CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Half-Yearly Examination

FORM VI

MATHEMATICS 2 UNIT

Thursday 25th February 2016

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 60 Marks

- All questions may be attempted.

Section I – 8 Marks

- Questions 1–8 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 52 Marks

- Questions 9–12 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Nine.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 87 boys

Examiner

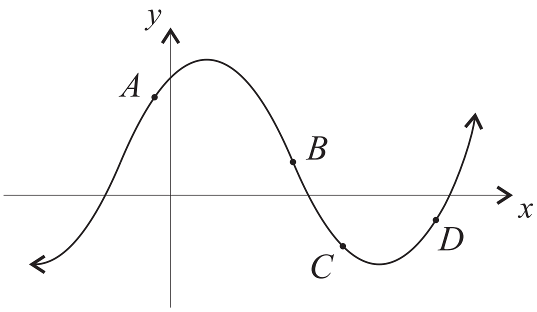
LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1



The graph of $y = f(x)$ is shown above. At which point is $f'(x) > 0$ and $f''(x) > 0$?

(A) A

(B) B

(C) C

(D) D

QUESTION TWO

What is the focal length of the parabola $x^2 = 4y$? 1

(A) $\frac{1}{4}$

(B) 1

(C) 4

(D) 8

QUESTION THREE

What is the domain of the curve $y = \log_e(x-1)$? 1

(A) $x > -1$

(B) $x > 0$

(C) $x > 1$

(D) all real x

Exam continues next page ...

QUESTION FOUR

Simpson's Rule is used with three function values to approximate $\displaystyle\int_1^2 xe^x\,dx$. **1**

Which expression is obtained?

(A) $\dfrac{1}{2}\left(e+3e^{1\cdot5}+2e^2\right)$

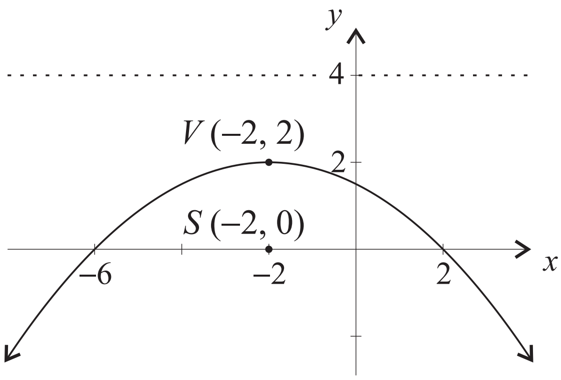
(B) $\dfrac{1}{4}\left(e+3e^{1\cdot5}+2e^2\right)$

(C) $\dfrac{1}{6}\left(e+6e^{1\cdot5}+2e^2\right)$

(D) $\dfrac{1}{12}\left(e+6e^{1\cdot5}+2e^2\right)$

QUESTION FIVE

1



A parabola with focus $S(-2,0)$ and directrix $y=4$ is shown above. Which of the following is the equation of the parabola?

(A) $(x+2)^2=-8(y-2)$

(B) $(x+2)^2=-8y$

(C) $(x-2)^2=-8(y+2)$

(D) $(x-2)^2=-8y$

Exam continues overleaf ...

QUESTION SIX

What is the value of k if $\displaystyle\int_1^k \frac{2}{x}\,dx = 4$? **1**

(A) e^2

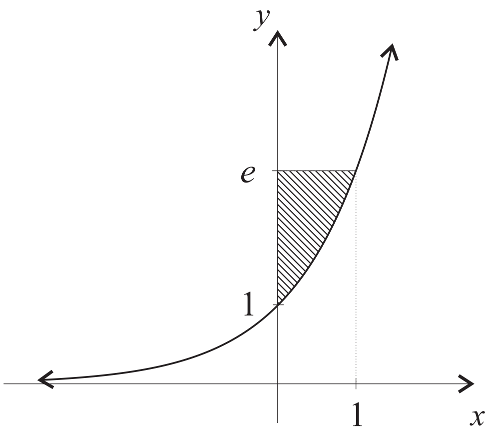
(B) e^3

(C) e^4

(D) e^5

QUESTION SEVEN

1



The graph of $y = e^x$ is shown above. Which of the following expressions does NOT represent the area of the shaded region?

