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BAULKHAM HILLS HIGH SCHOOL

YEAR 12

YEAR 12 HALF YEARLY EXAMINATION

2006

MATHEMATICS
EXTENSION 1

Time allowed - Two hours
(Plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown. All marks shown are a guide only
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided.

QUESTION 1(START A NEW PAGE)

(a) Solve $\frac{x}{x-3} \geq 2$ 3

(b) Differentiate

i) $\tan(e^{5x})$ 2

ii) $\ln\left(\frac{2x+1}{x^2}\right)$ 2

(c) Find the acute angle between the lines $y = 2x-1$ and $x + 3y = 6$ 2

(d) Use Simpson's rule once to estimate 3

$$\int_1^2 \ln x^2 dx$$

QUESTION 2(START A NEW PAGE)

(a) Find the Cartesian equation for 2

$x = 5\cos t$
$y = 5\sin t$

(b) Given that $\sin\alpha = \frac{4}{7}$ and $\sin\beta = \frac{5}{13}$ α, β are acute, find the exact value of $\cos(\alpha+\beta)$ 2

(c) Find the equation of the normal to the curve $y = 2\log_e x$ at the point $(e,2)$ 2

(d) (i) Show that $\frac{1+\cos 2A}{\sin 2A} = \cot A$ 2

(ii) Hence find the exact value of $\cot 15°$ 1

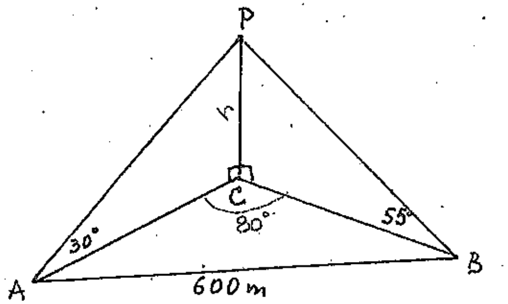
(e) Find the volume generated by $y = 2 + \cos x$ around the x-axis between $x = \frac{\pi}{4}$ and $\frac{\pi}{2}$ 4

QUESTION 3 (START A NEW PAGE)

(a) Prove by mathematical induction that **4**
$1\times 2^0+2\times 2^1+3\times 2^2+\ldots+n\times 2^{n-1}=1+(n-1)\times 2^n$ for all integers $n\geq 1$

(b) Differentiate $y=e^{x^3}$ and hence evaluate
$\int_{-1}^{1} x^2 e^{x^3}\,dx$ **3**

(c)



NOT TO SCALE

Two yachts A and B subtend an angle of 80° at the base C of a cliff. From yacht A the angle of elevation of the point P, h metres vertically above C, is 30°. From yacht B the angle of elevation of the point P is 55°. Yacht B is 600 metres from A.

(i) Show that $h^2=\dfrac{600^2}{\cot^2 30°+\cot^2 55°-2\cot 30°.\cot 55°.\cos 80°}$ **3**

(ii) Find h. **1**

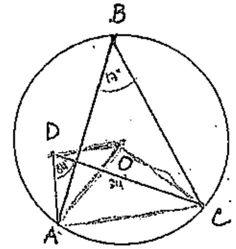
QUESTION 4 (START A NEW PAGE)

(a) $T(2t,t^2)$ is a point on the parabola $x^2=4y$ with focus S. P is the point which divides ST internally in the ratio 1:2.

(i) What are the coordinates of the focus S? **1**
(ii) Write down the coordinates of P in terms of t **2**
(iii) Hence show that as T moves on the parabola $x^2=4y$, the locus of P is the parabola $9x^2=12y-8$. **2**

QUESTION 4 continued

(b)



∠AOC = 34°
∠ADC = 34°
∴ ADOC lie on a circumference
∠'s subtended on equal arcs to circum one side
NOT TO SCALE
∴ ADOC must be concyclic

Points A, B and C lie on the circumference of a circle with centre O.
Point D lies inside the circle and $\angle ABC=17°$ and $\angle ADC=34°$. **3**
Prove that ADOC is a cyclic quadrilateral.

