CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Half-Yearly Examination

FORM VI

MATHEMATICS 2 UNIT

Friday 20th February 2015

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 94 boys

Examiner

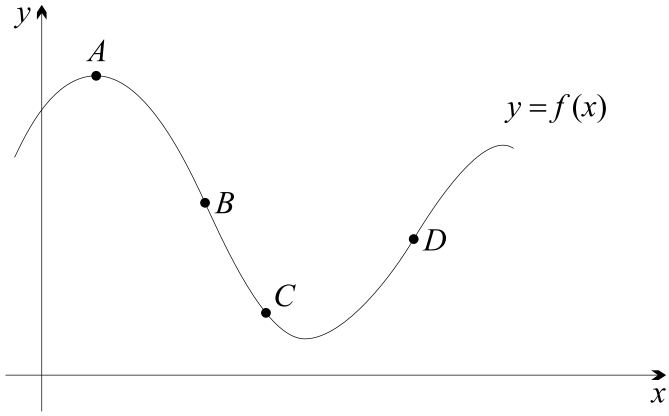
REJ

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The diagram below shows points A, B, C and D on the graph of $y = f(x)$. **1**



At which point is $f'(x) < 0$ and $f''(x) = 0$?

(A) A

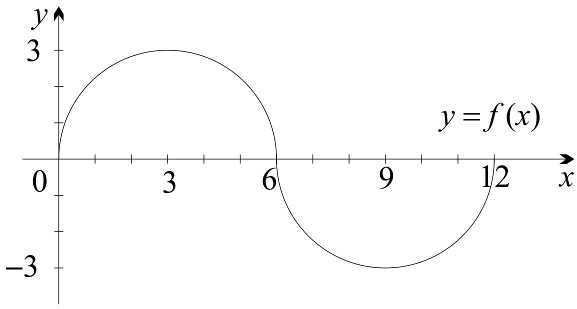
(B) B

(C) C

(D) D

QUESTION TWO

The graph of $y = f(x)$ is shown below. It consists of two semicircles. **1**



What is the value of $\displaystyle\int_0^{12} f(x)\,dx$?

(A) 9π

(B) -9π

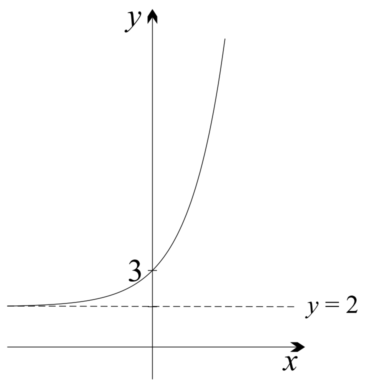
(C) 6π

(D) 0

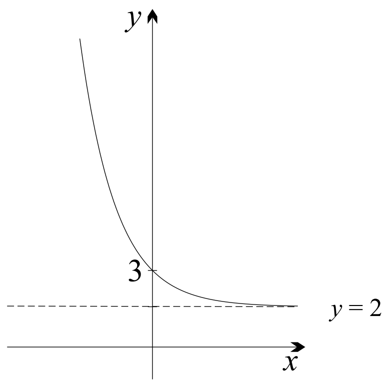
QUESTION THREE

Which of the following is the graph of $y = e^{-x} + 2$? **1**

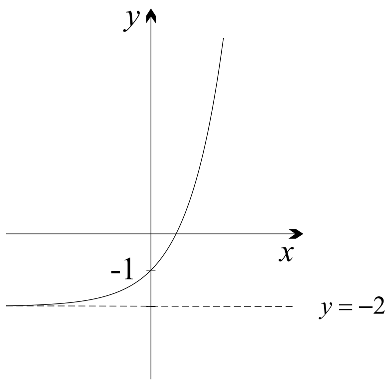
(A)