(A) $e - \displaystyle\int_0^1 e^x\,dx$

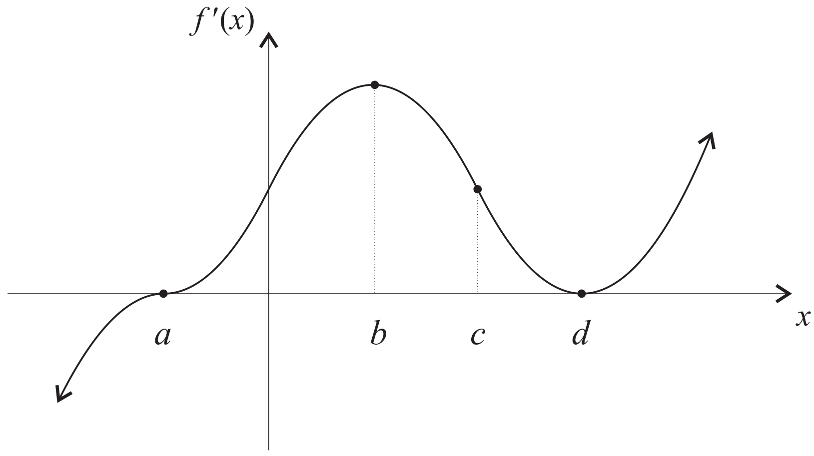
(B) $\displaystyle\int_1^e e^x\,dx$

(C) $\displaystyle\int_1^e \log_e y\,dy$

(D) $\displaystyle\int_0^1 (e - e^x)\,dx$

QUESTION EIGHT

1



The graph of the gradient function $y = f'(x)$ is shown above. Which of the labelled x-values corresponds to a stationary point of inflexion on the original curve $y = f(x)$?

(A) a

(B) b

(C) c

(D) d

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION NINE (13 marks) Use a separate writing booklet. **Marks**

(a) Calculate $\dfrac{3}{e^2}$ correct to 3 significant figures. **1**

(b) Expand and simplify $e^x\left(e^x + e^{-x}\right)$. **1**

(c) Differentiate:

(i) $\sqrt{x}$ **1**

(ii) e^{2x} **1**

(iii) $\log_e 3x$ **1**

(d) Find a primitive of:

(i) $6x^2$ **1**

(ii) e^{4x} **1**

(iii) $\dfrac{3}{x}$ **1**

(e) Evaluate $\displaystyle\int_2^3 (2x-1)\,dx$. **2**

(f) Find the equation of the parabola with vertex $(0,0)$ and focus $(4,0)$. **1**

(g) Consider the curve whose first derivative is given by $y' = x^2 + x - 6$. **2**
For what values of x is the curve increasing?

Exam continues next page ...

QUESTION TEN (13 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following with respect to x:

(i) $y = \log_e\left(x^2 + 3\right)$ **1**

(ii) $y = \dfrac{x+1}{x}$ **2**

(b) Find the equation of the tangent to the curve $y = e^{3x} + 1$ at the point $(0, 2)$. **2**

(c) Consider the parabola with equation $y^2 = -8\,(x+2)$.