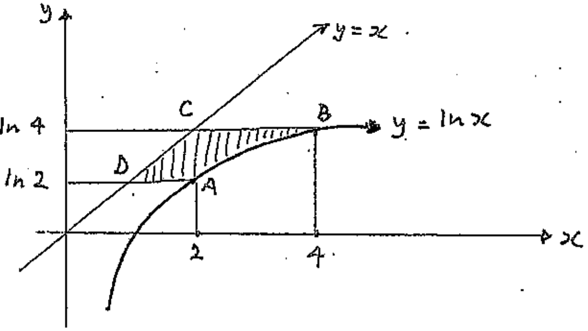
(c) Solve $3\sin\theta - 4\cos\theta = 3$ for $0°\leq\theta\leq 360°$. **4**

QUESTION 5 (START A NEW PAGE)

(a) A function is defined by the rule $F(x)=2x.e^{x}$

(i) Find the stationary point and determine its nature. **4**
(ii) Sketch the graph F(x) **2**

(b) The diagram below shows the graphs of $y=x$ and $y=\ln x$.
$A(2,\ln 2)$ and $B(4,\ln 4)$ are points on $y=\ln x$. C and D are points on $y=x$.
Find the shaded area ABCD. Assume BC and AD are parallel to the x-axis. **3**



NOT TO SCALE

(c) The rate of flow of water into, then out of a container is given by
$R=t(10-t)$ litres/minute.

(i) Find an expression for the volume, V litres, of water in the container at time t minutes assuming that the container is initially empty. **2**
(ii) Find the total time for the container to fill and then empty. **2**

QUESTION 6 (START A NEW PAGE)

(a) The velocity v m/s of an object at time t seconds is given by

$v = 3t^2 - 14t + 8.$

The object is initially 30 metres to the right of the origin.

(i) Find the initial acceleration of the particle 2
(ii) Find when the object is at rest 2
(iii) Find the minimum distance between the origin and the object during its motion 3

(b) The rate at which a body cools in air is proportional to the difference between the temperature, T, of the body and the constant surrounding temperature S. This can be expressed as

$$\frac{dT}{dt} = k(T - S)$$

where t is time in minutes and k is constant

(i) Show that $T = S + Be^{kt}$, where B is a constant is a solution of the above equation. 1

(ii) If a particular body cools from 100° Celsius to 80° Celsius in 30 minutes, find the temperature of the body after a further 30 minutes, given that the surrounding temperature remains constant at 25° Celsius. 3

QUESTION 7 (START A NEW PAGE)

(a) (i) State the largest positive domain for which $f(x) = x^2 - 2x + 3$ has an inverse function and sketch the curve for this domain. 2

(ii) Find the inverse function $f^{-1}(x)$ of $f(x) = x^2 - 2x + 3$ and state its domain 3

(iii) On the same set of axes draw a neat sketch of the inverse function 1

(iv) Evaluate $f\left(f^{-1}(8)\right)$ 1

QUESTION 7 continued

(b) A man borrows $15800 with monthly reducible interest of 8% p.a.
Let A_n be the amount owing at the end of n months, after the monthly repayment has been made.

(i) If the repayments are $M per month, show that after the second repayment he still owes $A_2 = 15800 \times \left(\frac{151}{150}\right)^2 - M\left(\frac{151}{150} + 1\right)$ 2

(ii) Show that after n repayments the amount owing is

$$A_n = 15800 \times \left(\frac{151}{150}\right)^n - M\left[\left(\frac{151}{150}\right)^{n-1} + \left(\frac{151}{150}\right)^{n-2} + \ldots + 1\right]$$ 1

(iii) If the repayments are $1260 per month, find the number of payments to repay all the loan. 3

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TOTAL (84)

Question 1 (12)

a) $\frac{x}{x-3} \ge 2 \quad \cdot (x-3)^2$ (3)

$x(x-3) \ge 2(x-3)^2$

$0 \ge (x-3)(x-6)$ (1)

$3 < x \le 6$

(1) (1)

b) i) $\frac{d}{dx}\tan(e^{5x}) = \sec^2(e^{5x}) \cdot 5e^{5x}$ (2) (1) (1)

ii) $\frac{d}{dx}\ln\left(\frac{2x+1}{x^2}\right) = \frac{x^2}{2x+1} \times \frac{2x^2-(2x-1)\cdot 2x}{x^4}$ (2) (1) (1)