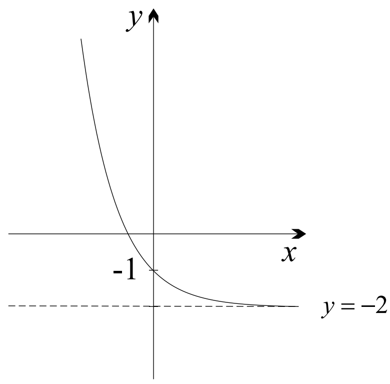
(C)



(B)



(D)



QUESTION FOUR

Which expression is equal to $\displaystyle\int \left(e^{3x} + 2\right) dx$? **1**

(A) $\dfrac{e^{3x}}{3} + 2x + C$

(B) $\dfrac{e^{3x+2}}{3} + C$

(C) $e^{3x} + C$

(D) $\dfrac{e^{3x+1}}{3x+1} + 2x + C$

Exam continues overleaf ...

QUESTION FIVE

Two fixed points A and B lie in the number plane. A point $P(x, y)$ moves so that PA is equal to PB. Which of the following best describes the locus of P? **1**

(A) A line

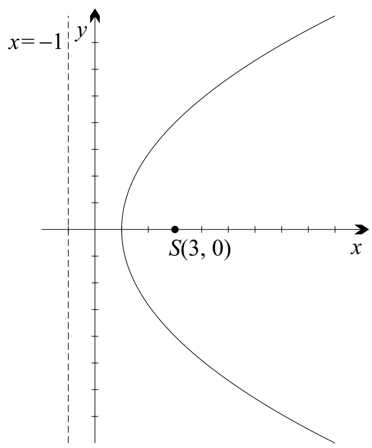
(B) A circle

(C) A parabola

(D) A hyperbola

QUESTION SIX

The parabola shown in the diagram below has focus $S(3, 0)$ and directrix $x = -1$. What is the equation of the parabola? **1**



(A) $y^2 = 16(x - 3)$

(B) $y^2 = 8(x - 1)$

(C) $y^2 = -16(x - 3)$

(D) $y^2 = -8(x - 1)$

QUESTION SEVEN

The gradient function of a curve $y = f(x)$ is given by $\dfrac{dy}{dx} = x^2 + x - 2$. Which of the following statements is false? **1**

(A) Stationary points occur at $x = -2$ and $x = 1$.

(B) The primitive of the function is $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + C$.

(C) There is a point of inflexion at $x = \dfrac{1}{2}$.

(D) The function is decreasing for $-2 < x < 1$.

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Calculate the value of $4e^2$ correct to two decimal places. **1**

(b) Differentiate the following with respect to x:

(i) $5x^2 + 10x - 1$ **1**

(ii) e^{5x} **1**

(iii) $\dfrac{e^x}{x}$ **2**

(c) Consider the parabola $x^2 = 12y$.

(i) What is the focal length? **1**

(ii) Write down the coordinates of the focus and the equation of the directrix. **2**

(iii) Sketch the graph of the parabola showing the above information. **1**

(d) Consider the curve $y = x^3 + 4x^2 - 3x$.

(i) Show that the gradient of the curve is 8 at the point where $x = 1$. **1**

(ii) Hence find the equation of the tangent to the curve at the point where $x = 1$. **2**

Exam continues next page ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

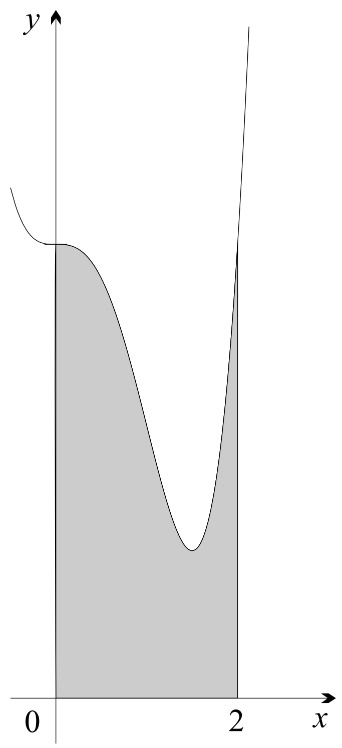
(a) Find:

(i) $\displaystyle\int \left(3x^2 + 2x\right) dx$ **1**

(ii) $\displaystyle\int \left(\frac{x^2 + 4}{x^2}\right) dx$ **2**

(b) If $f'(x) = 4x - 7$ and $f(2) = 5$, find $f(x)$. **3**

(c) **3**



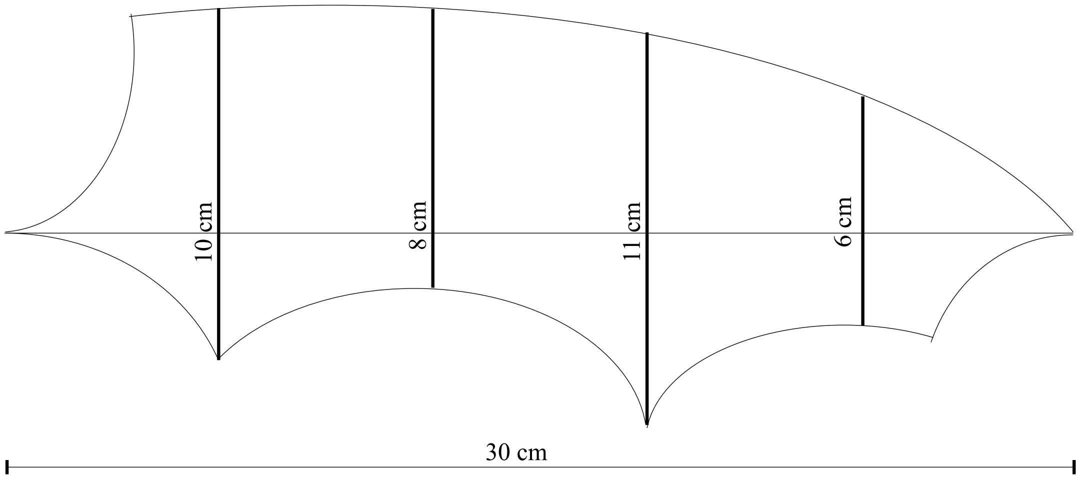
The shaded region in the diagram above is bounded by the curve $y = 2x^4 - 4x^3 + 5$, the x-axis, and the lines $x = 0$ and $x = 2$. Find the area of the shaded region.

(d) (i) Differentiate $(x^3 + 1)^5$ with respect to x. **1**

(ii) Hence find $\displaystyle\int x^2(x^3 + 1)^4\, dx$. **2**

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a)



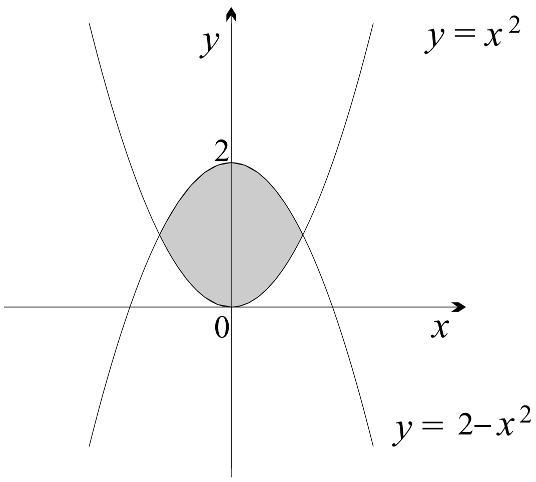
The diagram above shows the wing of a flying fox. The length of the wing is 30 cm and the width of the wing is shown at regular intervals.