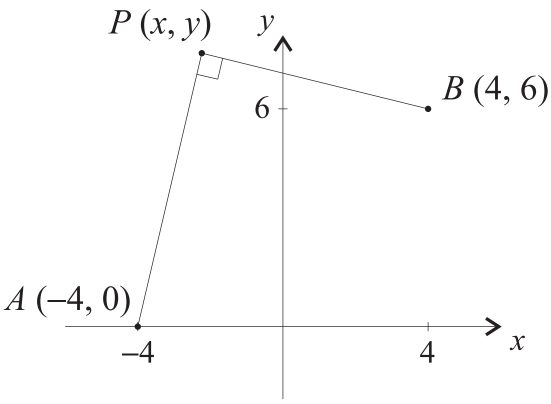
(i) Write down the coordinates of the vertex. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Find the equation of the directrix. **1**

(iv) Sketch the parabola clearly showing the vertex, focus and directrix. **1**

(d)



Consider the points $A\,(-4, 0)$ and $B\,(4, 6)$. The point $P\,(x, y)$ moves so that PA is always perpendicular to PB.

(i) Find expressions for the gradients of PA and PB. **1**

(ii) Hence show that the equation of the locus of P is $x^2 + y^2 - 6y - 16 = 0$. **1**

(iii) Express the above equation in the form $(x-h)^2 + (y-k)^2 = r^2$. **1**

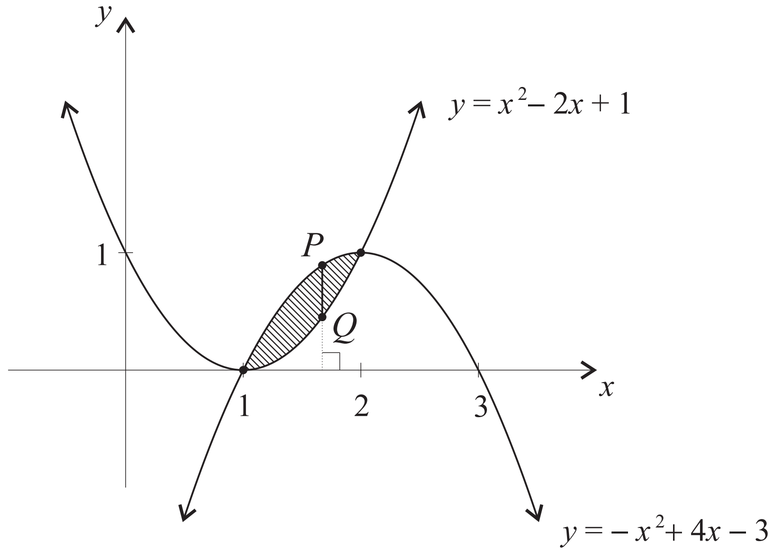
(iv) Hence, or otherwise, describe the locus of P geometrically. **1**

Exam continues overleaf ...

QUESTION ELEVEN (13 marks) Use a separate writing booklet. **Marks**

(a) If $f'(x) = \dfrac{2x-3}{x^2-3x+1}$ and $f(3) = 5$, find $f(x)$. **2**

(b)



The diagram above shows the curves $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 1$.

(i) Show that the curves intersect at the points where $x = 1$ and $x = 2$. **1**

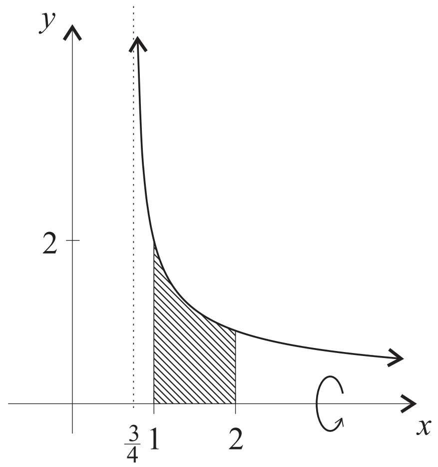
(ii) Find the area of the shaded region enclosed between the two curves. **3**

The points P and Q are located on the curves $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 1$ respectively, and share the same x-coordinate as shown.

(iii) If the length of PQ is ℓ units, show that $\ell = -2x^2 + 6x - 4$. **1**

(iv) Hence find the maximum length of PQ for $1 \le x \le 2$. **3**

(c) **3**



The region bounded by the curve $y = \dfrac{3}{\sqrt{4x-3}}$, the x-axis and the lines $x = 1$ and $x = 2$ is shown above. Find the volume of the solid generated when this region is rotated about the x-axis.

