correct to three decimal places:

(i) e^2 [1]

(ii) $\log_e 14$ [1]

(iii) $\sin 1{\cdot}5$ [1]

(b) Write down the exact value of $\cos \frac{5\pi}{6}$. [2]

(c) Differentiate the following:

(i) $y = 2x^{-1}$ [1]

(ii) $y = 4e^{2x}$ [1]

(iii) $y = 4 \log_e x$ [1]

(d) Find $\int e^{3x}\, dx$. [1]

(e) Let $\log_a 2 = x$ and $\log_a 5 = y$. Find an expression for $\log_a 50$ in terms of x and y. [2]

(f) Write down the period of $y = 5 \tan 3x$. [1]

QUESTION TWO (Start a new writing booklet)

Marks

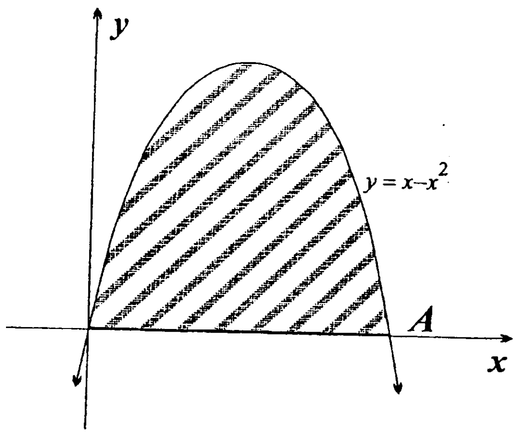
(a) A point moves so that its distance from the line $y = -2$ is equal to its distance from the point $(0, 6)$. The locus is a parabola of the form $(x-h)^2 = 4a(y-k)$.

(i) Write down the coordinates of the vertex of this parabola. **1**

(ii) Write down the focal length. **1**

(iii) Write down the equation of this parabola. **1**

(b)



The diagram above shows the graph of the function $y = x - x^2$. It crosses the x-axis at A and at the origin.

(i) Show that the x-coordinate of the point A is $x = 1$. **1**

(ii) Find the area of the shaded region contained by the curve $y = x - x^2$ and the x-axis. **3**

(iii) Find the volume obtained when this region is rotated about the x-axis. Leave your answer in terms of π. **3**

(c) The table below shows three values of the function $f(x)$. **2**

x	2	4	6
$f(x)$	8	15	20

Use these three function values to approximate the value of $\int_2^6 f(x)\,dx$ by the trapezoidal rule.

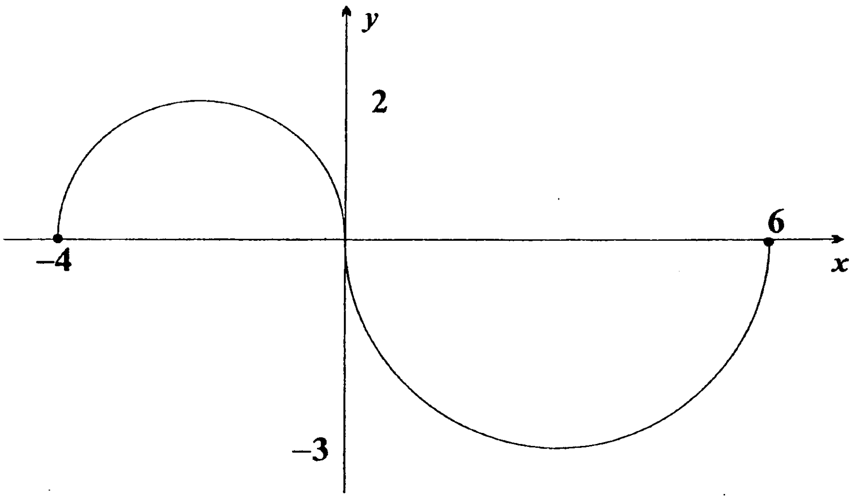
QUESTION THREE (Start a new writing booklet)

(a) Differentiate: Marks

(i) $f(x) = \cos(2x+1)$ 2

(ii) $f(x) = \dfrac{e^x}{\sin x}$ 3

(b) 2



The graph of the function $f(x)$ is shown in the diagram above. It consists of two semicircles.
Evaluate $\displaystyle\int_{-4}^{6} f(x)\,dx$.

