# BAULKHAM HILLS HIGH SCHOOL
# MARKING COVER SHEET

## YEAR 12 HALF-YEARLY
## EXTENSION 1
## 2011

**STUDENT NUMBER:____________________**

**TEACHER NAME:____________________**

| QUESTION | MARK |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| TOTAL | / 84 |
| PERCENTAGE | % |

# BAULKHAM HILLS HIGH SCHOOL

# YEAR 12
# HALF-YEARLY

# MATHEMATICS
# EXTENSION 1
# 2011

**General Instructions**

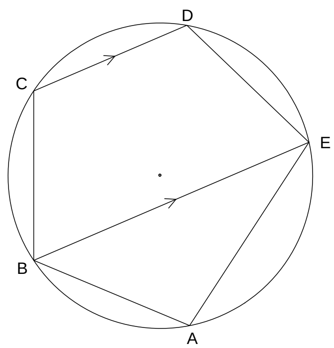
- Exam time – 2 hours
- Reading time – 5 minutes
- Start each question on a new page
- All necessary working should be shown
- Write your student number at the top of each page of your answers
- Board approved calculators may be used
- Write using black or blue pen

**Total Marks: 84**

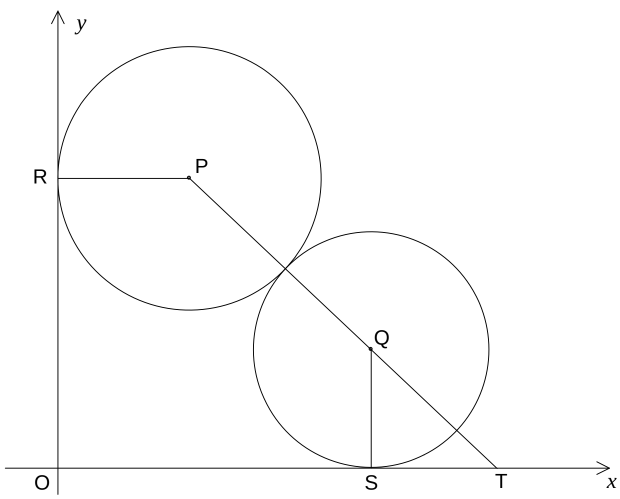
Attempt ALL questions

| | | |
|---|---|---|
| **Question 1** | **Start on a new page - (12 marks)** | **Marks** |
| a) | $\int_0^{1.5} \frac{dx}{\sqrt{9-2x^2}}$ leaving your answer in exact form | **3** |
| b) | Solve $\sin 2x = \tan x$ for $0 \leq x \leq \pi$ | **3** |
| c) | Solve $x^2 + x + \frac{12}{x^2+x} = 8$ | **3** |
| d) | i) Find the derivative of $f(x) = \tan(x^2)$ | **1** |
| | ii) Hence or otherwise evaluate $\int_0^1 x\sec^2(x^2)\,dx$ correct to two decimal places. | **2** |
| **Question 2** | **Start on a new page - (12 marks)** | |
| a) | Show that $\frac{d}{dx}\sin^{-1}(\sqrt{x}) = \frac{1}{\sqrt{4x-4x^2}}$ | **3** |
| b) | Find the size of the acute angle, in radians, between the two curves: $y = x^2$ and $y = 8 + \frac{x^2}{2}$ at their point of intersection in the first quadrant. Give your answer correct to two decimal places. | **4** |
| c) | For the function $f(x) = 2\cos^{-1} 3x$ | |
| | i) Write down the domain and range | **2** |
| | ii) Draw a neat sketch showing all important features | **1** |
| | iii) Calculate the area enclosed by $y = 2\cos^{-1} 3x$, the line $x = 0$ and the line $y = 0$ | **2** |
| **Question 3** | **(12 marks) - Start a new page** | |
| a) | If $y = x^n e^{ax}$ where $a, n$ are coordinates | |
| | i) find $\frac{dy}{dx}$ | **1** |
| | ii) show that $\frac{dy}{dx} - ay = \frac{ny}{x}$ | **2** |
| b) | Find $k$ if $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx = \log_e k$ | **3** |
| c) | Without the aid of a calculator find the exact value of $\tan\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ | **2** |
| d) | The arc of the curve $y = \cos 3x$ between the lines $x = 0$ and $x = \frac{\pi}{6}$ is rotated about the $x$-axis. Find the volume of the solid formed. | **4** |