Exam continues next page ...

QUESTION TWELVE (13 marks) Use a separate writing booklet. **Marks**

(a) Solve for x: **2**

$$\log_2\left(x^2 - 2x\right) = 3.$$

(b) (i) Show that the function $y = x\log_e x$ has derivative $\dfrac{dy}{dx} = \log_e x + 1$. **1**

(ii) Hence find $\displaystyle\int \log_e x\,dx$. **2**

(c) Consider the curve with equation $y = \dfrac{x}{e^x}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{1-x}{e^x}$ and $\dfrac{d^2y}{dx^2} = \dfrac{x-2}{e^x}$. **2**

(ii) Find the coordinates of the stationary point and determine its nature. **2**

(iii) Find the coordinates of the point of inflexion. You must show that it is a point of inflexion. **2**

(iv) Sketch the curve, clearly showing the stationary point, the point of inflexion and any intercepts with the coordinate axes. **2**

(You may use the fact that as $x \to \infty$, $\dfrac{x}{e^x} \to 0^+$.)

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

MATHEMATICS 2U

Half-Yearly Examination - SOLUTIONS

* Multiple Choice

Q1. D positive gradient
& concave up

Q2. B $a = 1$

Q3. C $x - 1 > 0$
$\therefore x > 1$

Q4. C

x	1	1.5	2
y	e	$\frac{3}{2}e^{1.5}$	$2e^2$

$\frac{1}{6}\left(e + 4 \times \frac{3}{2}e^{1.5} + 2e^2\right)$

Q5. A $(x-h)^2 = -4a(y-k)$
$(x-(-2))^2 = -4 \times 2(y-2)$

Q6. A $\int_1^k \frac{2}{x}\,dx = 4 \rightarrow \left[2\ln x\right]_1^k = 4$

$2\ln k - \cancel{2\ln 1}^{\,0} = 4$
$\ln k = 2$
$\therefore e^2 = k$

Q7. B

Q8. D for stationary point of inflexion either

+/ ‾o /+ OR ╲‾o ╲− on original curve

→ at d:

{ $f'(x)$ has a +ve value either side &
← 0 at point d.

* QUESTION 9:

a) $\frac{3}{e^2} = 0.4060058\ldots$

$= 0.406$ (to 3 sig. fig.) ✓

b) $e^x(e^x + e^{-x}) = e^{2x} + e^0$

$= e^{2x} + 1$ ✓

c) i) $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

✓ either

$= \frac{1}{2\sqrt{x}}$

ii) $\frac{d}{dx}(e^{2x}) = 2e^{2x}$ ✓

iii) $\frac{d}{dx}(\log_e 3x) = \frac{1}{3x} \times 3$

$= \frac{1}{x}$ ✓

d) i) $y' = 6x^2$

$y = \frac{6x^3}{3} (+C)$

$= 2x^3 (+C)$ ✓

ii) $y' = e^{4x}$

$y = \frac{e^{4x}}{4} (+C)$ ✓

iii) $y' = \frac{3}{x}$

$y = 3\ln x (+C)$ ✓

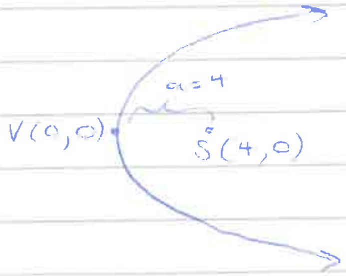
don't penalise missing +C.

e) $\int_2^3 (2x - 1)\,dx = \left[x^2 - x\right]_2^3$

$= 3^2 - 3 - (2^2 - 2)$

$= 4$

f)