(c) (i) Express 20° in radians as a multiple of π. 1

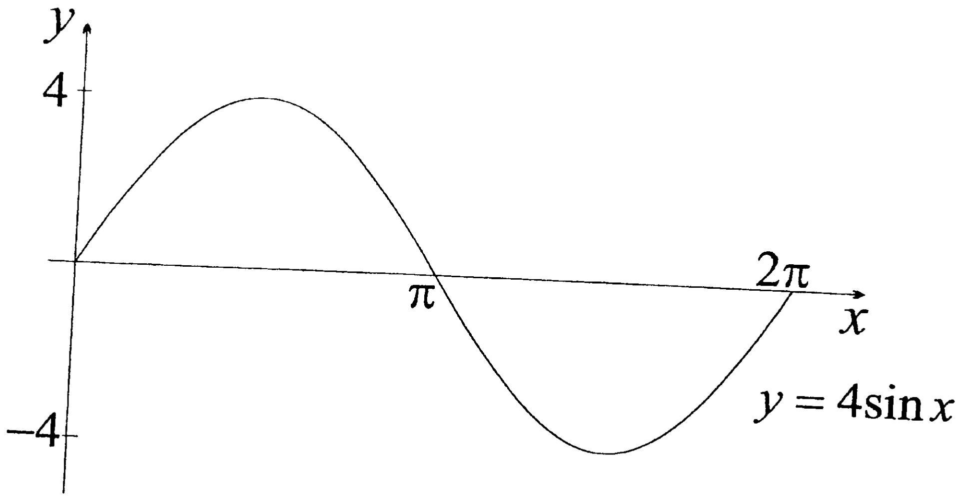
(ii) A circle has a radius of 5 cm. Find the length of the arc that subtends an angle of 20° at the centre. 1

(d) (i) If $f(x) = 27x - x^3$, find $f'(x)$. 1

(ii) For what values of x is the function $f(x) = 27x - x^3$ increasing? 2

QUESTION FOUR (Start a new writing booklet)

(a)



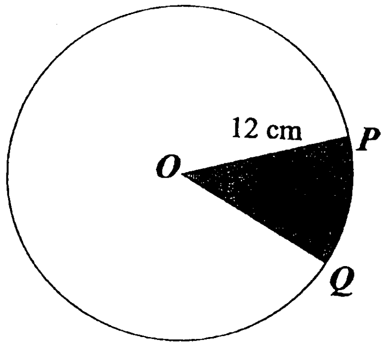
Above is a diagram of the function $y = 4\sin x$, for $0 \leq x \leq 2\pi$.

Marks

(i) Copy this sketch into your writing booklet and on the same diagram sketch the line $y = -2$. How many solutions does the equation $4\sin x = -2$ have for the domain $0 \leq x \leq 2\pi$? **1**

(ii) Solve $4\sin x = -2$, for $0 \leq x \leq 2\pi$. **3**

(b)



The above circle has centre O and radius 12 cm. The area of the shaded sector is $18\pi\ \text{cm}^2$. Let $\theta = \angle POQ$.

(i) Show that $\theta = \frac{\pi}{4}$. **1**

(ii) Find the exact area of the minor segment cut off by the chord PQ. **2**

(c) (i) Write down the period and amplitude of $y = 3\cos 2x$. **2**

(ii) Sketch $y = 3\cos 2x$, for $-\pi \leq x \leq \pi$. Show clearly the intercepts with the axes. **3**

QUESTION FIVE (Start a new writing booklet)

(a) Consider the function $f(x) = \sqrt{x+2}$.

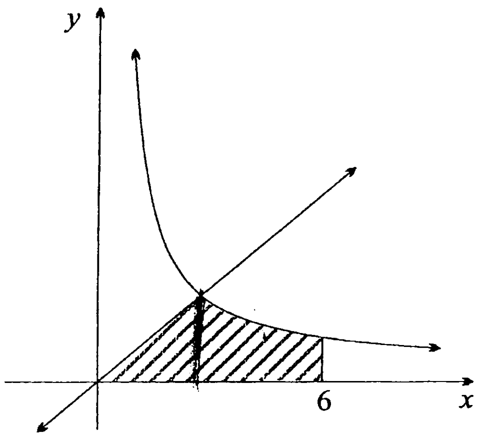
Marks

(i) Show that $f'(x) = \dfrac{1}{2\sqrt{x+2}}$. **1**

(ii) Show that the gradient of this function at the point $(7,3)$ is $\frac{1}{6}$. **1**

(iii) Find the equation of the normal to the curve $f(x) = \sqrt{x+2}$ at the point $(7,3)$. **2**

(b)



Above is a sketch of the functions $y = x$ and $y = \dfrac{9}{x}$.
The region between the functions and the x-axis for $0 \le x \le 6$ is shaded.

