NAME: ________________________________

TEACHER: ________________________________

BAULKHAM HILLS HIGH SCHOOL

YEAR 12

HALF YEARLY EXAMINATION

2008

MATHEMATICS

EXTENSION 1

GENERAL INSTRUCTIONS:

- Attempt **ALL** questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet.
- Marks indicated for each question are only a guide and could change.

QUESTION 1 **Marks**

(a) If $P(x) = x^3 - 2x^2 + ax + 4$ is divisible by $(x+2)$, what is the value of a? **1**

(b) Differentiate $\log_e x^2$. **2**

(c) Sketch the graph $y = \dfrac{4}{x-1}$ **4**

Hence or otherwise solve $x + 2 < \dfrac{4}{x-1}$.

(d) Find the solution for x if $\cos 2x = \cos x \quad 0 \le x \le 2\pi$ **3**

(e) Find $\int \sin^2 x \, dx$. **2**

QUESTION 2 **(Start a new page)**

(a) The equation $3x^3 - 4x^2 + 2x + 1 = 0$ has roots α, β and γ. **3**
Find:

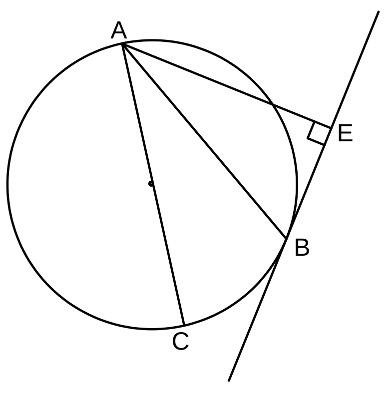
(i) $2\alpha + 2\beta + 2\gamma$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^2 x \, dx$. (Leave answer in exact form) **3**

(c) Two points A and B are on the circumference of a circle and AC is the **3**

diameter. AE is perpendicular to the tangent at B. Prove AB bisects $\angle CAE$.



(d) The graphs of $y = e^x$ and $y = e^{-2x}$ intersect at $x = 0$. **3**
Find the size of the acute angle between the curves at $x = 0$.

QUESTION 3 **(Start a new page)** **Marks**

(a) The volume of a sphere is increasing at a rate of $5cm^3/s$. **3**
At what rate is the surface area increasing when the radius is $20cm$.

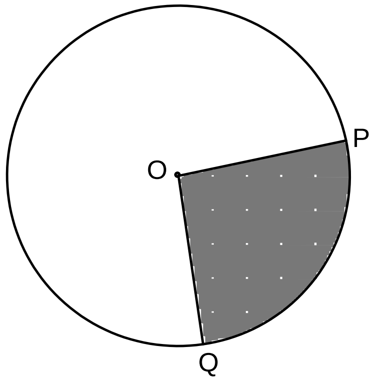
(b) Show that the point $A(\frac{1}{2}, 4)$ lies on the line joining the points **3**
$P(-3, -3)$ and $Q(1,5)$, and find the ratio in which it divides the line segment PQ.

(c) Evaluate $\lim_{x \to 0} \frac{5x\cos 2x}{\sin x}$ showing your reasoning. **2**

(d) Use the method of mathematical induction to show that the expression **4**
$9^n - 8n - 1$ is divisible by 64 for all integers $n \geq 2$.

QUESTION 4 **(Start a new page)**

(a) Find the area of sector POQ in the circle of radius $6cm$ and $\angle POQ = \frac{\pi}{3}$ **2**



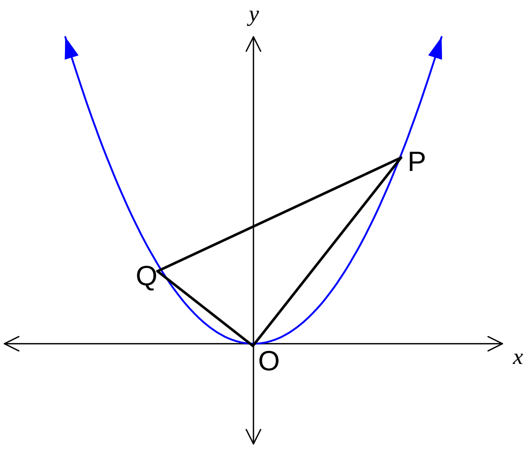
(b) **(ii)** Express $\cos 2x - \sin 2x$ in the form $R\cos(2x + \alpha)$ **2**
where α is acute and $R > 0$.