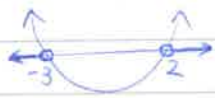
$y^2 = 4ax$

$y^2 = 16x$

g) increasing $\rightarrow y' > 0$

$x^2 + x - 6 > 0$

$(x+3)(x-2) > 0$



$x < -3$ or $x > 2$

/13

* QUESTION 10:

a) i) $\frac{dy}{dx} = \frac{1}{x^2+3} \times 2x$ ✓

$= \frac{2x}{x^2+3}$

ii) $y = \frac{x}{x} + \frac{1}{x}$

$= 1 + x^{-1}$ ✓

$\frac{dy}{dx} = -x^{-2}$ ✓

$= -\frac{1}{x^2}$

OR// Let $u = x+1$ $\quad v = x$

$u' = 1$ $\quad v' = 1$ ✓

$\frac{dy}{dx} = \frac{x(1) - (x+1)(1)}{x^2}$

$= -\frac{1}{x^2}$ ✓

b) $\frac{dy}{dx} = 3e^{3x}$

when $x = 0$, $\frac{dy}{dx} = 3$ ✓

$y - 2 = 3(x - 0)$

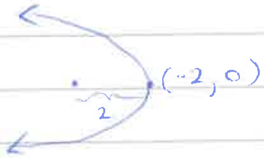
$\therefore y = 3x + 2$ ✓

c) $y^2 = -8(x+2)$

i) Vertex $(-2, 0)$

ii) $4a = 8$

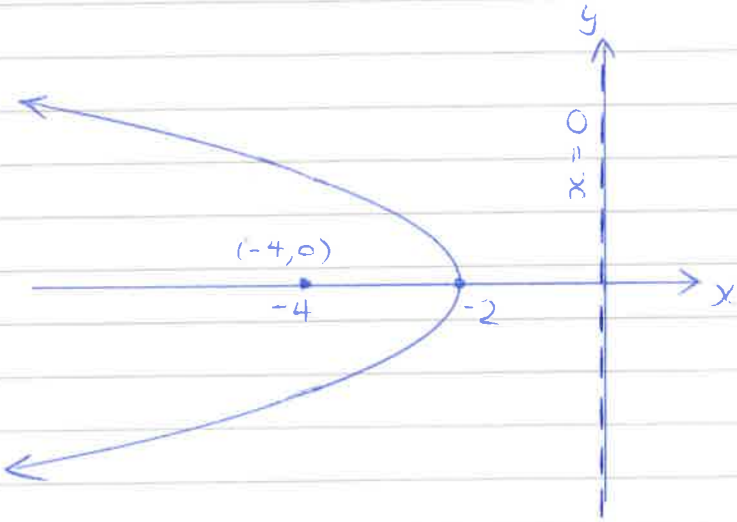
$a = 2$



Focus $(-4, 0)$

iii) Directrix $x = 0$

iv)



d) i) $m_{PA} = \dfrac{y}{x+4}$

$m_{PB} = \dfrac{y-6}{x-4}$

ii) $m_{PA} \times m_{PB} = -1$

$\dfrac{y}{x+4} \times \dfrac{y-6}{x-4} = -1$

show that...

$y^2 - 6y = -x^2 + 16$

$\therefore x^2 + y^2 - 6y - 16 = 0$

iii) $x^2 + y^2 - 6y + 9 = 16 + 9$

$x^2 + (y-3)^2 = 25$

$(x-0)^2 + (y-3)^2 = 5^2$

iv) The locus of P is a circle with centre $(0,3)$ & radius 5 units

13

* QUESTION 11:

a) $f(x) = \log_e(x^2 - 3x + 1) + C$

$5 = \log_e(9 - 9 + 1) + C$

$5 = \log_e 1 + C$ $\quad (\log_e 1 \to 0)$

$\therefore f(x) = \log_e(x^2 - 3x + 1) + 5$

b) i) $x^2 - 2x + 1 = -x^2 + 4x - 3$

$2x^2 - 6x + 4 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$\therefore x = 1$ or $x = 2$.