(i) Copy and complete the following table of values into your answer booklet. **1**

x	0					30
y						

(ii) Use the trapezoidal rule with 6 function values to find an approximate value for the area of the wing of the flying fox. **2**

(b)



The diagram above shows the curves $y = x^2$ and $y = 2 - x^2$. The region enclosed between the curves is shaded.

(i) Show that the curves intersect at the points where $x = -1$ and $x = 1$. **1**

(ii) Find the area enclosed between the two curves. **2**

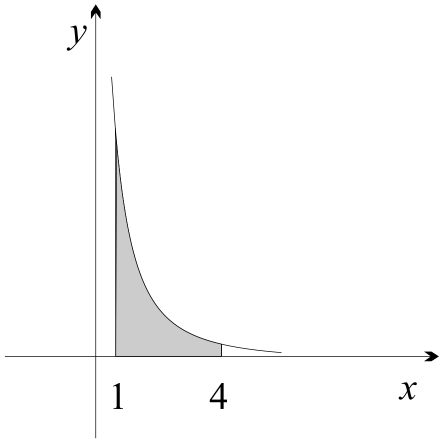
Exam continues next page ...

(c) Suppose $A(-2,4)$ and $B(4,1)$ are two points in the number plane. The point $P(x,y)$ moves so that $PA = 2 \times PB$.

(i) Write expressions for PA^2 and PB^2 in terms of x and y. **1**

(ii) Hence find the equation of the locus of P. **2**

(d)

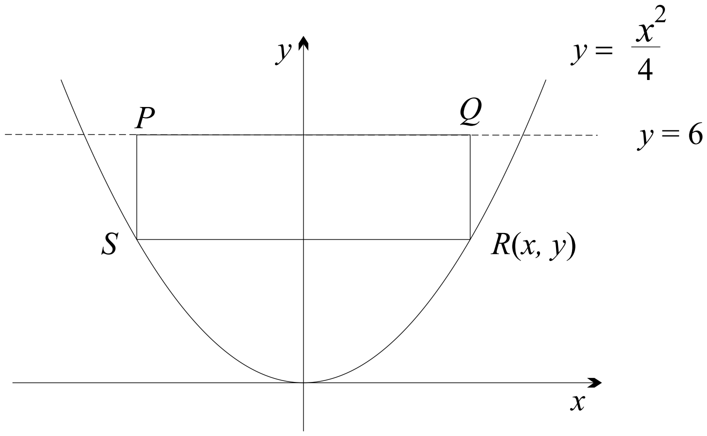


The shaded region above is bounded by the curve $y = \dfrac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 4$. Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis. Give your answer in exact form. **3**

Exam continues overleaf ...

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a rectangle $PQRS$ inscribed in the region bounded by the parabola $y = \dfrac{x^2}{4}$ and the line $y = 6$. Suppose the point R has coordinates (x, y).

(i) Show that the area of $PQRS$ is given by $A = 12x - \dfrac{x^3}{2}$. **2**

(ii) Find the dimensions of the rectangle so that its area is maximised. **3**

(b) Consider the curve $y = x^3 e^x$.

(i) Show that $\dfrac{dy}{dx} = 3e^x x^2 + x^3 e^x$. **1**

(ii) Find all stationary points on the curve. Give your answers in exact form. **2**

(iii) Using a table of gradients, or otherwise, determine the nature of each stationary point. **2**

(iv) What value does y approach as x approaches $-\infty$? **1**

(v) Sketch the curve, clearly indicating the stationary points. **1**

End of Section II

END OF EXAMINATION

B L A N K   P A G E

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE}:\ \ln x = \log_e x,\ \ x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ○ B ○ C ○ D ○