(ii) Sketch the curve $y = \cos 2x - \sin 2x$ for $0 \leq x \leq 2\pi$ **2**

(iii) Hence or otherwise solve: **2**
$\cos 2x - \sin 2x = 1$ for $0 \leq x \leq 2\pi$.

QUESTION 4 **(Continued)** **Marks**

(c)



P(2ap, ap^2) and *Q(2aq, aq*^{2}) are points on the parabola $x^2 = 4ay$. It is given that $\angle POQ = 90^\circ$. *O* is the origin.

(i) Prove that $pq = -4$. **1**

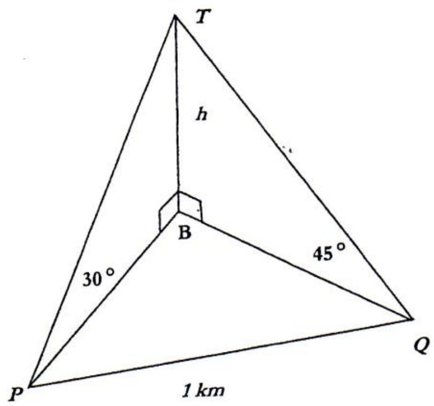
(ii) Find the co-ordinates of *M*, the midpoint of *PQ*. **1**

(iii) Prove that the Cartesian equation of the locus of *M* is $2ay = x^2 + 8a^2$. **2**

QUESTION 5 **(Start a new page)** **Marks**

(a) Evaluate $\int_0^{\ln 4} 3e^x \, dx$ **2**

(b)



The angle of elevation from a boat at *P* to a point *T* at the top of a vertical cliff is 30°. The boat sails 1*km* to a second point *Q*, from which the angle of elevation of T is 45°. B is the point at the base of the cliff directly below *T* and *h* is the height of the cliff in metres. The bearings of *B* from *P* and *Q* are 50° and 290° respectively.

(i) Show that $\angle PBQ = 120^\circ$. **1**

(ii) By finding expressions for *PB* and *QB* in terms of *h*, show that:

$$h = \frac{1000}{\sqrt{4+\sqrt{3}}}$$ **4**

QUESTION 5 (Continued) **Marks**

(c) A body cools according to the equation:

$$\frac{dT}{dt} = -k(T - S)$$

where T is the temperature of the body at time t,
S is the temperature of the surroundings and
k is a constant.

(i) Show that $T = S + Ae^{-kt}$ satisfies the equation, where A is a constant. **1**

(ii) A metal rod has an initial temperature of 470ºC and cools to 250ºC in 10 minutes.

The surrounding temperature is 30ºC. **2**

(α **)** Find the value of A and show that $k = \frac{1}{10}\log_e 2$.

(β **)** Find how much longer it will take the rod to cool to 70ºC, giving your answer to the nearest minute. **2**

QUESTION 6 (Start a new page) **Marks**

(a) **(i)** Find the polynomial $P(x)$, if $P(x)$ has: **3**

- degree 4.
- factors $(x + 3)^2$ and $(x - 3)^2$ and
- a remainder of –50 when divided by $(x + 2)$.

(ii) Sketch this curve.

(b) An arc of a circle subtends an angle θ radians at the centre. **3**
The length of the arc is ℓ and the length of the chord is d.
If $\ell : d = 4 : 3$, show that $3\theta - 8\sin\frac{\theta}{2} = 0$

(c) Find the exact value of x if $\log_e(2\log_e x) = 1$ **2**

(d) Find $\int 7^x \, dx$ **1**

(e) An amount of \$25,000 is borrowed and an interest rate of 6% pa is charged.
An amount M is repaid in monthly amounts.

