Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics

Term 1 Examination

March 2005

TIME ALLOWED: 70 minutes

Instructions:

- Write your name and class at the top of this page.
- At the end of the examination this question sheet must be attached to the front of your answers.
- Attempt all questions.
- Start all questions on a new page.
- All necessary working must be shown.

Q 1	Q 2	Q 3	Q 4	Q 5	TOTAL
/14	/15	/13	/13	/15	/70

QUESTION 1: (14 MARKS)

(a) If the nth term of a sequence is given by $T_n = 3^{2-n}$

2 (i) show that the sequence is geometric

1 (ii) Find its common ratio

1 (iii) Write an expression for the sum to n terms.

(b) Given the series $4 + 5\frac{1}{3} + 6\frac{2}{3} + \ldots\ldots$

2 (i) find the 19th Term

2 (ii) find the sum to 19 terms

3 (c) If the limiting sum of the series $1 + x + x^2 + \ldots\ldots\ldots\ldots$ is $\frac{3}{5}$ find the value of x.

3 (d) What is the value of the 11th term of the series

$$1\frac{1}{2} - 3 + 6 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

QUESTION 2: (15 MARKS) (Start a new page)

2 (a) Find the value of $\sum_{1}^{10} 2^n$

2 (b) Find the slope of the tangent to the curve $y = x - \frac{1}{x}$ at the point where $x=1$

2 (c) By using calculus, explain why the curve $y = 5 + 2x - 3x^2$ is always concave down.

3 (d) Find the value of x for which the curve $y = x^2 - 5x + 3$ is increasing.

6 (e) If α and β are the roots of the equation $x^2 - 2x - 7 = 0$ find the value of:

(i) $\alpha + \beta$ (ii) $\alpha\beta$ (iii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iv) $\alpha^2 + \beta^2$

QUESTION 3: (13 Marks) (Start a new page)

3 (a) Form a quadratic equation containing no fractions which has roots of $-\frac{2}{3}$ and $\frac{3}{2}$

7 (b) (i) Find the maximum and minimum turning points on the curve $y = x^3 - 3x + 3$

3 (ii) Hence sketch the curve showing all essential features.
DO NOT find inflexion points or the x intercepts.

QUESTION 4: (13 MARKS) (Start a new page)

2 (a) Find the acute angle x (in degress) which solves $\cos^2 x = \frac{1}{2}$

3 (b) Solve, to the nearest minute, the equation

$$3\sin\theta - 2 = 0 \quad \text{if} \quad 0^o \leq \theta \leq 360^o$$

(c) The adjacent sides of a parallelogram have lengths of 4cm and 6cm and include an angle of 60^o

(i) Draw the parallelogram

3 (ii) Calculate the length of the longer diagonal, giving your answer correct to 2 dec. places.

2 (iii) Calculate the area of the parallelogram, correct to 2 dec. places.

3 (d) By making a suitable substitution, or otherwise, solve the equation:

$$x^4 + x^2 - 2 = 0$$

QUESTION 5 (15 Marks) **(Start a new page)**

(a) If $y = ax^2 + bx + c$ passes through the point (0,7) and the curve has a turning point at (3,-2),

1 (i) find the value of c

2 (ii) by investigating $\frac{dy}{dx}$ prove that $6a + b = 0$

3 (iii) by finding another equation in a and b, and using your answer to part (ii) above, find the values of a and b.

(b) In the town of Mathsville, the population increases at a rate of 6% per year due to births and migration, but loses D persons per year due to deaths.

1 (i) The population as at January 1, 2005 is 100 000. Find an expression for the population on January 1, 2006

2 (ii) Show that the population on January 1, 2007 will be 112,360 – 2.06D

3 (iii) Show that the population after n years from January 1, 2005 is given by

$$P = 100\ 000(1.06)^n - \frac{50D}{3}(1.06^n - 1)$$

3 (iv) If the death rate (D) is a constant 500 persons per year, estimate the population as at January 1, 2025, to the nearest 100.

QUESTION 1:

(a) $T_n = 3^{2-n}$

(i) $T_1 = 3^1 = 3$

$T_2 = 3^0 = 1$

$T_3 = 3^{-1} = \frac{1}{3}$

$T_3/T_2 = T_2/T_1$

2

They may choose to say that there is a common ratio of $1/3$ between terms 2

(ii) $r = 1/3$ 1

(iii) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{3(\frac{1}{3}^n - 1)}{\frac{1}{3} - 1}$ ← this [or $\dfrac{3(1 - \frac{1}{3}^n)}{1 - \frac{1}{3}}$] is sufficient for the mark.