| **Question 4 (12 marks) - Start a new page** | | **Marks** |
|---|---|---|
| | | |
| a) | Joel puts \$500 into a Baulko Bank for 2 years where it earned interest at 6%pa paid twice a year. He withdrew all his money and immediately deposited it into Axel Credit Union where his money earned 8%pa paid quarterly. If he withdrew his savings and had \$633.75, how long was the money kept in the credit union? | 4 |
| b) | The rate at which a body warms in air is proportional to the difference between the temperature, $T$, and the constant temperature, $T_0$, of the surrounding air. This rate is represented by<br>$\frac{dT}{dt} = k(T - T_0)$ where $t$ is the time in minutes and $K$ is a constant. | |
| | i) Show that $T = T_0 + Ae^{kt}$ is a solution of $\frac{dT}{dt} = k(T - T_0)$ where $A$ is a constant. | 1 |
| | ii) For a particular body, when $t$=0, $T$= 5 and when $t$= 20, $T$ = 15. Given $T_0$ = 25, find the temperature after a further 30 minutes have elapsed. Give your answer to the nearest degree. | 3 |
| | iii) Briefly describe the behaviour of $T$ as $t$ becomes large. | 1 |
| c) | Use mathematical induction to prove that for all integers n≥1<br>$\cos(x + n\pi) = (-1)^n cosx$ | 3 |
| | | |
| **Question 5 (12 marks) - Start a new page** | | |
| a) | The following eight tiles are taken from a scrabble set:<br>A, A, B, B, C, D, E, F.<br>How many different 4 letter permutations can be formed from these eight tiles? | 3 |
| b) | Find $\lim_{x \to 0} \frac{sin\pi x^\circ}{x}$ | 3 |
| c) | Find the following indefinite integral $\int \frac{x^2+1}{x-1} dx$ | 2 |
| d) | Sketch the polynomial function $y = P(x)$ in the domain $-3 \leq x \leq 3$ given that:<br>$y = 0$ only when $x = 0$ and 2<br>$y' = 0$ only when $x = \pm 1$<br>$y'' = 0$ only when $x = -1$ and 0<br>$y'' < 0$ only when $-1 < x < 0$ | 4 |

| **Question 6** **(12 marks) - Start a new page** | | **Marks** |
|---|---|---|
| a) |  $A, B, C, D, E$ are points on the circumference of a circle such that $CD//BE$<br><br>Prove that $\angle CAB = \angle DAE$. | 4 |
| b) | $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$ | |
| | i) Show that the equation of the normal to the curve of the parabola at the point $P$ is $x + py = 2ap + ap^3$ | 2 |
| | ii) Find the coordinates of the point $Q$ where the normal at $P$ meets the $y$-axis | 1 |
| | iii) Find the coordinates of the point $R$ which divides $PQ$ externally in the ratio $2:1$ | 2 |
| | iv) Find the Cartesian equation of the locus of $R$ and describe the locus in geometrical terms | 3 |

QUESTION 7 ON NEXT PAGE

| Question 7 (12 marks) - Start a new page | | Marks |
|---|---|---|
| a) | Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ have values between 0 and $\frac{\pi}{2}$ | |
| | i) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ | 2 |
| | ii) hence or otherwise solve the equation $$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$$ | 2 |
| b) | The diagram shows two touching circles with centres $P$ and $Q$. The circle with centre $P$ has a radius of 4 units and touches the $y$-axis at $R$. The circle with centre $Q$ has a radius of 3 units and touches the $x$-axis at $S$.<br>$PQ$ produced meets the $x$-axis at $T$ and $\angle QTS = \theta$<br><br> | |
| | i) Show that $OR = 3 + 7\sin\theta$ and $OS = 4 + 7\cos\theta$ | 2 |
| | ii) Show that $RS^2 = 42\sin\theta + 56\cos\theta + 74$ | 2 |
| | iii) Hence express $RS^2$ in the form $74 + r\cos(\theta - \alpha)$<br>Clearly stating the values of $r$ and $\alpha$ | 3 |
| | iv) Find the maximum lengths of $RS$ and the value of $\theta$ for which this occurs. | 1 |