(i) Show that the amount owing after n months is: **2**

$$A_n = 25000(1.005)^n - M\left(\frac{1.005^n - 1}{0.005}\right)$$

(ii) If the loan is paid off in 5 years, calculate the repayment M to the nearest dollar. **1**

QUESTION 7 **(Start a new page)** **Marks**

(a) A particle is moving in a straight line and its velocity v metres/second at time t seconds is given by:

$$v = \frac{dx}{dt} = 1 - 2\sin 2t \qquad t \geq 0$$

Initially the particle is at the origin.

(i) Express the displacement x, as a function of t. **2**

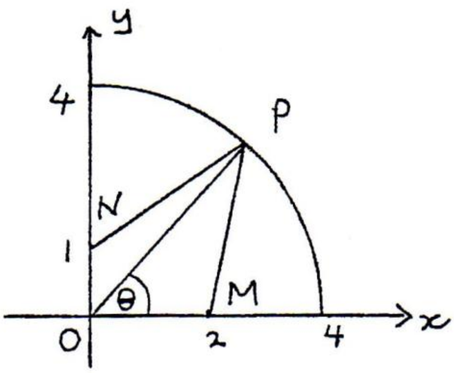
(ii) Find the position of the particle when $t = \frac{\pi}{6}$. **1**

(iii) Find an expression for the acceleration $a = \frac{d^2x}{dt^2}$. **1**

(iv) Sketch the graph of the acceleration as a function of time, $0 \leq t \leq \pi$. **1**

(v) What is the maximum acceleration of the particle? **1**

(b)



The diagram shows the part of the circle $x^2 + y^2 = 16$ that lies in the first quadrant. The point P (x, y) is on the circle with centre at the origin O. M is on the x-axis at $x = 2$ and N is on the y-axis at $y = 1$. The size of angle MOP is θ radians.

(i) Show that the area, A of the quadrilateral $OMPN$ is given by **2**

$$A = 4\sin\theta + 2\cos\theta$$

(ii) Find the value of $\tan\theta$ for which A is maximum. **2**

(iii) Hence determine in surd form the coordinates of P for which A is a maximum. **2**

2008 ½ yrly yr 12 Ext 1. Solns.

Quest 1. /12

a) $P(x) = x^3 - 2x^2 + ax + 4$

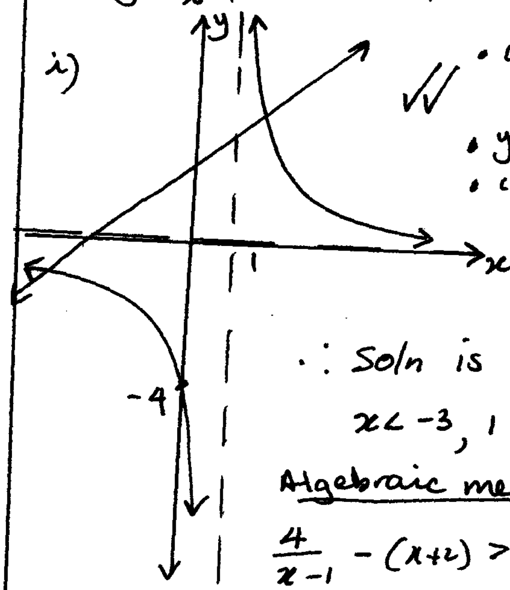
$P(-2) = -8 - 8 - 2a + 4 = 0$

$2a = -12$

$a = -6$ ✓ (1)

c) $y = \frac{4}{x-1}$ $x \neq 1$ $y \neq 0$

i)



✓✓
- correct asymptotes labelled
- y = -4 shown
- correct shape

$\therefore$ Soln is

$x < -3, \ 1 < x < 2$

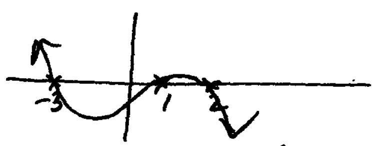
Algebraic method.