(b) $4 + 5\frac{1}{3} + 6\frac{2}{3} +$

(i) $T_{19} = a + 18d$

$= 4 + 18(1\frac{1}{3})$

$= 28$

basically there are 2 marks for 28 or 1 for identifying $a = 4$ and $d = 1\frac{1}{3}$ and 1 for the formula $a + 18d$.

(ii) $S_{19} = \frac{19}{2}[4 + 28]$ ← 1 OR $S_{19} = \frac{19}{2}[8 + 18(1\frac{1}{3})]$ 1

$= 304$ ← 1 for the answer

(c) $1 + x + x^2 + \quad = 3/5$

$a = 1,\ r = x \qquad S_\infty = \dfrac{1}{1-x} = 3/5$ ← 1

1 OR

$3 - 3x = 5$

$3x = -2$

$x = -2/3$ 1

(d) $a = 1\frac{1}{2}\ r = -2$

1

$T \quad 1\frac{1}{2}(-2)^{10}$ ← 1] if this becomes $(-3)^{10}$ they

QUESTION 2:

(a) $\sum_{1}^{10} 2^n = 2 + 4 + \ldots + 2^{10}$ ← ①

$= \dfrac{2(2^{10}-1)}{2-1}$

$= 2046$ } ①

(b) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$ (OR $1 + x^{-2}$) ← ①

At $x = 1$ $m_T = 1 + \frac{1}{1}$

$= 2$ ← ①

(c) $y = 5 + 2x - 3x^2$

$\dfrac{dy}{dx} = 2 - 6x$

$\dfrac{d^2y}{dx^2} = -6$ ← ① for getting to here.

$\therefore$ always concave down because $\dfrac{d^2y}{dx^2} < 0$ ← ① for a statement like this (they must make this statement)

(d) $y = x^2 - 5x + 3$

$\dfrac{dy}{dx} = 2x - 5$ ← ①

For increasing, $\dfrac{dy}{dx} > 0$ ← ① for making this statement or showing they know they have to make $\dfrac{dy}{dx} > 0$

$\therefore 2x - 5 > 0$

$\therefore x > \frac{5}{2}$ ← ①

(e) $f(x) = x^2 - 2x - 7$

(i) $\alpha + \beta = 2$ ← ①

(ii) $\alpha\beta = -7$ ← ①

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$ ← ① for this step

$= -\frac{2}{7}$ ← ①

QUESTION 3.

(a) (i) EITHER

$(x + 2/3)(x - 3/2) = 0 \leftarrow$ 1 or

$\therefore (3x + 2)(2x - 3) = 0 \leftarrow$ 2

$6x^2 - 5x - 6 = 0 \leftarrow$ this line is NOT necessary

OR

$\alpha\beta = -1 \leftarrow$ 1

$\alpha + \beta = 5/6 \leftarrow$ 1

$\left.\begin{array}{l} x^2 - 5/6\, x + 1 = 0 \\ 6x^2 - 5x + 6 = 0 \end{array}\right\}$ 1

if doing it this way the last line IS necessary

(b) (i) $y = x^3 - 3x + 3$

$\dfrac{dy}{dx} = 3x^2 - 3 \leftarrow$ 1

$\dfrac{d^2y}{dx^2} = 6x \leftarrow$ 1

At S.P.s $\left.\begin{array}{l} \dfrac{dy}{dx} = 0 \\ 3x^2 - 3 = 0 \end{array}\right\}$ 1 for stating this

$\therefore$ $x = 1$ or $x = -1$

1 $\left\{\begin{array}{l} y = 1 \\ y'' = 6 > 0 \end{array}\right.$ $\left\{\begin{array}{l} y = 5 \leftarrow 1 \\ y'' = -6 < 0 \end{array}\right.$