(i) Show that the point of intersection of these functions in the first quadrant is the point $(3,3)$. **1**

(ii) Find the exact area of the shaded region. **4**

(c) By completing squares, find the centre and radius of the circle **3**

$$x^2 - 6x + y^2 + 8y = 0.$$

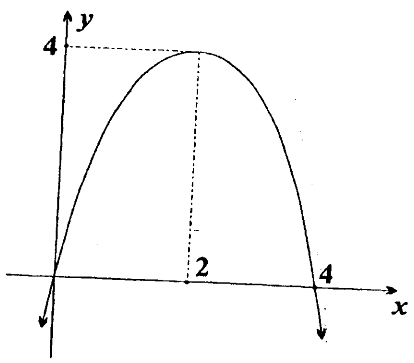
QUESTION SIX (Start a new writing booklet)

Marks

(a) Solve $\log_e(7x - 12) = 2\log_e x$. **3**

(b) If $f(x) = 4\sin x + \cos x + 1$, find the exact value of $f'(x)$ when $x = \frac{\pi}{4}$. **3**

(c) **2**



The diagram above shows the graph of the curve $y = f'(x)$. For what value of x does $f(x)$ have a maximum turning point? Justify your answer.

(d) (i) Find $f(x)$, given that $f'(x) = \dfrac{2x-5}{x^2 - 5x + 6}$ and $f(1) = 2 + \log_e 2$. **2**

(ii) Hence show that $f(-1) = 2 + 2\log_e 2 + \log_e 3$. **2**

QUESTION SEVEN (Start a new writing booklet)

Marks

(a) Find the minimum value of $y = 1 + \sqrt{3}\sin x$. **1**

(b) Consider the curve given by

$$y = x^3 - 6x^2 + 9x + 4.$$

(i) Show that $\dfrac{dy}{dx} = 3(x-3)(x-1)$. **1**

(ii) Find the coordinates of the two stationary points. **2**

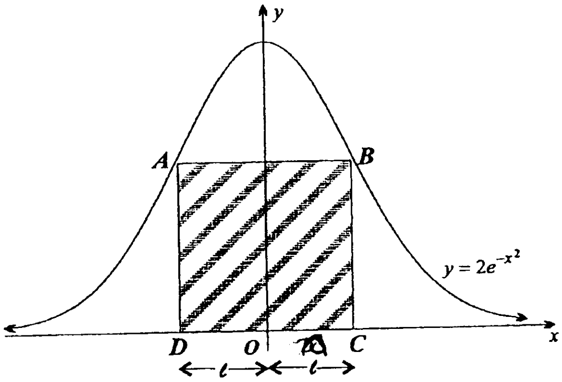
(iii) Determine the nature of the stationary points. **2**

(iv) Find the coordinates of the point of inflection. (Remember to justify that it is a point of inflection.) **2**

(v) Using half a page, sketch this function, showing clearly all important features (You need not find any x-intercepts.) **4**

QUESTION EIGHT (Start a new writing booklet)

(a)



The diagram above shows the curve $y = 2e^{-x^2}$. Let D and C lie on the x-axis with $OC = OD = \ell$. Complete the rectangle $ABCD$ as shown.

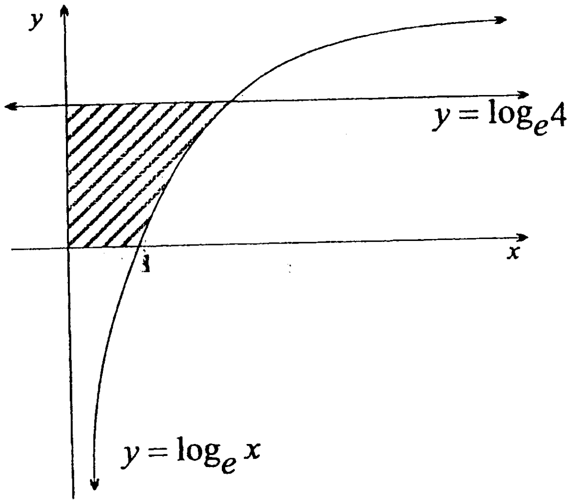
Marks

(i) Explain why the length of BC is $2e^{-\ell^2}$. **1**

(ii) Show that the area of the rectangle is $A = 4\ell e^{-\ell^2}$. **1**

(iii) Find the value of ℓ for which $ABCD$ has maximum area. **3**

(b)



The diagram above shows the curve $y = \log_e x$ and the line $y = \log_e 4$.