$\frac{4}{x-1} - (x+2) > 0$

$\frac{4 - (x-1)(x+2)}{x-1} > 0$

$(x-1)(6 - x - x^2) > 0$

$(x-1)((3+x)(2-x)) > 0$



$\therefore x < -3, \ 1 < x < 2$ ✓✓ (4)

b) $\frac{d}{dx} \log_e x^2 = \frac{2x}{x^2}$ ✓

$= \frac{2}{x}$. ✓ (2)

d) $\cos 2x = \cos x$ $0 \leq x \leq 2\pi$

$2\cos^2 x - 1 - \cos x = 0$ ✓

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2}$ $\cos x = 1$ ✓

$x = \pi + \frac{\pi}{3}, \pi - \frac{\pi}{3}$ $x = 0, 2\pi$

$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$. ✓ (3)

e) $\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx$ ✓

$= \frac{1}{2}(x - \frac{1}{2}\sin 2x) + C$ ✓ (2)

Question 2. /12

a) $3x^3 - 4x^2 + 2x + 1 = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$ $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= \frac{4}{3}$ $= \frac{2}{3}$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -\frac{1}{3}$

$\therefore$ i) $2(\alpha + \beta + \gamma) = 2 \times \frac{4}{3}$

✓ $= \frac{8}{3}$ ✓

ii) $\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{\frac{2}{3}}{-\frac{1}{3}} = -2$ ✓ (3)

b) $\int_0^{\frac{\pi}{3}} \tan^2 x \, dx = \int_0^{\frac{\pi}{3}} (\sec^2 x - 1)\, dx$ ✓

$= \left[\tan x - x\right]_0^{\frac{\pi}{3}}$ ✓

$= (\tan\frac{\pi}{3} - \frac{\pi}{3}) - 0$

$= \sqrt{3} - \frac{\pi}{3}$ ✓ or $\frac{3\sqrt{3} - \pi}{3}$ (3)

c) Construct CB

Proof: $A\hat{B}E = A\hat{C}B$ (angle formed between chord and tangent at pt of contact is equal to the angle in the alternate segment)

✓ $= x$

✓ $A\hat{B}C = 90°$ (angle in a semicircle)

$\therefore E\hat{A}B = 180 - (90 + x)$ (angle sum of △)

$= 90 - x$

✓ $C\hat{A}B = 90 - x$ (angle sum of △)

$= E\hat{A}B$

$\therefore$ AB bisects $C\hat{A}E$ (3)

quest 2 cont.

d) $y = e^{x}$ $\quad y = e^{-2x}$

$\frac{dy}{dx} = e^{x}$ $\quad \frac{dy}{dx} = -2e^{-2x}$

$m_1 = e^{0} = 1$ ✓ $\quad m_2 = -2 \cdot e^{0} = -2$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{1+2}{1-2}\right| = 3$ ✓

$\theta = 71°34'$ ✓ (3)

Question 3 /12

a) $V = \frac{4}{3}\pi r^3$ $\quad SA = 4\pi r^2$ $\quad$ [r = 20]

$\frac{dV}{dt} = 5$ $\quad \frac{dV}{dr} = 4\pi r^2$ $\quad \frac{dA}{dr} = 8\pi r$.

$\therefore \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$5 = 4\pi r^2 \times \frac{dr}{dt}$ ✓

$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ ✓

$= 8\pi r \cdot \frac{5}{4\pi r^2}$

$= \frac{10}{r}$

at $r = 20$

$\frac{dA}{dt} = \frac{10}{20} = 0.5$

$\therefore$ A increasing at $0.5\,m^2/sec$ ✓ (3)

b) $P(-3,-3)$ $\quad Q(1,5)$

$m_{PQ} = \frac{5+3}{1+3} = \frac{8}{4} = 2$ $\quad$ line PQ.

