**BAULKHAM HILLS HIGH SCHOOL**

**YEAR 12**

**HALF YEARLY EXAMINATION**

# 2007

# MATHEMATICS
## EXTENSION 1

**GENERAL INSTRUCTIONS:**

- Attempt **ALL** questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet.
- Marks indicated for each question are only a guide and could change.



**QUESTION 1**

**Marks**

(a) Solve $2\log x = \log 6x$ — **2**

(b) (i) Differentiate $\log_e(\cos x)$ — **1**

(ii) Hence evaluate $\int_0^{\frac{\pi}{3}} \tan x \, dx$ — **3**

(c) Given the points A(−1,4) and B(2,−3), find the coordinates of the point P(x,y) which divides the interval AB externally in the ratio 2 : 3. — **3**

(d) Solve $\frac{x}{x+2} \geq 4$. — **3**

**QUESTION 2**

(a) Solve for $0 \leq \vartheta \leq 360°$ :

(i) $\sin 2\vartheta - \cos\vartheta = 0$. — **3**

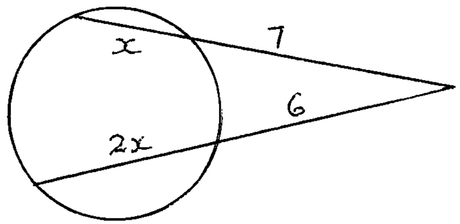
(ii) $\sin\vartheta + \sqrt{3}\cos\vartheta = 1$. — **3**

(b) (i) Show that the derivative of $y = \sec x$ is $\tan x \sec x$. — **3**

(ii) Hence find the equation of the tangent to the curve $y = \sec x$ at $\frac{\pi}{6}$. — **3**

**QUESTION 3**

(a) Find x in the following: — **2**



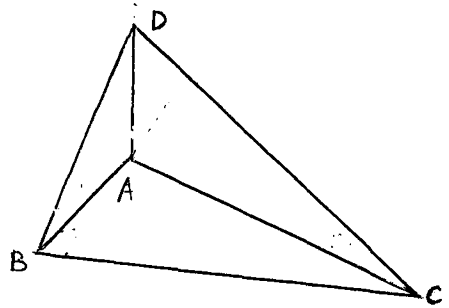

(b) Find the volume of the solid generated when the area between the curve $y = \sin x$ the x axis and the lines $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x axis. — **3**

(c) The polynomial $P(x) = 2x^3 + ax + b$ has a root at $x = 1$ and when divided by $x + 2$ the remainder is 4. Find the values of a and b. — **3**

**QUESTION 3 (Continued)**

**Marks**

**(d)** From a point B on a wharf it is noted that the cruise liner Queen Mary II bears due north and its flagpole at D has an angle of elevation of 17°. From another point C 280 metres due east of B the angle of elevation to the flagpole is 12°.




**(i)** Show that the height (h) of the flagpole above the wharf is given by:

$$h = 280\,(\cot^2 12 - \cot^2 17)^{-1/2}$$ **3**

**(ii)** Find h the height of the flagpole above the wharf. **1**

**QUESTION 4**

**(a)** Find $\int \sin x \cos^3 x \, dx$. **2**

**(b)** If $\log_a 3 = x$ and $\log_a 4 = y$, find $\log_a 6$ in terms of x and y. **2**

**(c)** Prove by mathematical induction that $7^n - 3^n$ is divisible by 4 for all positive integers n where $n \geq 1$. **3**

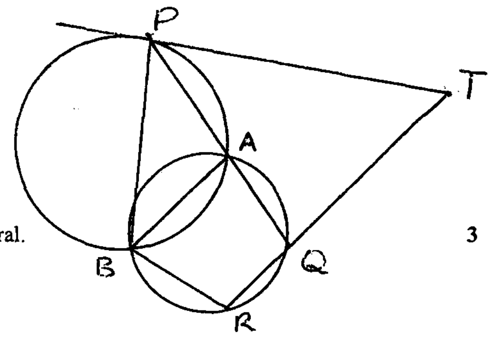
**(d)** Use Newton's method once to find a better approximation for the root of $f(x) = \ln x - x^3 + 2$. Let $x = 2$ be the first estimate for the root. **3**