[Alternatively show the substitutions of both x values into both equations → four substitutions]

ii) $A = \int_1^2 \left(-x^2 + 4x - 3 - (x^2 - 2x + 1)\right) dx$

$= \int_1^2 (-2x^2 + 6x - 4)\, dx$

$= -2\int_1^2 (x^2 - 3x + 2)\, dx$

$= -2\left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$

$= -2\left(\frac{8}{3} - 6 + 4 - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right)$

$= \frac{1}{3}\ u^2$

iii) $l = y_P - y_Q$

$= -x^2 + 4x - 3 - (x^2 - 2x + 1)$ ✓ show that...

$= -2x^2 + 6x - 4$

iv) $\frac{dl}{dx} = -4x + 6$

$= 0$ when $x = \frac{3}{2}$ ✓

$\frac{d^2l}{dx^2} = -4$

< 0 $\therefore$ ✓ $\therefore$ max occurs when $x = \frac{3}{2}$

$\therefore l_{max} = -2\left(\frac{3}{2}\right)^2 + 6\left(\frac{3}{2}\right) - 4$

$= \frac{1}{2}$ units ✓

Alternatively, for middle mark:

x	1	$\frac{3}{2}$	2
$\frac{dl}{dx}$	2	0	-2
	/	—	\

✓

$\therefore$ max occurs when $x = \frac{3}{2}$

c) $V = \pi \int_1^2 \frac{9}{4x-3}\, dx$ ✓

$= \frac{9\pi}{4} \int_1^2 \frac{4}{4x-3}\, dx$

$= \frac{9\pi}{4} \left[\log_e(4x-3)\right]_1^2$ ✓

$= \frac{9\pi}{4}\left(\log_e 5 - \cancel{\log_e 1}\right)$

$= \frac{9\pi}{4} \log_e 5 \ u^3$ ✓

/13

* QUESTION 12:

a) $2^3 = x^2 - 2x$ ✓

$x^2 - 2x - 8 = 0$
$(x-4)(x+2) = 0$

$\therefore x = -2$ or $x = 4$ ✓

Check $x^2 - 2x > 0$:
when $x = -2$: $(-2)^2 - 2(-2) = 8 > 0$

$x = 4$: $(4)^2 - 2(4) = 8 > 0$ $\therefore$ both valid

validity check not required for full marks

b) i) $y = x \log_e x$

Let $u = x$ $\quad v = \log_e x$
$u' = 1$ $\quad v' = \frac{1}{x}$

$y' = \log_e x \times 1 + x \times \frac{1}{x}$ ✓ Show that...

$= \log_e x + 1$

ii) $\frac{d}{dx}(x \log_e x) = \log_e x + 1$

$\therefore \int (\log_e x + 1)\,dx = x \log_e x + C_1$ ✓

$\int \log_e x\,dx + \int 1\,dx = x \log_e x + C_1$

$\therefore \int \log_e x\,dx = x \log_e x - \int 1\,dx + C_1$

$= x \log_e x - x + C_2$ ✓

c) $y = \frac{x}{e^x}$

i) Let $u = x$ $v = e^x$

$u' = 1$ $v' = e^x$

$\frac{dy}{dx} = \frac{e^x \times 1 - x \times e^x}{(e^x)^2}$

$= \frac{\cancel{e^x}(1-x)}{(e^x)^{\cancel{2}}}$

$= \frac{1-x}{e^x}$

Let $u = 1-x$ $v = e^x$

$u' = -1$ $v' = e^x$

$\frac{d^2y}{dx^2} = \frac{e^x \times -1 - (1-x)e^x}{(e^x)^2}$

$= \frac{\cancel{e^x}(-1-1+x)}{(e^x)^{\cancel{2}}}$

$= \frac{x-2}{e^x}$

show that...