$= \frac{-2x^2+2x}{(2x+1)\cdot x^2}$

c) $y = 2x-1 \qquad x+3y=6$ (2)

$m_1 = 2 \qquad m_2 = \frac{-1}{3}$

$\tan\theta = \frac{|m_1 - m_2|}{|1+m_1 \cdot m_2|} = \frac{|2+\frac{1}{3}|}{|1-\frac{2}{3}|} = 7$ (1)

$\therefore \theta = 81^\circ 52'$ (1)

d) $\int_1^2 \ln x^2 \, dx$ (3)

x	1	1.5	2
$\ln x^2$	0	0.8109	1.3863

(1)

$\therefore \int_1^2 \ln x^2 \, dx \doteq \frac{0.5}{3}(0+4\times 0.8109+1.3863)$ (1)

$\doteq 0.77165$ (1)

Question 2 (13)

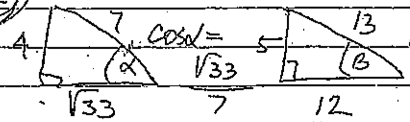
a) $x = 5\cos t$ (2)

$y = 5\sin t$

$x^2+y^2 = 5^2(\sin^2 t+\cos^2 t)$ (1)

$\therefore x^2+y^2 = 25$ (1)

b) Given $\sin\alpha = \frac{4}{7}$ $\sin\beta = \frac{5}{13}$ (2)



$\cos\alpha = \frac{\sqrt{33}}{7}$ $\cos\beta = \frac{12}{13}$

$\therefore \cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$= \frac{\sqrt{33}}{7} \cdot \frac{12}{13} - \frac{4}{7} \cdot \frac{5}{13}$ (1)

$= \frac{12\sqrt{33} - 20}{91}$ (1)

c) $y = 2\log_e x$ at $(e, 2)$ (2)

$\frac{dy}{dx} = \frac{2}{x} \therefore m = \frac{2}{e} \therefore m_{NORMAL} = -\frac{e}{2}$ (1)

$\therefore$ normal: $y-2 = -\frac{e}{2}(x-e)$ (1)

d) i) $LHS = \frac{1+\cos 2A}{\sin 2A} = \frac{1+2\cos^2 A - 1}{2\sin A \cdot \cos A}$ (1)

$= \frac{2\cos^2 A}{2\sin A\cos A} = \frac{\cos A}{\sin A} = \cot A$ (1) (3)

$= RHS \therefore$ proven

ii) $\cot 15^\circ = \frac{1+\cos 30^\circ}{\sin 30^\circ} = \frac{1+\frac{\sqrt{3}}{2}}{\ldots}$ (1)

Question 2 - cont.

e) $y = 2+\cos x$ (4)

$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2+\cos x)^2 dx$

$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4+4\cos x+\cos^2 x \, dx$ (1)

$\doteq \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4+4\cos x+\underbrace{\tfrac{1}{2}\cos 2x+\tfrac{1}{2}}_{(1)} dx$

$= \pi\left[4x+4\sin x+\frac{1}{4}\sin 2x+\frac{1}{2}x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ (1)

$= \pi\left[2\pi+4+0+\frac{\pi}{4} - \pi - \frac{4}{\sqrt{2}} - \frac{1}{4} - \frac{\pi}{8}\right]$

$= \pi\left(\frac{9\pi}{8} - 2\sqrt{2} + 3\frac{3}{4}\right)$ (1)

Question 3 (11)

a) ~~STEP 1 prove true for $n=1$~~ (4)

$LHS = 1\times 2^{1-1} = 1$

$RHS = 1+(1-1)\times 2^1 = 1$

$LHS = RHS \therefore$ true (1)

STEP 2 $\therefore$ assume true for $n=k$

$\therefore 1\times 2^0+2\times 2^1+3\times 2^2+\ldots+k(2^{k-1}) = 1+(k-1)\cdot 2^k$ (1)

then prove for $n = k+1$

$\therefore 1\times 2^0 \ldots (k+1)2^{k+1-1} = 1+(k+1-1)\cdot 2^{k+1}$

$= \boxed{1+k\cdot 2^{k+1}}$

Proof

$LHS = \underbrace{1\times 2^0+2\times 2^1+\ldots+k\times 2^{k-1}}_{\text{assumption}} + (k+1)\cdot 2^{k+1-1}$