$y + 3 = 2(x+3)$

$y = 2x + 3$

does $A(\frac{1}{2}, 4)$ satisfy line ✓

ie. $4 = 2 \times \frac{1}{2} + 3$

$= 4$ True

ratio

$x = \frac{mx_2 + nx_1}{m+n}$ ✓

$\frac{1}{2} = \frac{m \times 1 + n \times -3}{m+n}$

$m + n = 2m - 6n$ ✓

$7n = m$ $\quad \therefore \frac{m}{n} = \frac{7}{1}$ or $m : n = 7 : 1$ (3)

c) $\lim_{x\to 0} \cos 2x \cdot \frac{5x}{\sin x} = \lim_{x\to 0} \cos 2x \cdot 5 \lim_{x\to 0} \frac{x}{\sin x}$ ✓ – some working

$= 5$ ✓ (2)

d) $9^n - 8n - 1$

let $n = 2$

$9^2 - 16 - 1 = 81 - 17$

$= 64$ which is divisible by 64 ✓

$\therefore$ true for $n = 2$.

assume true for $n = k$

$\therefore \frac{9^k - 8k - 1}{64} = P$ (some integer)

$9^k - 8k - 1 = 64P$

prove true for $n = k+1$

ie. $9^{k+1} - 8(k+1) - 1$

$= 9^k \times 9 - 8k - 9$ ✓ ✓✓

[but $9^k = 64P + 8k + 1$]

$\therefore = (64P + 8k + 1) \times 9 - 8k - 9$

$= 9 \times 64P + 72k + 9 - 8k - 9$

$= 64(9P + k)$ which is divisible by 64 ✓

$\therefore$ if true for $n = k$ then true for $n = k+1$

Since true for $n = 2$ then by M.I. also true for $n = 2+1 = 3$, $n = 3+1 = 4$ for all integers $n \geq 2$. ✓ (4)

Question 4. /12

a) $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 36 \times \frac{\pi}{3}$ ✓

$= 6\pi\,cm^2$ ✓ (2)

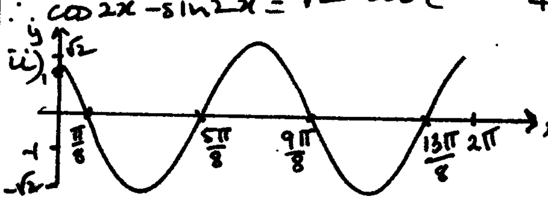
b) $\cos 2x - \sin 2x = \cos 2x \cos\alpha - \sin 2x \sin\alpha$

$\therefore \tan\alpha = 1$ ✓ $\quad R = \sqrt{2}$.

$\alpha = \frac{\pi}{4}$

$\therefore \cos 2x - \sin 2x = \sqrt{2}\cos(2x + \frac{\pi}{4})$ ✓

ii)



✓✓

quest 4 cont.

iii) $\cos 2x - \sin 2x = 1 \qquad 0 \le x \le 2\pi$

$\sqrt{2}\cos\left(2x + \frac{\pi}{4}\right) = 1 \qquad \frac{\pi}{4} \le 2x + \frac{\pi}{4} \le 4\pi + \frac{\pi}{4} \left(\frac{17\pi}{4}\right)$

$\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ ✓

$2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

$2x = 0, \frac{6\pi}{4}, \frac{8\pi}{4}, \frac{14\pi}{4}, \frac{16\pi}{4}$ (6) marks

$x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$ ✓

c) i) $M_{PO} = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2} \qquad M_{QO} = \frac{q}{2}$

since $M_{PO} \times M_{QO} = -1$ — $Q\hat{O}P = 90°$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$ ✓

$pq = -4.$

ii) mid pt PQ

$x = \frac{2ap + 2aq}{2}$ ✓ $\quad y = \frac{ap^2 + aq^2}{2}$

$= a(p+q) \qquad = \frac{a}{2}(p^2 + q^2)$

$\therefore M = \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$ ✓

iii) now

$\frac{x}{a} = p + q \qquad y = \frac{a}{2}\left((p+q)^2 - 2pq\right)$

$= \frac{a}{2}\left(\left(\frac{x}{a}\right)^2 - 2x - 4\right)$

$= \frac{a}{2}\left(\frac{x^2}{a^2} + 8\right)$ ✓

$= \frac{x^2}{2a} + 4a$

$2ay = x^2 + 8a^2$ (4)

/12

Question 5

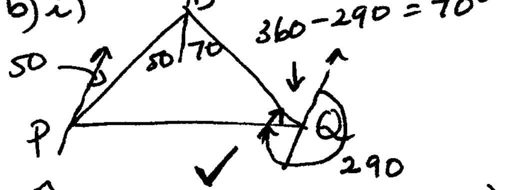
a) $\int_0^{\ln 4} 3e^x\,dx = \left[3e^x\right]_0^{\ln 4}$ ✓