(i) Solve the equation $\log_e x = \log_e 4$. Hence write down the coordinates of the point of intersection of the two functions. **1**

(ii) Show that the area of the shaded region is 3 square units. **2**

(iii) Find the exact value of $\int_1^4 \log_e x \, dx$. **1**

(c) (i) If $y = \dfrac{\sin x}{x}$, find $\dfrac{dy}{dx}$. **1**

(ii) Show that $y = \dfrac{\sin x}{x}$ is a solution of $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{\cos x}{x}$. **2**

Solutions. 2U 2003 HY.

Q1. (a) (i) 7·389 ✓ (penalise for 3 d.pl. here only).

(ii) 2·639 ✓

(iii) 0·997 ✓

(b) $-\frac{\sqrt{3}}{2}$ ✓✓

(c) (i) $-2x^{-2}$ ✓

(ii) $8e^{2x}$ ✓

(iii) $\frac{4}{x}$ ✓

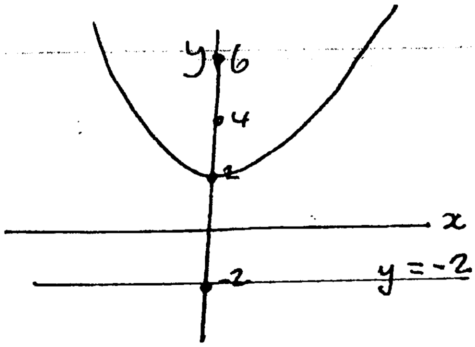
(d) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$ ✓ (don't worry about c)

(e) $\log_a 50 = \log_a(2 \times 5 \times 5)$
$= \log_a 2 + \log_a 5 + \log_a 5$ ✓
$= x + 2y$ ✓

(f) $\frac{\pi}{3}$ ✓

Q2.

(a)



(i) $(0,2)$ ✓

(ii) 4 ✓

(iii) $x^2 = 16(y-2)$ ✓

(b)(i) A is on the x axis where $y=0$

So solve $x - x^2 = 0$

$$x(1-x) = 0$$

$$x = 0, 1$$

So $x = 1$ is one of the solutions ✓

(ii) $$\text{Area} = \int_0^1 (x - x^2)\,dx \quad ✓$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \quad ✓$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0)$$

$$= \frac{1}{6}\ u^2 \quad ✓$$

(iii) $$\text{Volume} = \pi \int_0^1 y^2\,dx$$

$$= \pi \int_0^1 (x - x^2)^2\,dx \quad ✓$$

$$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 \quad ✓$$

$$= \pi \left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) - (0)\right]$$

$$= \frac{\pi}{30}\ u^3 \quad ✓$$

(c) $\int_2^6 f(x)\,dx \doteqdot \frac{h}{2}\left[f(2) + f(4)\right] + \frac{h}{2}\left[f(4) + f(6)\right]$

$= \frac{2}{2}\left[8 + 15 + 15 + 20\right]$ ✓ for correct substitution into correct formula

$= 58$ ✓

Q3.

(a)(i) $f(x) = \cos(2x+1)$

$f'(x) = -2\sin(2x+1)$

(ii) $f(x) = \dfrac{e^x}{\sin x}$

$$f'(x) = \frac{e^x \sin x - e^x \cos x}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