$= 1+(k-1)\cdot 2^k + (k+1)\cdot 2^{k+1-1}$ (1)

$= 1+k\cdot 2^k - 2^k + k\cdot 2^k + 2^k$

$= 1+2\times k\cdot 2^k = 1+k\cdot 2^{k+1}$

$= RHS \therefore$ proven

Proven true for $n=1$, if true for $n=k$ and proven true for $n=k+1$, since proven true for $n=1 \therefore$ true for $n=2,3,\ldots$ (1)

$\therefore$ true for all $n \ge 1$.

b) $y = e^{x^3} \therefore \frac{dy}{dx} = 3x^2 \cdot e^{x^3}$ (1) (3)

$\therefore y = \int 3x^2 \cdot e^{x^3} dx$

$\therefore \int_{-1}^{1} x^2 \cdot e^{x^3} dx = \frac{1}{3}y = \frac{1}{3}e^{x^3}$ (1)

$= \frac{1}{3}e - \frac{1}{3}e^{-1}$ (1)

c) i) $\cot 30^\circ = \frac{AC}{h} \therefore AC = h\cdot\cot 30^\circ$ (4)

$\cot 55^\circ = \frac{BC}{h} \therefore BC = h\cdot\cot 55^\circ$ (1)

and $600^2 = AC^2+BC^2-2\cdot AC\cdot BC\cdot\cos 80^\circ$ (1)

$600^2 = h^2\cot^2 30^\circ + h^2\cot^2 55^\circ - 2\cdot h\cot 30^\circ \cot 55^\circ \cos 80^\circ$ (1)

$\therefore h^2 = \frac{600^2}{\cot^2 30^\circ+\cot^2 55^\circ - 2\cot 30^\circ \cot 55^\circ \cos 80^\circ}$

ii) $h = 342.489\text{m}$ (1)

Question 4 (11)

a) $T(2t, t^2)$ $x^2 = 4y$

i) $x^2 = 4ay \therefore a = 1$

$\therefore S(0,1)$ (1)

ii) $S(0,1)$ $T(2t, t^2)$

1 : 2

$P\left(\frac{2t}{3}, \frac{2+t^2}{3}\right)$ (1) (1)

Locus of P

$x = \frac{2t}{3}$

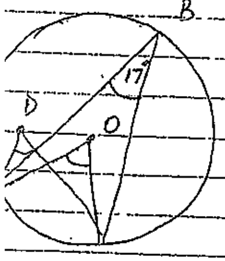
$y = \frac{2+t^2}{3}$

$\therefore t = \frac{3x}{2}$ (1) $\therefore y = \frac{2+\left(\frac{3x}{2}\right)^2}{3}$ (1)

$\therefore 3y = 2 + \frac{9x^2}{4}$

$12y - 8 = 9x^2 \therefore$ shown

b)



(Since) $\angle ABC = 17° \therefore \angle AOC = 34°$

(angle at the centre is double (1) the angle at the circumf.)

But $\angle ADC = 34°$ and both $\angle ADC$ and AOC are standing (1) on the same arc and they are equal $\therefore ADOC$ is cyclic

c) Solve $3\sin\theta - 4\cos\theta = 3$

$0° \le \theta \le 360°$

$A = \sqrt{3^2+4^2} = 5$ (1)

$3\sin\theta - 4\cos\theta = A\sin(\theta - \alpha)$

$\alpha \therefore \tan\alpha = \frac{4}{3} \therefore \alpha = 53°8'$ (1)

$\therefore 5\sin(\theta - 53°8') = 3$

$\sin(\theta - 53°8') = \frac{3}{5}$

$\theta - 53°8' = 36°52'$ $\theta - 53°8' = 143°8'$

$\therefore \theta = 90°$ (1) or $\theta = 196°16'$ (1)

Question 5 (13)

a) $F(x) = 2x \cdot e^x$

i) $F'(x) = 2e^x + 2xe^x$ (1)

$0 = 2e^x(1+x)$

$2e^x \neq 0 \therefore$ at $x = -1$ (1)