ii) $\frac{dy}{dx} = 0$ when $x = 1$ $y = \frac{1}{e}$

when $x = 1$: $\frac{d^2y}{dx^2} = \frac{1-2}{e^1}$

$= -\frac{1}{e}$

< 0 $\therefore$ ∩ $\therefore$ max t.p. occurs at $(1, \frac{1}{e})$

iii) $\frac{d^2y}{dx^2} = 0$ when $x = 2$ ✓ $y = \frac{2}{e^2}$

x	1	2	3
$\frac{d^2y}{dx^2}$	$-\frac{1}{e}$	0	$\frac{1}{e^3}$
	$\frown$	$\cdot$	$\smile$

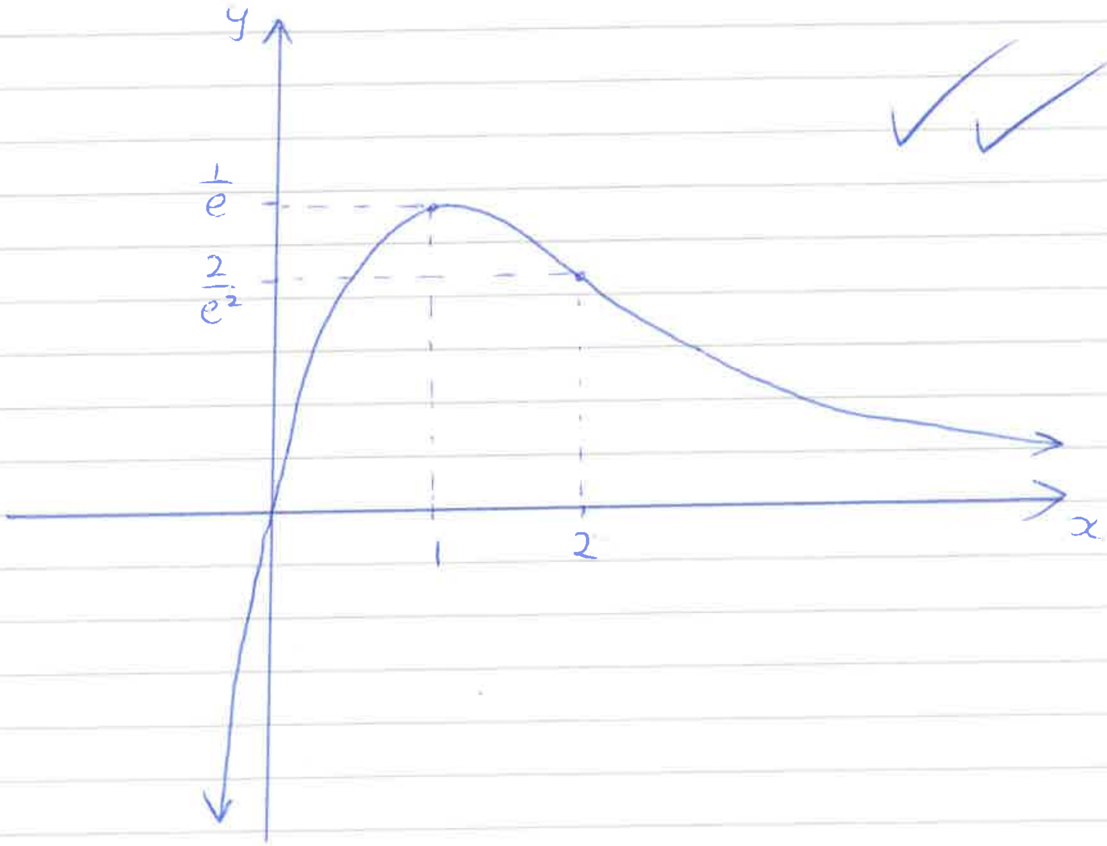
∴ change in concavity ✓
∗ point of inflexion
at $\left(2, \frac{2}{e^2}\right)$

iv) when $x = 0$, $y = 0$

max t.p. $\left(1, \frac{1}{e}\right)$ $\doteqdot 0.36\ldots$

pt of infl. $\left(2, \frac{2}{e^2}\right)$ $\doteqdot 0.27\ldots$

as $x \to \infty$, $y \to 0^+$



✓✓

/13