**End of Paper**

2011 EXT 1 HY SOLUTIONS TOMANOVIC Q7

a) $= \int_0^{1.5} \frac{1}{\sqrt{2}} \frac{dx}{\sqrt{\frac{9}{2}-x^2}}$

$= \frac{1}{\sqrt{2}} \left[\sin^{-1}\left(\frac{\sqrt{2}x}{3}\right)\right]_0^{1.5}$ ✓

$= \frac{1}{\sqrt{2}}\left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0\right)$

$= \frac{\pi}{4\sqrt{2}}$ or $\frac{\pi\sqrt{2}}{8}$ ✓

b) $2\sin x \cos x = \frac{\sin x}{\cos x}$

$2\sin x \cos^2 x = \sin x$ ✓

$\sin x(2\cos^2 x - 1) = 0$

$\therefore \sin x = 0$ $\quad 2\cos^2 x = 1$

$\sin x = 0$ $\quad \cos^2 x = \frac{1}{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$.

$\therefore x = 0, \pi$ ✓ OR $x = \frac{\pi}{4}, \frac{3\pi}{4}$ ✓

$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

c) let $m = x^2 + x$ $\quad$ NB $x^2 + x \neq 0$

$\therefore m + \frac{12}{m} = 8$ $\quad$ ie $x(x+1) \neq 0$

$\therefore x \neq 0$ or $-1$

$m^2 - 8m + 12 = 0$

$(m-6)(m-2) = 0$

$m = 2$ or $m = 6$ ✓

$x^2 + x - 2 = 0$ $\quad x^2 + x - 6 = 0$

$(x+2)(x-1) = 0$ $\quad (x+3)(x-2) = 0$

$x = 1, -2$ ✓ $\quad x = -3, 2$

$\therefore x = -3, -2, 1, 2$ ✓

d i) $f'(x) = 2x\sec^2(x^2)$ ✓

ii) $\frac{1}{2}\int_0^1 2x\sec^2(x^2)\,dx$

$= \frac{1}{2}\left[\tan(x^2)\right]_0^1$ ✓

$= \frac{1}{2}(\tan 1 - \tan 0)$

$= 0.78$ (2dp) ✓

Q2 a) $\frac{d}{dx}\left(\sin^{-1}(x^{\frac{1}{2}})\right)$

$= \frac{1}{\sqrt{1-(x^{\frac{1}{2}})^2}} \times \frac{1}{2}x^{-\frac{1}{2}}$ ✓ ✓

$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$= \frac{1}{\sqrt{4x(1-x)}}$

$= \frac{1}{\sqrt{4x-4x^2}}$ ✓

b) $x^2 = 8 + \frac{x^2}{2}$

$2x^2 = 16 + x^2$

$x^2 = 16$

$x = \pm 4$. ✓

$\frac{d}{dx}(x^2) = 2x$ $\quad \frac{d}{dx}\left(8 + \frac{x^2}{2}\right) = x$

$m_1 = 8$ $\quad m_2 = 4$ ✓

$\therefore$ angle $\tan\theta = \left|\frac{8-4}{1+8\times 4}\right|$ ✓

$\tan\theta = \frac{4}{33}$

$\theta = \tan^{-1}\frac{4}{33}$

$= 0.12$ radians ✓

2 c) i) $\frac{y}{2} = \cos^{-1} 3x$

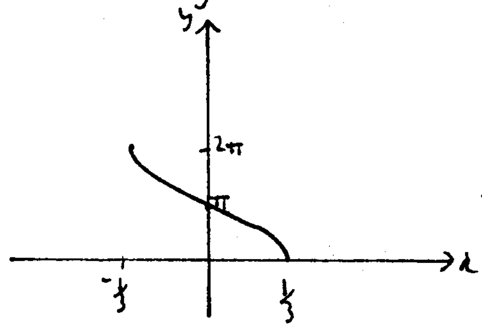
Domain: $-1 \le 3x \le 1$

Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $0 \le \frac{y}{2} \le \pi$

ie $0 \le y \le 2\pi$

(ii)