(b) $$\int_{-4}^{6} f(x)\,dx = \frac{\pi}{2}2^2 - \frac{\pi}{2}3^2$$

$$= \frac{\pi}{2}(4-9)$$

$$= -\frac{5\pi}{2}$$

(c) (i) $180° = \pi$ radians

$20° = \frac{\pi}{9}$ radians

(ii) $l = r\theta$

$= 5 \times \frac{\pi}{9}$ cm

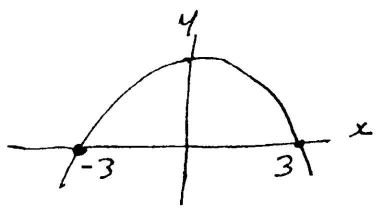
$= \frac{5\pi}{9}$ cm

(d) (i) $f(x) = 27x - x^3$

$f'(x) = 27 - 3x^2$

$= 3(3-x)(3+x)$

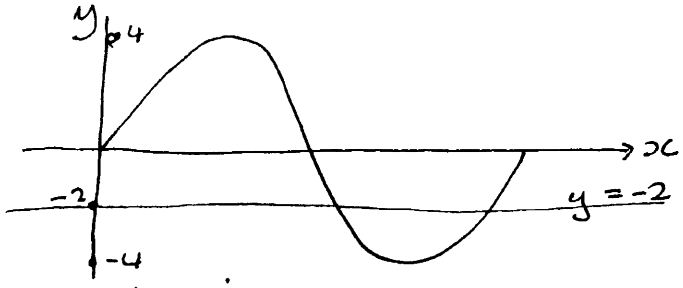
(ii) Solve $3(3-x)(3+x) > 0$



$-3 < x < 3$

Q4.

a)

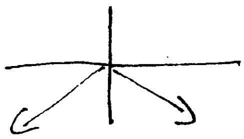


(need line $y=-2$ and 2 soln for this mk)

(i) 2 solutions

(ii) $4\sin x = -2$

$\sin x = -\frac{1}{2}$



related angle is $\frac{\pi}{6}$

$x = \pi + \frac{\pi}{6}, \quad 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}, \quad \frac{11\pi}{6}$

b) (i) Area $= \frac{1}{2}r^2\theta$

$18\pi = \frac{1}{2} \times 12^2 \times \theta$

$\theta = \frac{18\pi \times 2}{144}$

$= \frac{\pi}{4}$

(ii) Area $\Delta OPQ = \frac{1}{2} \times 12 \times 12 \times \sin\frac{\pi}{4}$

$= \frac{72}{\sqrt{2}}$ cm^2

Area of segment $= 18\pi - \frac{72}{\sqrt{2}}$ cm^2

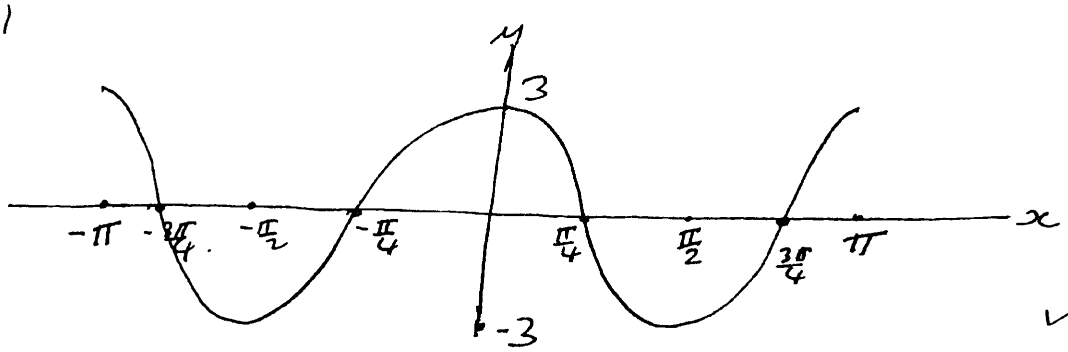
or/ Area segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= 72\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$ cm^2

(c) (i) period is $\frac{2\pi}{2} = \pi$

amplitude is 3.

(ii)



5 (a) (i) $f(x) = (x+2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x+2)^{-\frac{1}{2}}$ ✓

$= \frac{1}{2\sqrt{x+2}}$

(ii) $f'(7) = \frac{1}{2\sqrt{7+2}}$ is gradient ✓

$= \frac{1}{6}$

(iii) gradient of normal is -6 ✓

equation of normal is

$y - 3 = -6(x-7)$

$y - 3 = -6x + 42$

$y + 6x - 45 = 0$ ✓

(b)(i) $(3,3)$ clearly satisfies $y = x$.