$\therefore y = \frac{-2}{e}$ (1) st. point

$\boxed{\text{nature}}$ $F''(x) = 2e^x + 2e^x + 2xe^x$

$F''(x) = 4e^x + 2xe^x$ (1)

$F''(\text{at } x=-1) = 0.74 > 0 \therefore \text{min}$

iii) Sketch: x-int: (0,0)

$e^x > 0 \therefore y = 0$ asympt when $x \to -\infty$

ii) pt. of infl. $F''(x) = 0$

$4e^x + 2xe^x = 0$

$2e^x(2+x) = 0$

$x = -2$

x	-3	-2	-1.5
F''	-0.04	0	$+0.22$

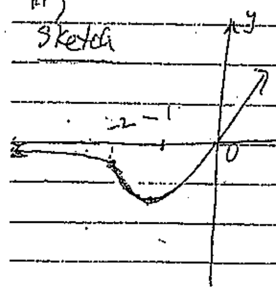
$\therefore$ pt. of infl $\left(-2, \frac{-4}{e^2}\right)$

not necessary to show

iii) Sketch

x-int. + shape (1)

asymptote (1)



b) $y = \ln x \therefore e^y = x$

$A_{shaded} = \int_{\ln 2}^{\ln 4} e^y\,dy - \int_{\ln 2}^{\ln 4} y\,dy$ (1)

$= \left[e^y\right]_{\ln 2}^{\ln 4}$ (1) $- \left[\frac{y^2}{2}\right]_{\ln 2}^{\ln 4}$ (1)

$= (4-2) - (0.72) = 1.28$ (2d.p)

or in exact $A = 2 - \frac{1}{2} \times \ln 2 \times \ln 8$

or $A = 2 - \frac{3}{2}(\ln 2)^2$

(by using $A = \int_{\ln 2}^{\ln 4} e^y\,dy - A_{\substack{TRAPEZIUM \\ ADCB}}$)

c) $R = t(10-t)$ in Litres/min

i) $R = \frac{dV}{dt} \therefore V = \int t(10-t)\,dt$

$V = \int 10t - t^2\,dt = 5t^2 - \frac{t^3}{3} + C$ (1)

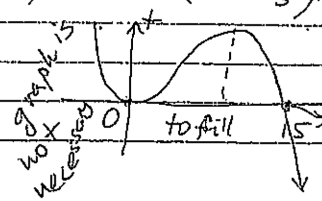
when $t = 0 \therefore V = 0 \therefore C = 0$ (1)

$\therefore \boxed{V = 5t^2 - \frac{t^3}{3}}$

ii) $V = t^2\left(5 - \frac{t}{3}\right)$ $V = 0 \therefore t = 0$, $t = 15$ (1)

$\therefore t = 15$ (1)

graph is not necessary



Question 6 (11)

a) $v = 3t^2 - 14t + 8$

at $t = 0$, $x = +30m$

i) $a = \dot{v} = 6t - 14$ (1)

$a(t=0) = -14\, m/s^2$ (1)

ii) rest ∴ $v = 0 = 3t^2 - 14t + 8$ (1)

$0 = (3t-2)(t-4)$

$t = \frac{2}{3}$, $t = 4$ (1)

iii) ~~min distance ∴ v = 0~~

~~∴ $t = \frac{2}{3}$, $t = 4$~~

~~$x = \int v\,dt = \int 3t^2 - 14t + 8\,dt$~~

$x = t^3 - 7t^2 + 8t + c$

$30 = 0 - 0 + 0 + c$ ∴ $c = 30$

∴ $x = t^3 - 7t^2 + 8t + 30$ (1)

$x(t = \frac{2}{3}) = 35\frac{23}{27}$

$x(t = 4) = 14$ (1)

∴ min distance is 14m (1)

b) $\frac{dT}{dt} = k(T - S)$

i) if $T = S + Be^{kt}$ is a solution

∴ $\frac{dT}{dt} = B \times k \times e^{kt}$ (1) but from $T - S = Be^{kt}$

∴ $\frac{dT}{dt} = k(T - S)$ ∴ shown

ii) $t = 0$ $T = 100°C$

$t = 30$ mins $T = 80°C$

$t = 60$ (further 30) ∴ $T = ?$

$T = S + Be^{kt}$ & $S = 25°C$

$100 = 25 + B \cdot e^{0}$ ∴ $B = 75$ (1)