min T.P. at (1, 1) 1

MAX T.P. at (−1, 5) 1

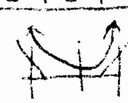
OR

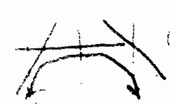
$x = 1$ or $x = -1$

$y = 1$ $y = 5$

x	0	1	2
y'	−	0	+

x	−2	−1	0
y'	+	0	−

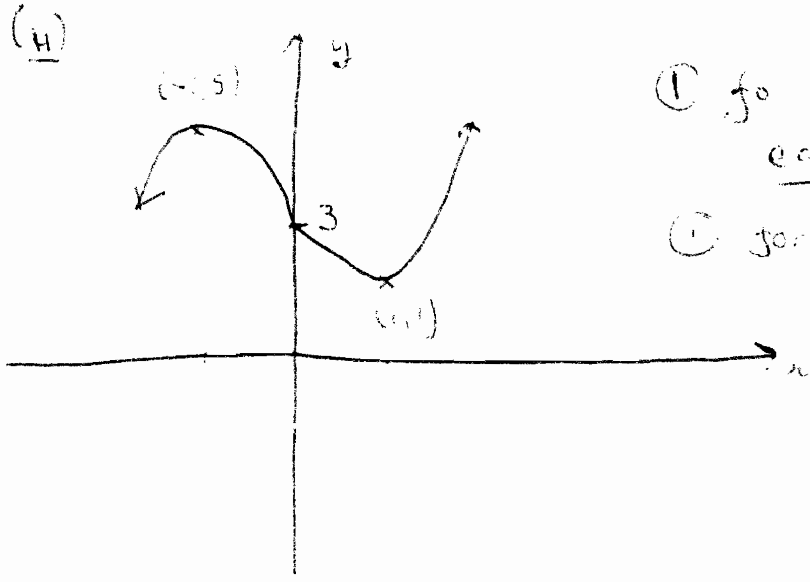

$\leftarrow$ 1


1

min T.P. at (1, 1) 1

MAX T.P. at (−1, 5) 1

(ii)



1 for showing and labelling each T.P.

1 for the point (0, 3)

QUESTION 4:

(a) $\cos^2 x = \frac{1}{2}$

$\cos x = \pm \frac{1}{\sqrt{2}}$

$\therefore x = 45^\circ$

award (2) for correct answer

award (1) only for including 135°

award (1) only for getting to step 2 and not getting 45°

(b)

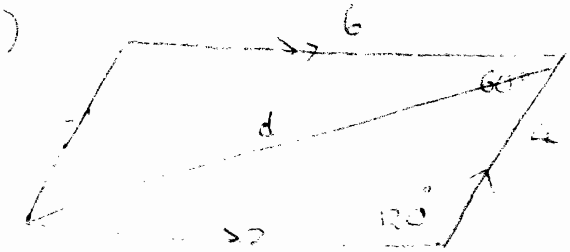
$3\sin\theta - 2 = 0$

$3\sin\theta = 2$

$\sin\theta = 2/3$ ← (1)

$\therefore \theta =$ or ← (2) 1 for each answer

(c) (i)



no MARKS FOR THIS

(ii) $d^2 = 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 120^\circ$ ← (1) for this

(1) for cos rule correct

$d = \sqrt{76} \approx$ ← (1)

(iii) $A = \frac{1}{2} \cdot 6 \cdot 4 \cdot \sin 120^\circ$ ← (1)

$= 6\sqrt{3}$

$A \doteqdot$ ← (1) for double area

ie only 1 mark if only half the pllgm

(d) EITHER

$(x^2 + 2)(x^2 - 1) = 0$ ← (1)

$\therefore x^2 = -2$ or $x^2 = 1$

no soln $x = \pm 1$ (2)

lose (1) if ... to ...

OR Let $u = x^2$

$\therefore u^2 + u - 2 = 0$

$\therefore (u + 2)(u - 1) = 0$ (1)

$\therefore x^2 = -2$ or $x^2 = 1$

no soln $x = \pm 1$

QUESTION 5

(a) $y = ax^2 + bx + c$

(i) Passes through $(0, 7)$

$\therefore \; c = 7$ (1)

(ii) $\dfrac{dy}{dx} = 2ax + b$

Turning Point at $(3, -2)$

$\therefore \; 2a(3) + b = 0$ (1)

$6a + b = 0$ — (1)

(iii) Passes through $(3, -2)$

$\therefore \; -2 = 9a + 3b + 7$ (1)

$\therefore \; 9a + 3b + 9 = 0$

$3a + b + 3 = 0 \quad (1)$

$6a + b = 0 \quad (2)$

$(2) - (1) \quad 3a - 3 = 0$

$a = 1$

$b = -6$

(2) 1 each

the process is important only penalise once for an incorrect step, not for losing a mark in both way

(b) (i) $P = 100{,}000(1.06) - D$ (1) MARK – not required to write $106{,}000 - D$

(ii) $P = (100{,}000(1.06) - D)1.06 - D$

$= 100{,}000(1.06)^2 - D(1 + 1.06)$ ← (1)

$= 112{,}360 - 2.06D$ ← (1)

(iii) $P_n = 100{,}000(1.06)^n - D(\underbrace{1 + 1.06 + \ldots + 1.06^{n-1}}_{S_n = \frac{1[(1.06)^n - 1]}{1.06 - 1}})$ ← (1) ← (1)

$= 100{,}000(1.06)^n - \dfrac{50D}{3}(1.06^n - 1)$ ← (1)

(iv) $P_{20} = 100{,}000(1.06)^{20} - \dfrac{75000}{3}(1.06^{20} - 1)$

1 MARK for $n = 20$

1 MARK for expression of ...

$= 320713.5472 - 18392.796$

$=$ [illegible]