Question Two

A ○ B ○ C ○ D ○

Question Three

A ○ B ○ C ○ D ○

Question Four

A ○ B ○ C ○ D ○

Question Five

A ○ B ○ C ○ D ○

Question Six

A ○ B ○ C ○ D ○

Question Seven

A ○ B ○ C ○ D ○

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Question One

A ○ B ● C ○ D ○

Question Two

A ○ B ○ C ○ D ●

Question Three

A ○ B ○ C ● D ○

Question Four

A ● B ○ C ○ D ○

Question Five

A ● B ○ C ○ D ○

Question Six

A ○ B ● C ○ D ○

Question Seven

A ○ B ○ C ● D ○

Half-yearly examination Solutions

Question 1 (B)

Question 2 (D)

Question 3 (C)

Question 4

$\int e^{3x} + 2 \, dx$

$= \frac{e^{3x}}{3} + 2x + C$ (A)

Question 5 (A)

Question 6

$a = 2$ vertex $(1,0)$ concave right

$\cancel{x}$ $(y-k)^2 = 4a(x-h)$

$(y-0)^2 = 4 \times 2(x-1)$

$y^2 = 8(x-1)$ (B)

Question 7

$\frac{dy}{dx} = x^2 + x - 2$

stat pts $\frac{dy}{dx} = 0$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$ ✓

$y = \int x^2 + x - 2$

$= \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$ ✓

inflex. $\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = 2x + 1$

at $x = \frac{1}{2}$ $\frac{d^2y}{dx^2} = 2 \times \left(\frac{1}{2}\right) + 1$

$= 2$

$\neq 0$

Answer (C)

Question 8

a) $4e^2$
$= 29.556...$
≈ 29.56 (2dp)

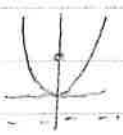
b) i) $\frac{d}{dx}\ 5x^2 + 10x - 1$
$= 10x + 10$

ii) $\frac{d}{dx}\ e^{5x}$
$= 5e^{5x}$

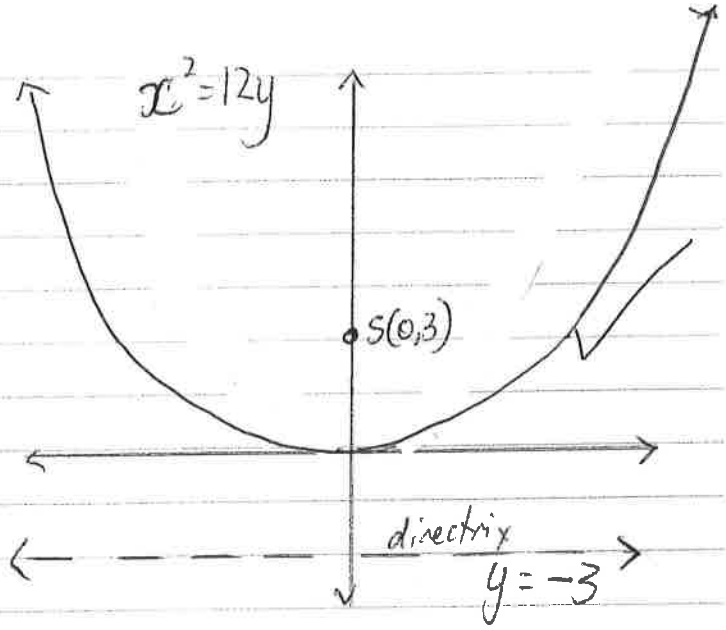
iii) $u = e^x$ $\quad v = x$
$u' = e^x$ $\quad v' = 1$
$v^2 = x^2$

$\frac{d}{dx}\frac{e^x}{x} = \frac{vu' - uv'}{v^2}$ progress toward.
$= \frac{xe^x - e^x}{x^2}$
$= \frac{e^x(x-1)}{x^2}$

c) $x^2 = 12y$
$4a = 12$
$a = 3$
focal length is 3 units

ii)  vertex $(0,0)$

focus: $(0,3)$
directrix: $y = -3$



d) i) $y = x^3 + 4x^2 - 3x$
$\frac{dy}{dx} = 3x^2 + 8x - 3$

at $x = 1$ $\quad m_T = 3(1)^2 + 8(1) - 3$
$= 8$ as required.