**(e)** Find the exact value of cos 75°. **2**



**QUESTION 5**

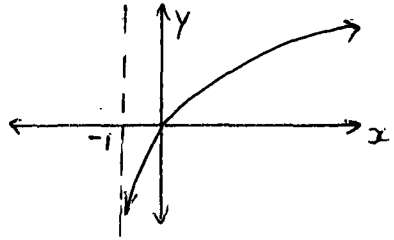
**Marks**




**(a)** TP is a tangent at P.

Prove TRBP is a cyclic quadrilateral. **3**

**(b)** Below is the graph of $y = \ln(x + 1)$.




**(i)** Find the gradient of the tangent to the curve $y = \ln(x + 1)$ at the origin. **1**

**(ii)** Hence find the acute angle between $y = \ln(x + 1)$ and the line $y = 4x$ at $x = 0$. **2**

**(iii)** For what range of values for m (where $m>0$) will $mx - \ln(x+1) = 0$ have 2 solutions. **1**

**(c)** When the temperature T of a certain body is 65°C, it is cooling at a rate of 1°C per minute. Assuming Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - S),$$

where T is the temperature of the body at time t minutes and S is that of the surrounding medium and k is a constant.

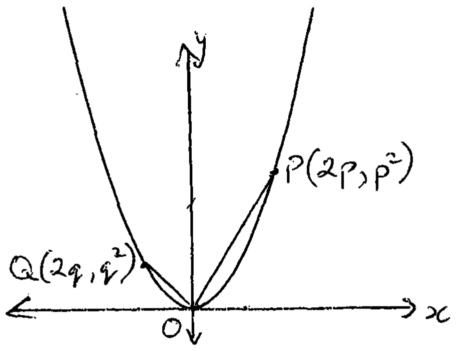
**(i)** Verify that $T = S + Ae^{-kt}$ is a solution of the differential equation above. **1**

**(ii)** Show that $k = 0.02$ given that $S = 15°C$. **1**

**(iii)** Find the temperature of the body after 20 minutes. **1**

**(iv)** How much more time must elapse before the temperature of the body reaches 35°C. **2**

**QUESTION 6**




Marks

**(a)** **(i)** What is the cartesian equation represented by the parametric equations:
$x = 2t \quad y = t^2$ **1**

**(ii)** For 2 points $P(2p,p^2)$ and $Q(2q,q^2)$ on the parabola the chord joining them subtends an angle of 90° at the origin (i.e. $\angle QOP = 90°$). By finding the gradients of OP and OQ show that $pq = -4$. **2**

**(iii)** Show clearly that the coordinates of R such that OPRQ is a rectangle is given by $(2p + 2q, p^2 + q^2)$. **2**

**(iv)** Find the locus of R as P and Q move on the parabola such that OPRQ is always a rectangle. **2**

**(b)** **(i)** Sketch $y = \dfrac{1}{|x-2|}$ without using calculus. **2**

**(ii)** Hence or otherwise solve $\dfrac{1}{|x-2|} < 1$. **2**

**QUESTION 7**

**(a)** If $f(x) = -x^2 e^{-x}$

**(i)** Find the stationary points on the curve and determine their nature. **4**

**(ii)** Sketch the curve. **2**

**(iii)** If $f''(x) = -e^{-x}(x^2 - 4x + 2)$ for what value(s) of x will $f'(x) > 0$ and $f''(x) < 0$. **3**

**(b)** **(i)** Factorise $a^3 - b^3$. **1**

**(ii)** Hence show $\lim_{x \to 0} \dfrac{(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x} = \dfrac{\sqrt[3]{2}}{6}$ **3**

Solutions /84

## Question 1 - 12 marks

1(a) $2\log x = \log 6x$ [1]

$\therefore \log x^2 = \log 6x$

$x^2 = 6x$ ---- (1)