iii) $y = 2\cos^{-1} 3x$

$\frac{y}{2} = \cos^{-1} 3x$

$3x = \cos \frac{y}{2}$

Area $= \int_0^{\pi} \frac{1}{3} \cos \frac{y}{2}\, dy$

$= \frac{1}{3}\left[2 \sin \frac{y}{2}\right]_0^{\pi}$

$= \frac{2}{3}\left(\sin \frac{\pi}{2} - \sin 0\right)$

$= \frac{2}{3}(1-0)$

$\therefore$ Area $= \frac{2}{3}$ unit$^2$

3 a) i) $\frac{dy}{dx} = e^{ax}.nx^{n-1} + x^n.ae^{ax}$

$= x^{n-1}e^{ax}(n+ax)$

ii) $\frac{dy}{dx} - ay$

$= x^{n-1}e^{ax}(n+ax) - ax^n e^{ax}$

$= x^{n-1}e^{ax}(n+ax-ax)$

$= nx^{n-1}e^{ax}$

$= \frac{ny}{x}$

$\therefore \frac{dy}{dx} - ay = \frac{ny}{x}$

3b) $\int_0^{\pi/6} \frac{\cos x}{1+\sin x} dx$

$= \left[\log(1+\sin x)\right]_0^{\pi/6}$

$= \log\left(1+\sin\frac{\pi}{6}\right) - \log(1+\sin 0)$

$= \log\left(1+\frac{1}{2}\right) - \log 1$

$= \log(1.5)$

$\therefore k = 1.5$

3 c) $\tan\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \tan\left(\pi - \frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right)$

$= \frac{-1}{\sqrt{3}}$ or $\frac{-\sqrt{3}}{3}$

3 d) $V = \pi \int_0^{\pi/6} y^2 dx$

$= \pi \int_0^{\pi/6} \cos^2 3x\, dx$

$= \pi \int_0^{\pi/6} \frac{\cos 6x + 1}{2} dx$

$= \frac{\pi}{2}\left[\frac{\sin 6x}{6} + x\right]_0^{\pi/6}$

$= \frac{\pi}{12}\left[\sin 6x + 6x\right]_0^{\pi/6}$

$= \frac{\pi}{12}\left[(\sin \pi + \pi) - (\sin 0 + 0)\right]$

$= \frac{\pi^2}{12}$

4 a) let $A_n$ be amount after $n$ 6 months

$r = 3\%$ per period

$A_4 = 500(1.03)^4$

$A_4 = \$562.75$

$633.75 = 562.75(1.02)^n$

$\frac{633.75}{562.75} = (1.02)^n$

$\ln\left(\frac{633.75}{562.75}\right) = \ln(1.02)^n$

$n = \frac{\ln\left(\frac{633.75}{562.75}\right)}{\ln(1.02)}$

$n \doteqdot 6$

$\therefore$ money is in credit union for 1.5yrs.

4(b) i) $\frac{dT}{dt} = kAe^{kt}$

$= k(T_0 + Ae^{kt} - T_0)$

$= k(T - T_0)$

ii) when $t=0$, $T_0 = 5°C$

$T = 25 + Ae^{kt}$

$5 = 25 + Ae^0$

$A = -20$

$\therefore T = 25 - 20e^{kt}$

when $t = 20$, $T = 15$

$15 = 25 - 20e^{20k}$

$20e^{20k} = 10$

$e^{20k} = \frac{1}{2}$

$\ln e^{20k} = \ln(\frac{1}{2})$

$20k = \ln \frac{1}{2}$

$k = \ln(0.5)$ ( $\div$ ... )

When $t = 50$

$T = 25 - 20e^{50k}$

$= 21.46...$

$T = 21°C$

(iii) As $t \to \infty$ $T = 25 - 20e^{kt}$

approaches $T = 25 - 0$ since $e^{kt} \to 0$

$T = 25°C$

$\therefore$ Temperature approaches 25°C which is the temperature of the surrounding air.

c) Test $n=1$

LHS $= \cos(x + \pi)$

$= \cos x \cos\pi - \sin x \sin\pi$

$= -\cos x$

RHS $= (-1)^1 \cos x$

$= -\cos x$

$=$ LHS

$\therefore$ True for $n=1$

Assume true for $n=k$

~~ie~~ $\cos(x + k\pi) = (-1)^k \cos x$

We wish to prove $\cos(x + (k+1)\pi) = (-1)^{k+1} \cos x$ for $n = k+1$

$\cos(x + (k+1)\pi)$

$= \cos((x + k\pi) + \pi)$

$= \cos(x + k\pi)\cos\pi - \sin(x + k\pi)\sin\pi$

$= -\cos(x + k\pi) - 0$

$= -(-1)^k \cos x$

$= (-1)^{k+1} \cos x$ as req'd.