Consider $y = \frac{9}{x}$, if $x=3$ then $y = \frac{9}{3} = 3$

So $(3,3)$ lies on both curves, so it is a point of intersection. ✓

(ii) Area = area triangle + area under curve

$= \frac{1}{2}\times 3\times 3 + \int_3^6 \frac{9}{x}\,dx$ ✓

$= \frac{9}{2} + \left[9\log_e x\right]_3^6$ ✓

$= \frac{9}{2} + 9\{\log 6 - \log 3]$

$= \frac{9}{2} + 9\log 2$ ✓ ✓

(c) $x^2 - 6x + 9 + y^2 + 8y + 16 = 9 + 16$ ✓

$(x-3)^2 + (y+4)^2 = 25$

centre is $(3,-4)$, radius is 5 ✓✓

Q6. (a)

$$\log_e(7x-12) = 2\log_e x$$
$$\log_e(7x-12) = \log_e x^2$$
$$7x-12 = x^2$$
$$x^2 - 7x + 12 = 0$$
$$(x-3)(x-4) = 0$$
$$x = 3, 4.$$

(b)

$$f(x) = 4\sin x + \cos x + 1$$
$$f'(x) = 4\cos x - \sin x$$
$$f'(\tfrac{\pi}{4}) = 4\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}$$
$$= \tfrac{4}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}$$
$$= \tfrac{3}{\sqrt{2}}$$

(c) There is a maximum turning point at $x = 4$.
The gradient, $f'(x)$, changes from positive to negative at $x = 4$.

x	3	4	5
$f'(x)$	+ve	0	−ve

(d) (i)

$$f(x) = \int \frac{2x-5}{x^2-5x+6}\,dx$$
$$= \log_e(x^2-5x+6) + c$$
$$f(1) = \log_e(1-5+6) + c = 2 + \log_e 2$$
$$\log_e 2 + c = \log_e 2 + 2$$
$$c = 2$$

so $f(x) = \log_e(x^2-5x+6) + 2$

(ii)

$$f(-1) = \log_e(1+5+6) + 2$$
$$= \log_e 12 + 2$$
$$= \log(2^2 \times 3) + 2$$
$$= 2\log 2 + \log 3 + 2$$

Q7. (a) Minimum value is when $\sin x = -1$.
Min. $y = 1 - \sqrt{3}$.

(b) (i) $y = x^3 - 6x^2 + 9x + 4$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-1)(x-3)$$

(ii) At stationary point $\frac{dy}{dx} = 0$.

$$3(x-1)(x-3) = 0$$
$$x = 1, 3$$

$x = 1,\ y = 8$ and $x = 3,\ y = 4$.
Stationary points are $(1, 8)$ and $(3, 4)$

(iii) $\frac{d^2y}{dx^2} = 6x - 12$.

when $x = 1$, $\frac{d^2y}{dx^2} = 6 - 12 = -6$, negative $\cap$
so $(1, 8)$ is a maximum turning point.

when $x = 3$, $\frac{d^2y}{dx^2} = 18 - 12 = 6$, positive $\cup$
so $(3, 4)$ is a minimum turning point.