$x(x-6) = 0$

$\therefore x = 0, 6$ but $x > 0$

$\therefore x = 6$ ..... (1)

b)(i) $y = \log_e(\cos x)$ [4]

$y' = \dfrac{-\sin x}{\cos x}$ } (1)

$= -\tan x$

(ii) (1) ... $\int_0^{\frac{\pi}{3}} \tan x = \left[-\log_e(\cos x)\right]_0^{\frac{\pi}{3}}$

(1) ... $= -\log_e(\cos\frac{\pi}{3}) + \log_e(\cos 0)$

$= -\log_e(\frac{1}{2}) + \log_e 1$

$= -\log(\frac{1}{2})$

$= \log 2$ } (1)

$= 0.693...$

c) $A(-1,4)$ $B(2,-3)$ [3] (1) ← $-2:3$

$\left(\dfrac{-2(2)+3(-1)}{-2+3}, \dfrac{-2(-3)+3(4)}{-2+3}\right)$

$= (-7, 18)$ (1) (1)

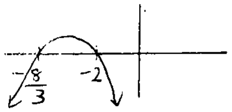
d) $\dfrac{x}{x+2} \geq 4$ $x \neq -2$

$\dfrac{x}{x+2} - 4 \geq 0$

(1) ... $x(x+2) - 4(x+2)^2 \geq 0$ [3]

$(x+2)(x - 4(x+2)) \geq 0$

$(x+2)(-3x-8) \geq 0$ ..(1)




$\therefore -\frac{8}{3} \leq x < -2$ --(1)

[lose (1) if $-\frac{8}{3} \leq x \leq -2$.]

## Question 2 - 12 marks

a) (i) $\sin 2\theta - \cos\theta = 0$ [3]

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ $\sin\theta = \frac{1}{2}$

$\therefore \theta = 90°, 270°, 30°, 150°$

(1) (1)

(ii) [3] $\sin\theta + \sqrt{3}\cos\theta = 1$

let $A\sin(\theta + \alpha) = \sin\theta + \sqrt{3}\cos\theta$

$A\sin\theta\cos\alpha + A\cos\theta\sin\alpha =$ "

$\therefore A\cos\alpha = 1$ $A\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}$ $A = \sqrt{1^2 + (\sqrt{3})^2}$

(1) ... $\alpha = 60°$ $A = 2$ --(1)

$\therefore 2\sin(\theta + 60°) = 1$

$\sin(\theta + 60°) = \frac{1}{2}$

$\theta + 60° = 30°, 150°$

$\theta = -30°, 90°$

$\theta = 330°, 90°$ -- (1)

$\therefore h = \dfrac{280}{\sqrt{(\cot^2 12 - \cot^2 17)}}$

(1) $h = 280(\cot^2 12 - \cot^2 17)^{-\frac{1}{2}}$

(ii) $82.8$ m. (1)

## Question 4 12 marks

a) [2] $\int \sin x \cos^3 x \, dx$

$\dfrac{d}{dx}(\cos x)^4 = -4\sin x(\cos x)^3$ (1)

$\therefore \int \sin x \cos^3 x \, dx$

$= -\frac{1}{4}\cos^4 x + c$ --(1)

b) $\log_a 3 = x$ $\log_a 4 = y$

$\log_a 6 = \log_a 3 \times \sqrt{4}$ ---(1)

$= \log_a 3 + \log_a 4^{\frac{1}{2}}$

$= \log_a 3 + \frac{1}{2}\log_a 4$

$= x + \frac{y}{2}$ (1)

c) $7^n - 3^n = 4M$

Step 1. prove true for $n=1$

$\therefore 7^1 - 3^1 = 4$ ✓ ---(1)

Step 2 Assume true for $n = k$ ie $\dfrac{7^k - 3^k}{4} = M$ (integer)

ie $7^k = 4M + 3^k$ --(A)

Step 3. Prove true for $n = k+1$

ie now $7^{k+1} - 3^{k+1} = 7(7^k) - 3(3^k)$ (1)

from (A) $= 7(4M + 3^k) - 3(3^k)$

$= 28M + 7(3^k) - 3(3^k)$

(1) { $= 28M + 4(3^k)$

$= 4(7M + 3^k)$ }

which is ÷ by 4.