$\therefore$ If true for $n=k$ its true for $n=k+1$. But true for $n=1$, therefore true for $n = 1+1 = 2$ and $n = 2+1 = 3$ and so ... for all $n \geq 1$

5 a) Case 1 all letters different

A, B, C, D, E, F

$^6P_4 = 360$ ✓

Case 2 Only 2 the same, 2 different

$2 \times {}^5C_2 \times \frac{4!}{2} = 240$ ✓

Case 3 Both As and Bs

$\frac{4!}{2!\,2!} = 6$

∴ number of ways $= 360 + 240 + 6$

$= 606$ ✓

5 b) $\lim_{x\to 0} \sin \frac{\pi x^\circ}{x}$

$= \lim_{x\to 0} \sin \frac{\pi\left(\frac{\pi x}{180}\right)}{x}$ ✓

$= \lim_{x\to 0} \frac{\sin\left(\frac{\pi^2 x}{180}\right)}{\frac{\pi^2 x}{180}} \times \frac{\pi^2}{180}$ ✓

$= \frac{\pi^2}{180} \lim_{x\to 0} \frac{\sin \frac{\pi^2 x}{180}}{\frac{\pi^2 x}{180}}$

$= \frac{\pi^2}{180} \times 1$

$= \frac{\pi^2}{180}$ ✓

5 c) $\int \frac{x^2+1}{x-1}\,dx$

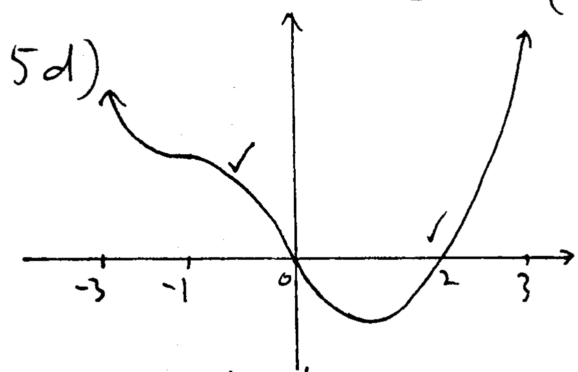
$= \int \frac{x^2-1+2}{x-1}\,dx$

$= \int \frac{x^2-1}{x-1} + \frac{2}{x-1}\,dx$

$= \int (x+1) + \frac{2}{x-1}\,dx$ ✓

$= \frac{x^2}{2} + x + 2\log_e(x-1) + c$ ✓ (ignore c

5d)




(1) $x$ intercepts

(1) concave down between $-1$ and $0$

(1) stationary points

(1) concave up $-3 \le x < -1$ and $x \ge 0$

6a) Join AC, AD

Let $\angle DCB = x^\circ$ ✓

$\angle CBE = 180^\circ - x^\circ$ (co-interior angles, CD∥

$\angle CDE = x^\circ$ (opposite angles in cyclic quad CDEB are supplementary ✓

$\angle DAB = 180^\circ - x^\circ$ (opp ∠'s in cyclic quad ABCD are supplementary

$\angle EAC = 180^\circ - x^\circ$ (∠'s at circumference standing on arc CE) ✓

Since $\angle DAB = \angle EAC = 180^\circ - x^\circ$ then

$\angle CAB = \angle DAE$ (as $\angle DAC$ is common) ✓

6 b) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

Grad of tangent at $(2ap, ap^2)$ is

$\frac{dy}{dx} = p$ ✓

∴ Grad of normal is $-1$

Equation of normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ ✓

(ii) When $x = 0$

$y = \frac{2ap + ap^3}{p}$

$y = 2a + ap^2$

$\therefore Q$ is $(0, 2a + ap^2)$ ✓

(iii) $P$ $Q$

$(2ap, ap^2)$ $(0, a(2+p^2))$

$-2 : 1$

$x = \frac{2ap + 0}{-1}$ $y = \frac{ap^2 - 2a(2+p^2)}{-1}$

$x = -2ap$ $y = \frac{-4a - ap^2}{-1}$ ① $x$

$x = -2ap, \; y = 4a + ap^2$ ① $y$

$\therefore R$ is $(-2ap, a(4+p^2))$ or internal division ①

(iv) $x = -2ap \rightarrow p = \frac{x}{-2a}$ ✓ attempts to eliminate parameter

sub in $y = a(4+p^2)$

$y = a\left(4 + \frac{x^2}{4a^2}\right)$

$y = 4a + \frac{x^2}{4a}$

$4ay = 16a^2 + x^2$

$x^2 = 4a(y - 4a)$ ✓

Locus is a parabola with vertex ✓

$(0, 4a)$, (focus at $0, 5a$ and

directrix at $y = 3a$)