$= 3e^{\ln 4} - 3e^0$

$= 3 \times 4 - 3$

$= 9$ ✓ (2)

b) i)



✓

$P\hat{B}Q = 50 + 70$ (alt ∠s)

$= 120$

ii) In $\triangle TBP$ $\qquad$ In $\triangle TBQ$

$\tan 30 = \frac{h}{PB} \qquad \tan 45 = \frac{h}{BQ}$

$PB = \frac{h}{\tan 30} \qquad BQ = \frac{h}{\tan 45}$

$= \sqrt{3}h$ ✓ $\qquad = h$ ✓

$\therefore$ in $\triangle PBQ$.

$1000m^2 = PB^2 + BQ^2 - 2\,PB.BQ.\cos 120$

$= (\sqrt{3}h)^2 + (h)^2 - 2(\sqrt{3}h . h) . -\frac{1}{2}$ ✓

$1000 = 3h^2 + h^2 + \sqrt{3}h^2$

$= h^2(4 + \sqrt{3})$

$h^2 = \frac{1000}{4+\sqrt{3}}$ ✓

$h = \frac{10\sqrt{10}}{\sqrt{4+\sqrt{3}}}.$ (5)

c) i) $T = S + Ae^{-kt}$

then $\frac{dT}{dt} = -k.Ae^{-kt}$ ✓

but $Ae^{-kt} = T - S$

$\therefore \frac{dT}{dt} = -k(T - S)$ as required

ii) $t = 0, T = 470° \qquad t = 10, T = 250°$

$S = 30$

$\therefore T = 30 + Ae^{-kt}$

(α) $470 = 30 + Ae^0$ ✓

$A = 440$

$T = 30 + 440e^{-kt}$

$250 = 30 + 440e^{-10k}$

$\frac{220}{440} = e^{-10k}$

$\ln 0.5 = -10k \ln e$ → $\ln 0.5 = \ln 2^{-1} = -\ln 2$

$k = -\frac{\ln 0.5}{10}$ ✓

$= -\frac{(-\ln 2)}{10}$

$k = \frac{1}{10}\log_e 2$

(β) $70 = 30 + 440e^{-kt}$

$\frac{40}{440} = e^{-kt}$ ✓

$\ln\frac{1}{11} = -\left(\frac{1}{10}\ln 2\right)t$

$t = 34.59$

$\doteq 35$ min ✓ (5)

4

Question 6.

/12.

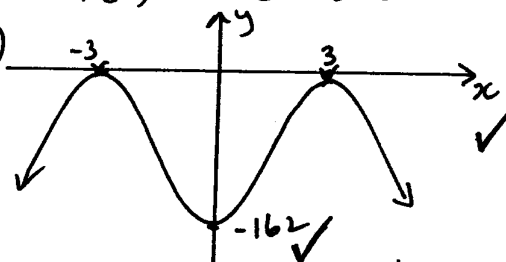
a) i) $P(x) = a(x+3)^2(x-3)^2$

$P(-2) = a(1)(-5)^2 = -50$

$\therefore a = -2$ ✓

$\therefore P(x) = -2(x+3)^2(x-3)^2$

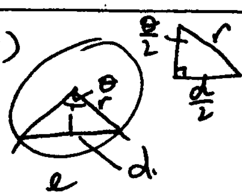
ii)



✓ correct shape ✓

(3)

b)



$\sin\frac{\theta}{2} = \frac{\frac{d}{2}}{r}$

$r\sin\frac{\theta}{2} = \frac{d}{2}$

$\Rightarrow d = 2r\sin\frac{\theta}{2}$. ✓

now $\Rightarrow l = r\theta$

$l : d = r\theta : 2r\sin\frac{\theta}{2} = 4 : 3$ ✓

$\frac{r\theta}{2r\sin\frac{\theta}{2}} = \frac{4}{3}$ ✓

$3\theta = 8\sin\frac{\theta}{2}$

or $3\theta - 8\sin\frac{\theta}{2} = 0$.