Step 4 Proved true for $n=1$ assume true for $n = k$ & proven true for $n = k+1$ $\therefore$ If true for $n=1$ true for $n=2$, $n=3$ & for all $n$ by mathematical ind'n.

d) $a_1 = 2 - \dfrac{f(2)}{f'(2)}$

$f'(x) = \frac{1}{x} - 3x^2$

$f(2) = \ln 2 - 8 + 2$

$= \ln 2 - 6$

$= -5.3$ ....... (1)

$f'(2) = \frac{1}{2} - 12$

$= -11.5$ .... (1)

lose 1 otherwise $\therefore a_1 = 2 - \dfrac{-5.3}{-11.5}$

$= 1.54$ (1)

e) $\cos(75°) = \cos(45 + 30)$

(1) $= \cos 45\cos 30 - \sin 45\sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ or $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ } (1)

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2 b) (i) $y = \sec x$ [3]

$y = (\cos x)^{-1}$ ---(1)

$y' = -1(\cos x)^{-2} \times -\sin x$ (1)

(1) { $= \dfrac{\sin x}{\cos^2 x}$

$= \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x}$

$= \tan x \cdot \sec x$ }

(ii) [3] $y = \sec x$

at $x = \frac{\pi}{6}$ $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$ --(1)

at $x = \frac{\pi}{6}$ $y' = \tan\frac{\pi}{6}\sec\frac{\pi}{6}$

$= \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}$

$= \frac{2}{3}$ --- (1)

Equation of tangent

$y - \frac{2}{\sqrt{3}} = \frac{2}{3}(x - \frac{\pi}{6})$ } (1)

$y = 2x - \pi + 2$

## Question 3 - 12 marks

a) [2] $7(7+x) = 6(2x+6)$ (1)

$49 + 7x = 12x + 36$

$13 = 5x$

$x = \frac{13}{5}$ --- (1)

b) [3] $V = \pi\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

$V = \pi\int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos 2x$ -(1)

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$ (1)

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{\sin 2(\frac{\pi}{4})}{2} - 0\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{\pi^2}{8} - \frac{\pi}{4}$ ..... (1)

(or $0.448...$)

c) $P(x) = 2x^3 + ax + b$ [3]

$P(1) = 0$

(1) ... $2 + a + b = 0$ --- (A)

$P(-2) = 4$

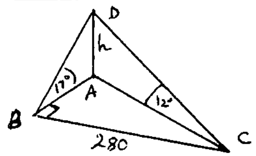
(1) { $-16 - 2a + b = 4$

$-2a + b = 20$ -- (B) }

Solving (A) + (B) simult.

$a = -\frac{22}{3}$, $b = \frac{16}{3}$ --(1)

3d)




(1) { $\cot 17° = \frac{AB}{h}$ → $h\cot 17° = AB$

$\cot 12° = \frac{AC}{h}$ → $AC = h\cot 12°$ }

$h^2\cot^2 12° = h^2\cot^2 17 + 280$

(1) $\therefore h^2(\cot^2 12 - \cot^2 17) = 280$

## Question 5 - 12 marks

a) Let $\angle AQT = x°$

$\therefore \angle ABR = x°$ (Exterior $\angle$ of a cyclic quad = interior opposite $\angle$) (1)

let $\angle TPA = y°$

$\therefore \angle PBA = y°$ (Angle between tangent & chord = angle in alternate segment) ---(1)

$\therefore \angle PBR = x° + y°$

$\angle PTQ = 180 - (x + y)$ (1) Angle sum of $\Delta$

$\therefore \angle PBR + \angle PTQ$

$= x + y + 180 - (x + y)$

$= 180°$

$\therefore TPBR$ is a cyclic quad as opposite $\angle$'s are supplementary (lose 1 if no conclusion)

b)(i) [4] $y = \ln(x+1)$

$y' = \dfrac{1}{x+1}$

at $x = 0$ $y' = \frac{1}{0+1}$

$y' = 1$ --- (1)