The gradient of the tangent at $x = 1$ is 8
proof.

ii) at $x = 1$ $\quad y = 1 + 4 - 3$
$y = 2$ matching y.

$y - y_1 = m(x - x_1)$
$y - 2 = 8(x - 1)$
$y - 2 = 8x - 8$
$\boxed{8x - y - 6 = 0}$
or $y = 8x - 6$

Question 9

i) $\int 3x^2 + 2x \; dx$

$= \frac{3x^3}{3} + \frac{2x^2}{2} + C$

$= x^3 + x^2 + C$ ✓

ii) $\int \frac{x^2+4}{x^2} \; dx$

$= \int 1 + 4x^{-2} \; dx$ ✓

$= x + \frac{4x^{-1}}{-1} + C$

$= x - \frac{4}{x} + C$ ✓

b) $f'(x) = 4x - 7$

$f(x) = \frac{4x^2}{2} - 7x + C$

$f(x) = 2x^2 - 7x + C$ ✓

$f(2) = 5$ ✓ substitution

$5 = 2 \times 2^2 - 7 \times 2 + C$

$5 = -6 + C$

$C = 11$

$f(x) = 2x^2 - 7x + 11$

✓ finding C + writing Result.

c) $y = 2x^4 - 4x^3 + 5$

$A = \int_0^2 2x^4 - 4x^3 + 5 \; dx$

$= \left[\frac{2x^5}{5} - \frac{4x^4}{4} + 5x\right]_0^2$ ✓

$= \left[\frac{2 \times 32}{5} - 16 + 10\right] - [0]$ ✓ substitution

$= 6.8$ units^2 ✓

d) i) $\frac{d}{dx} (x^3+1)^5$

$= 5(x^3+1)^4 \times 3x^2$

$= 15x^2(x^3+1)^4$ ✓

ii) $\frac{1}{15} \int 15x^2(x^3+1)^4$

$= \frac{1}{15}(x^3+1)^5 + C$.

✓ ✓

Question 10

a) i)

x	0	6	12	18	24	30
y	0	10	8	11	6	0

ii) $A \simeq \frac{6}{2}\left[0 + 2\times(10+8+11+6) + 0\right]$

$\simeq 210\text{ cm}^2$ (1 mark progress toward)

The area is approximately 210 cm^2.

b) Either:

$x = -1$

$y = x^2$
$= (-1)^2$
$= 1$

$y = 2 - x^2$
$= 2 - (-1)^2$
$= 1$

intersect at $x = -1$

$x = 1$

$y = 1^2$
$y = 1$

$y = 2 - 1^2$
$= 1$

intersect at $x = 1$ as required

OR

equating
$x^2 = 2 - x^2$
$2x^2 = 2$
$x^2 = 1$
$x = \pm 1$ as required

Appropriate proof.