7a) i) let $\alpha = \sin^{-1} x$ $\sin \alpha = x$

let $\beta = \cos^{-1} x$ $\cos \beta = x$

NB $0 \le x \le 1$

$\sin(\sin^{-1} x - \cos^{-1} x)$

$= \sin(\alpha - \beta)$

$= \sin\alpha \cos\beta - \cos\alpha \sin\beta$ ✓

$= x \cdot x - \sqrt{1-x^2} \cdot \sqrt{1-x^2}$

$= x^2 - (1 - x^2)$ ✓

$= 2x^2 - 1$

ii) $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$

$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(2x^2 - 1)$

then if $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1 - x)$ ✓

$2x^2 - 1 = 1 - x$

$2x^2 + x - 2 = 0$

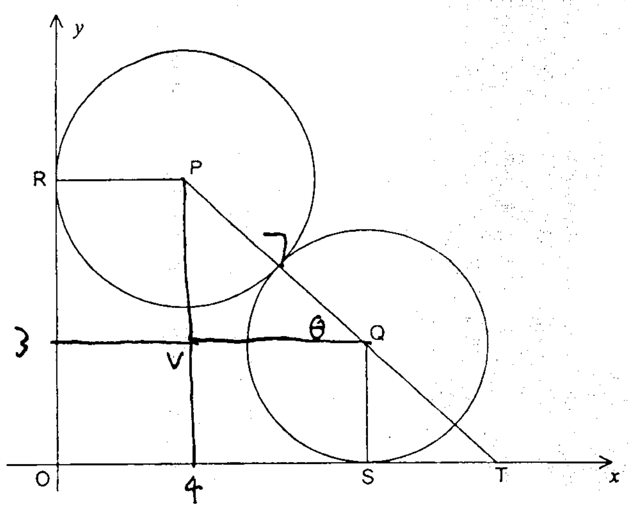
$x = \frac{-1 \pm \sqrt{1+16}}{4}$

$x = \frac{-1 \pm \sqrt{17}}{4}$

$\therefore x = \frac{-1 + \sqrt{17}}{4}$ as $0 \le x \le 1$ ✓

7b) next page

7b)




i) $OR = 3 + PV$

$= 3 + 7\sin\theta$ ✓

$OS = 4 + VQ$

$= 4 + 7\cos\theta$ ✓

ii) $RS^2 = OR^2 + OS^2$

$= (3+7\sin\theta)^2 + (4+7\cos\theta)^2$ ✓

$= 9 + 42\sin\theta + 49\sin^2\theta + 16 + 56\cos\theta + 49\cos^2\theta$

$= \underbrace{25 + 49(\sin^2\theta + \cos^2\theta)}_{1} + 42\sin\theta + 56\cos\theta$

$= 74 + 42\sin\theta + 56\cos\theta$ ✓

iii) $r\cos(\theta - \alpha) = 42\sin\theta + 56\cos\theta$

$r = \sqrt{42^2 + 56^2}$

$r = 70$ ✓

$\cos\theta\cos\alpha + \sin\theta\sin\alpha = \frac{42}{70}\sin\theta + \frac{56}{70}\cos\theta$

$\therefore \sin\alpha = \frac{42}{70} \quad \cos\alpha = \frac{56}{70}$

$\tan\alpha = \frac{3}{4}$ ✓

$\alpha = 36°52'$ (nearest min)

$\therefore RS^2 = 74 + 70\cos(\theta - 36°52')$ ✓

iv) max value occurs when

$\cos(\theta - 36°52') = 1$

$\therefore RS^2 = 74 + 70$

$= 144$

$\therefore RS = 12$ gives max value. ✓

$\therefore 12^2 = 74 + 70\cos(\theta - 36$

$\boxed{\theta = 36°52'}$