(ii) $m_1 = 1$ $m_2 = 4$

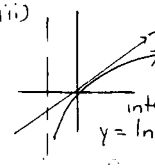
$\tan\theta = \dfrac{|m_1 - m_2|}{1 + m_1 m_2}$

$= \dfrac{|4-1|}{1 + 4 \times 1}$ (1)

$= \frac{3}{5}$

$\theta = 30° 58'$ --(1)

(iii)




$y = mx$ will intersect twice with $y = \ln(x+1)$ if (1)

$m \neq 1$

c) (i) $\dfrac{dT}{dt} = -k(T - S)$ --(A)

$T = S + Ae^{-kt}$ --(B)

show (B) is a solution to (A)

$\dfrac{dT}{dt} = -kAe^{-kt}$ ... } (1)

from (B) $T - S = Ae^{-kt}$

$\therefore \dfrac{dT}{dt} = -k(T - S)$

$\therefore$ it's a solution

(ii) when $T = 65$ $\dfrac{dT}{dt} = -1$

$\therefore -1 = -k(65 - 15)$

$\therefore k = \frac{1}{50}$ (1)

$k = 0.02$.

(iii) When $T = 20$

$T = 15 + Ae^{-0.02t}$

$65 = 15 + Ae^0$

$\therefore A = 50$ (1)

$T = 15 + 50e^{-0.02t}$

$= 48.5°C$

(iv) find t when I = 35

$35 = 15 + 50e^{-0.02t}$

$\frac{20}{50} = e^{-0.02t}$

$\therefore \ln(0.4) = -0.02t$

$t = \frac{\ln(0.4)}{-0.02}$ (1)

$= 45.8$ min

$\therefore$ A further 25.8 mins. (1)

## Question 6 - 11 marks

a) (i) $x = 2t \quad y = t^2$

[1] $t = \frac{x}{2} \quad \therefore y = \left(\frac{x}{2}\right)^2$

$y = \frac{x^2}{4}$ (1)

$x^2 = 4y$

(ii) $m_{OP} = \frac{p^2 - 0}{2p - 0} = \frac{p}{2}$

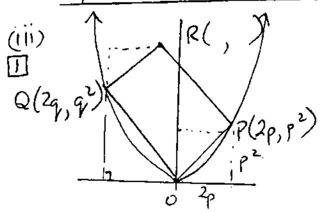
[2] similarly $m_{OQ} = \frac{q}{2}$ (1)

$m_{OP} \times m_{OQ} = -1 \quad \therefore \frac{p}{2} \times \frac{q}{2} = -1$ (1)

$\therefore pq = -4$

(iii)

[1]




From Q $\uparrow p^2$ then $\rightarrow 2p$ (could use midpoint)

ie $R(2q + 2p,\ q^2 + p^2)$
(1) (1)

(iv) $x = 2p + 2q \quad y = p^2 + q^2$

$\therefore p + q = \frac{x}{2}$ (1) $\quad y = (p+q)^2 - 2pq$

$\therefore y = \left(\frac{x}{2}\right)^2 - 2(-4)$ (1)

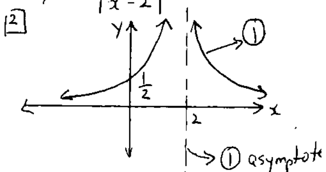
$y = \frac{x^2}{4} + 8$

$4y = x^2 + 32$

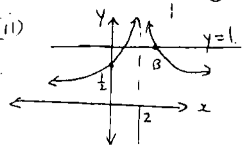
$x^2 - 4y + 32 = 0$

b) $y = \frac{1}{|x-2|}$

[2]