$80 = 25 + 75 \cdot e^{k \times 30}$

$\frac{11}{15} = e^{30k}$

$\ln\left(\frac{11}{15}\right) = 30k$

∴ $k = -0.010338$ (1)

∴ $T = 25 + 75 \cdot e^{-0.010338 \times 60}$

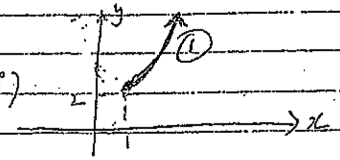
$T = 65.33°C$ (1)

Question 7 (13)

a) $f(x) = x^2 - 2x + 3$ (7)

axis ∴ $x = \frac{2}{2} = 1$

∴ vertex $(1, 2)$

i) 

i) largest domain ∴ $x \geq 1$ (1)

ii) f: $y = x^2 - 2x + 3$

$y - 3 + 1 = x^2 - 2x + 1$

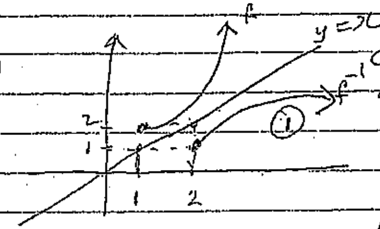
$y - 2 = (x - 1)^2$ (1)

$\sqrt{y - 2} = x - 1$

∴ $x = \sqrt{y-2} + 1$

∴ f^{-1}: $y = \sqrt{x - 2} + 1$ (1)

Domain of f^{-1}: $x \geq 2$ (1)

iii) 

could be drawn on the number plane in part i)

iv) $f(f^{-1}(8)) = 8$ (1)

b) $A_1 = 15800\left(1 + \frac{8 \div 12}{100}\right) - M$

(2) $A_1 = 15800\left(1 + \frac{1}{150}\right) - M$ (1)

∴ $A_1 = 15800 \times \frac{151}{150} - M$

∴ $A_2 = A_1 \times \frac{151}{150} - M$ (1)

$= 15800 \times \left(\frac{151}{150}\right)^2 - M\left(\frac{151}{150} + 1\right)$

∴ $A_3 = A_2 \times \frac{151}{150} - M = 15800\left(\frac{151}{150}\right)^3 - M($

(1) ii) $A_n = A_{n-1} \times \frac{151}{150} - M$ (1)

$= 15800 \times \left(\frac{151}{150}\right)^n - M \times \left(\frac{151}{150}\right)^{n-1} - \dots - M$

$= 15800 \left(\frac{151}{150}\right)^n - M\left[\left(\frac{151}{150}\right)^{n-1} + \dots + 1\right]$

(3) iii) $A_n = 0$, $M = \$1260$

$0 = 15800\left(\frac{151}{150}\right)^n - 1260\left[1 \times \frac{\left(\frac{151}{150}\right)^n - 1}{\frac{151}{150} - 1}\right]$ (1)

$0 = 15800\left(\frac{151}{150}\right)^n - 1260 \times S_n$

$0 = 15800\left(\frac{151}{150}\right)^n - 1260 \times \frac{\left(\frac{151}{150}\right)^n - 1}{\frac{1}{150}}$

$1260 \times 150 \times \left[\left(\frac{151}{150}\right)^n - 1\right] = 15800\left(\frac{151}{150}\right)^n$

$189000\left(\frac{151}{150}\right)^n - 15800\left(\frac{151}{150}\right)^n = 189000$ (1)

$\left(\frac{151}{150}\right)^n \left[189000 - 15800\right] = 189000$

$\left(\frac{151}{150}\right)^n = 1\frac{79}{866}$

$\ln\left[\frac{151}{150}\right]^n = \ln 1\frac{79}{866}$

$n = \frac{\ln\left(1\frac{79}{866}\right)}{\ln\left(\frac{151}{150}\right)}$ $\frac{945}{866}$ $\frac{\ln 1.0912}{\ln\left(\frac{151}{150}\right)}$ (1)

∴ $n = 13.1386$ repayments