ii) $A = \int_{-1}^{1} 2 - x^2 - x^2 \, dx$

$= \int_{-1}^{1} 2 - 2x^2 \, dx$

$= \left[2x - \frac{2x^3}{3}\right]_{-1}^{1}$

$= \left[2 - \frac{2}{3}\right] - \left[-2 + \frac{2}{3}\right]$

$= 2\frac{2}{3}\text{ units}^2$

c) $A(-2,4)$ $B(4,1)$ $P(x,y)$

i) $PA^2 = (x+2)^2 + (y-4)^2$

$PB^2 = (x-4)^2 + (y-1)^2$

ii) $PA = 2PB$

$PA^2 = 4PB^2$

$(x+2)^2 + (y-4)^2 = 4\left[(x-4)^2 + (y-1)^2\right]$

$x^2 + 4x + 4 + y^2 - 8y + 16 = 4\left[x^2 - 8x + 16 + y^2 - 2y\right.$

$x^2 + 4x + y^2 - 8y + 20 = 4x^2 - 32x + 4y^2 - 8y + 6$

$3x^2 - 36x + 3y^2 + 48 = 0$

$x^2 - 12x + y^2 + 16 = 0$

d) $y = \frac{1}{x^2}$

$y^2 = \frac{1}{x^4}$

$V = \pi \int_1^4 \frac{1}{x^4} \, dx$

$= \pi \int_1^4 x^{-4} \, dx$ ✓

$= \pi \left[\frac{x^{-3}}{-3} \right]_1^4$ ✓

$= \pi \left[-\frac{1}{3 \times 4^3} - \frac{1}{-3 \times 1} \right]$

$= \pi \times \frac{21}{64}$

$= \frac{21\pi}{64}$ units^3 ✓

Question 11

a) i) $R(x, y)$

$R\left(x, \frac{x^2}{4}\right)$

$Q(x, 6)$

$RS: 2x$ units

$QR = 6 - \frac{x^2}{4}$

$\text{Area} = 2x(6-y)$ ✓

$\text{Area} = 2x\left(6 - \frac{x^2}{4}\right)$

$= 12x - \frac{x^3}{2}$ as required ✓

ii) $A = 12x - \frac{x^3}{2}$

$\frac{dA}{dx} = 12 - \frac{3x^2}{2}$

max when $\frac{dA}{dx} = 0$

$12 - \frac{3x^2}{2} = 0$ ✓

$24 - 3x^2 = 0$

$3x^2 = 24$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

~~since~~ $x > 0$

So $x = 2\sqrt{2}$

$\frac{d^2A}{dx^2} = -\frac{6x}{2}$ ✓ test concavity

$= -3x$

at $x = 2\sqrt{2}$ $\frac{d^2A}{dx^2} = -6\sqrt{2}$

< 0

concave down

So at $x = 2\sqrt{2}$ the area is a maximum.

when $x = 2\sqrt{2}$

dimensions:

$RS = 2 \times 2\sqrt{2}$

$= 4\sqrt{2}$

$QR = 6 - \frac{(2\sqrt{2})^2}{4}$

$= 6 - \frac{8}{4}$

$= 4$

Dimensions: ✓

$4\sqrt{2} \times 4$ gives a maximum area

b) i) $y = x^3 e^x$

$\frac{dy}{dx} = vu' + uv'$

$u = x^3$ $\quad v = e^x$

$u' = 3x^2$ $\quad v' = e^x$

✓ show.

$\frac{dy}{dx} = 3x^2 e^x + x^3 e^x$ as required.

ii) stationary points $\frac{dy}{dx} = 0$

$3x^2 e^x + x^3 e^x = 0$

$x^2 e^x (3 + x) = 0$

~~yuxa~~ $e^x \neq 0$

$x^2 (3 + x) = 0$

$x = 0 \quad x = -3$ ✓

at $x = 0$ $y = 0$ $\quad$ at $x = -3$ $y = -27e^{-3}$

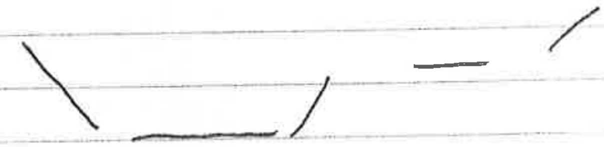
$= -\frac{27}{e^3}$ ✓

penalise not exact

Stationary points: $(0, 0)$ $\quad \left(-3, \frac{-27}{e^3}\right)$

iii)

x	-4	-3	-1	0	1
$\frac{dy}{dx}$	$-0.29..$	0	$0.735...$	0	10.873



needs numeric values

$\left(-3, \frac{-27}{e^3}\right)$ is a minimum turning point

conclusion

$(0,0)$ is a horizontal inflexion point.

penalise inflexion point

iv) as $x \to -\infty$ $y \to 0$

v)