(3)

c) $\log_e(2\log_e x) = 1$

$e^1 = 2\log_e x$ ✓

$\frac{e}{2} = \log_e x$

$\therefore e^{e/2} = x$ ✓

(2)

d) $\int 7^x\,dx = \frac{1}{\ln 7}\cdot 7^x + c$ ✓

must have (↑ $+c$)

(1)

e) $r = \frac{0.06}{12} = 0.005$

end of 1st month

$A_1 = 25000(1.005) - m$

$A_2 = A_1(1.005) - m$

$= 25000(1.005)^2 - m(1.005) - m$ ✓

$A_3 = A_2(1.005) - m$

$= 25000(1.005)^3 - m(1.005)^2 - m(1.005) - m$

$\vdots$

$A_n = 25000(1.005)^n - m(1.005)^{n-1} - m(1.005)^{n-2} \ldots$ ✓

$= 25000(1.005)^n - M(1 + 1.005 + 1.005^2 \ldots)$

$= 25000(1.005)^n - m\left(\frac{1.005^n - 1}{0.005}\right)$

ii) 5 yrs = 60 f or n

$M = 25000(1.005)^{60} \times \frac{0.005}{1.005^{60} - 1}$

$= \$483.32$

$\doteqdot \$483$ ✓

(3)

Question 7.

/12

a) $V = \frac{dx}{dt} = 1 - 2\sin 2t$

i) $x = t + \cos 2t + c$ ✓

at $t = 0$ $x = 0$

$0 = 0 + \cos 0 + c$

$c = -1$

$\therefore x = t + \cos 2t - 1$ ✓

ii) $t = \frac{\pi}{6}$

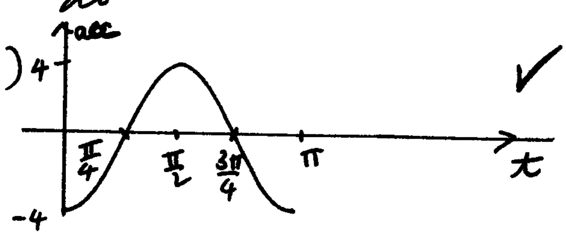
$x = \frac{\pi}{6} + \cos\frac{\pi}{3} - 1$

$= \left(\frac{\pi}{6} - \frac{1}{2}\right)m$ or $0.02m$ ✓ to the right

iii) $\frac{dx}{dt} = 1 - 2\sin 2t$

$\frac{d^2x}{dt^2} = -4\cos 2t$ ✓

iv)



✓

v) max accel. = $4\,m/sec^2$ ✓

(6)

Quest 7. cont

b. Area of OMPN

$= A_{OPM} + A_{OPN}$

$\therefore A_{\Delta OPM} = \frac{1}{2} \times 4 \times 2 \times \sin\theta$ ✓

$= 4\sin\theta$

$A_{\Delta OPN} = \frac{1}{2} \times 1 \times 4 \times \sin(90-\theta)$ ✓

$= 2\cos\theta$

$\therefore$ Total Area $= 4\sin\theta + 2\cos\theta$

ii) $\frac{dA}{d\theta} = 4\cos\theta - 2\sin\theta = 0$

$4\cos\theta = 2\sin\theta$

$2 = \tan\theta$ ✓

$\frac{d^2A}{d\theta^2} = -4\sin\theta - 2\cos\theta$

$= -(4\sin\theta + 2\cos\theta)$ ✓

$\therefore \frac{d^2A}{d\theta^2} < 0$ for any value of θ (-4.47 when using $\theta = 63.4°$)

$\therefore$ for A maximum $\tan\theta = 2$

iii) let P be (x, y)

$x^2 + y^2 = 16$

$\tan\theta = \frac{y}{x} = 2$ ✓

$y = 2x$

$\therefore x^2 + 4x^2 = 16$

$5x^2 = 16$

$x = \frac{4}{\sqrt{5}}$

$y = 2 \times \frac{4}{\sqrt{5}} = \frac{8}{\sqrt{5}}$

$\therefore$ P is $\left(\frac{4}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$ ✓

(6)