(iv) At a possible point of inflection, $\frac{d^2y}{dx^2} = 0$

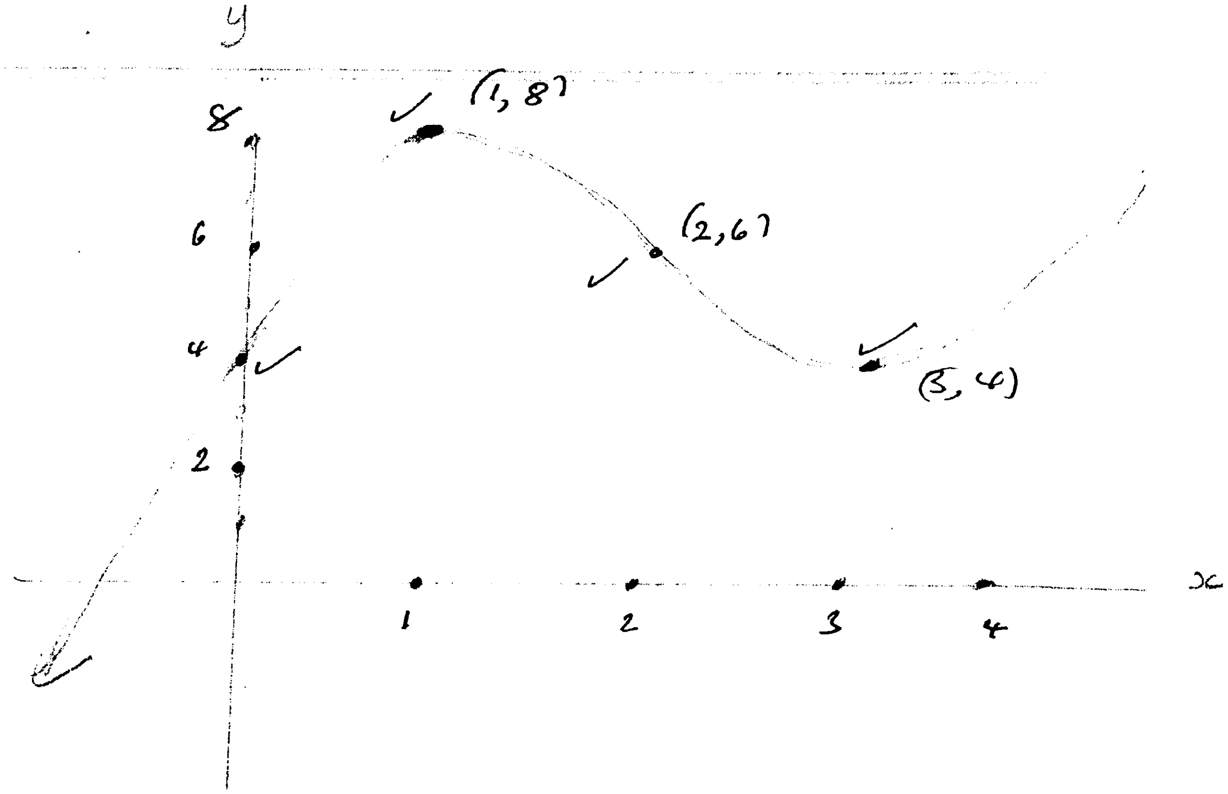
$$6x - 12 = 0$$
$$x = 2$$
$y = 6$, $(2, 6)$ may be a pt of inflection

x	1	2	3
$\frac{d^2y}{dx^2}$	-6	0	6
	$\cap$	$-$	$\cup$

ie concavity changes at $(2, 6)$

So $(2, 6)$ is a point of inflection.

(Q7)



Q8.

a) (i) at C, $x = \ell$

So B is $(\ell, 2e^{-\ell^2})$

and BC is $2e^{-\ell^2}$ units

(ii) $\text{Area} = DC \times BC$

$= 2\ell \times 2e^{-\ell^2}$

$= 4\ell e^{-\ell^2}$

(iii) $\frac{dA}{d\ell} = 4\ell \times (-2\ell e^{-\ell^2}) + 4e^{-\ell^2}$

$= 4e^{-\ell^2}(1 - 2\ell^2)$

now, $4e^{-\ell^2}(1-2\ell^2) = 0$ at a stationary point

$\ell^2 = \frac{1}{2}$

$\ell = \pm \frac{1}{\sqrt{2}}$

choose $\ell = \frac{1}{\sqrt{2}}$ as ℓ is a length.

To check for maximum

ℓ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1
$\frac{dA}{d\ell}$	$4e^{-\frac{1}{4}} \times \frac{1}{2}$ +ve	0	$-4e^{-1}$ -ve
	/	—	\

so we have a maximum A when $\ell = \frac{1}{\sqrt{2}}$.

(b) (i) $\log x = \log 4$

$x = 4$

Pt of intersection is $(4, \log 4)$

(ii) $y = \log_e x$

$e^y = x$

$A = \int_0^{\log 4} e^y \, dy$

$= \left[e^y \right]_0^{\log 4}$

$= [4 - e^0]$

$= 4 - 1 = 3 \text{ u}^2$

(iii) $\int_1^4 \log_e x \, dx$ = area of rectangle − shaded area
$= 4\log 4 - 3$

(c) (i) $y = \dfrac{\sin x}{x}$

$\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}$

$= \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$

(ii) $LHS = \dfrac{dy}{dx} + \dfrac{y}{x}$

$= \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} + \dfrac{\sin x}{x}\dfrac{1}{x}$

$= \dfrac{\cos x}{x}$

$= RHS$