(ii)




need A & B

at A $\quad -\frac{1}{x-2} = 1$

$-1 = x - 2$

$x = 1$

at B $\quad \frac{1}{x-2} = 1$

$x - 2 = 1$

$x = 3$

$\therefore \frac{1}{|x-2|} < 1$

$\therefore x < 1 \quad x > 3$
(1) (1)

7a) Question 7 - 13 marks

$f(x) = -x^2 e^{-x}$

(i) $f'(x) = -2xe^{-x} + -e^{-x}\cdot(-x^2)$

$= -2xe^{-x} + x^2e^{-x}$

$= -xe^{-x}(2 - x)$ (1)

$x = 0 \qquad x = 2$

$y = 0 \qquad y = \frac{-4}{e^2}$ (1)

test

$x = 0$

| $x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $f'(x)$ | $e_{>0}^{(3)}$ | $0$ | $-1e^{-1}(1)$ $<0$ |
| | ↗ | ↔ | ↘ |

(1)

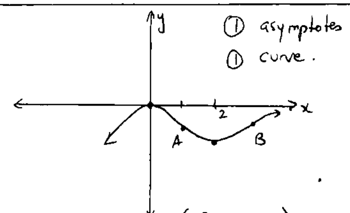
$\therefore$ Max.

test $x = 2$

| $x$ | $1$ | $2$ | $3$ |
|---|---|---|---|
| $f'(x)$ | $<0$ | $0$ | $-3e^{-3}(-1)$ $>0$ |
| | ↘ | ↔ | ↗ |

(1)

$\therefore$ Min.




(1) asymptotes
(1) curve.

(ii) $f''(x) = -e^{-x}(x^2 - 4x + 2)$

$f''(x) = 0$ when $x^2 - 4x + 2 = 0$

$\therefore x = \frac{4 \pm \sqrt{16 - 4\times1\times2}}{2\times1}$

$= \frac{4 \pm 2\sqrt{2}}{2}$

$= 2 \pm \sqrt{2}$ (1)

at A $x = 2 - \sqrt{2}$ $\quad$ at B $x = 2 + \sqrt{2}$

$f'(x) > 0$ when $f(x)$ is increasing

& $f''(x) < 0$ when $f(x)$ is concave down

This occurs when

$x < 0$ and $x > 2 + \sqrt{2}$
(1) (1)

b (i) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ (1)

(ii) $a - b = \frac{a^3 - b^3}{a^2 + ab + b^2}$

let $a = (2+x)^{\frac{1}{3}}$ and $b = 2^{\frac{1}{3}}$

$(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}} = \frac{(2+x) - 2}{(2+x)^{\frac{2}{3}} + (2+x)^{\frac{1}{3}}2^{\frac{1}{3}} + 2^{\frac{2}{3}}}$ (1)

$(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}} = \frac{x}{(2+x)^{2/3} + (2+x)^{\frac{1}{3}}2^{\frac{1}{3}} + 2^{\frac{2}{3}}}$

$\therefore \frac{(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x} = \frac{1}{(2+x)^{2/3} + (2+x)^{\frac{1}{3}}2^{\frac{1}{3}} + 2^{\frac{2}{3}}}$

$\therefore \lim_{x\to0} \frac{(2+x)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x}$

$= \lim_{x\to0} \frac{1}{(2+x)^{2/3} + (2+x)^{\frac{1}{3}}\cdot2^{\frac{1}{3}} + 2^{2/3}}$ (1)

$= \frac{1}{(2+0)^{\frac{2}{3}} + (2+0)^{\frac{1}{3}}\cdot2^{\frac{1}{3}} + 2^{\frac{2}{3}}}$

$= \frac{1}{2^{\frac{2}{3}} + 2^{\frac{2}{3}} + 2^{2/3}}$

$= \frac{1}{3\times2^{\frac{2}{3}}}$

$= \frac{1}{3} \times 2^{-2/3}$

$= \frac{1}{3} \times 2^{-\frac{2}{3}} \times \frac{2}{2}$ -- (1)

$= \frac{...}{3 \quad 2^{\frac{1}{3}} \quad \sqrt[3]{2}}$

# STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad if \quad n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$