BAULKHAM HILLS HIGH SCHOOL

2012
YEAR 12 HALF-YEARLY

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

Total marks – 70
Exam consists of 9 pages.

This paper consists of TWO sections.

Section 1 – Pages 4-5
Multiple Choice
Question 1-10 (10 marks)

Section 2 – Pages 5-8
Extended Response
Question 11- 14 (60 marks)

Standard integrals provided on page 9

Section I - 10 marks

Attempt questions 1-10
Allow about 15 minutes for this section
Use the multiple choice answer sheet for question 1-10

Marks

1) A polynomial $P(x)$ has a relative maximum at $(-2,4)$, a relative minimum at $(1,1)$ and a relative maximum at $(5,7)$ and no other critical points. How many real zeros does $P(x)$ have? **1**

(A) one (B) two (C) three (D) four

2) If $\frac{dy}{dx} = \cos x \, . \sin^2 x$ then **1**

(A) $y = \sin^3 x + c$ (B) $y = \cos^2 x + c$ (C) $y = \sin^2 x + c$ (D) $y = \frac{1}{3}\sin^3 x + c$

3) What are the solutions of the equation $\sin 2\theta = \cos\theta$ in the domain $-\pi \leq \theta \leq \pi$ **1**

(A) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ (B) $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ (C) $\frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$ (D) $-\frac{\pi}{4}, \frac{\pi}{4}$

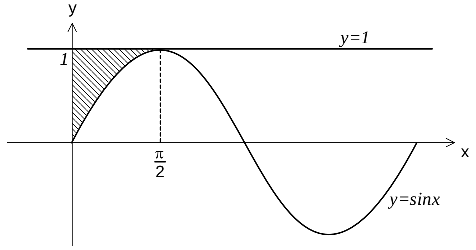
4) Three boys and four girls are to sit around a table. **1**
How many arrangements are there if 2 specific girls sit next to each other?

(A) $5!$ (B) $5! \times 2!$ (C) $6! \times 2!$ (D) $7! \times 2!$

5) $\frac{d}{dx}\ln[\cos(2x)]$ is **1**

(A) $\frac{1}{\cos 2x}$ (B) $\frac{-\sin 2x}{\cos 2x}$ (C) $-2\tan 2x$ (D) $2\sec 2x$

6) Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ **1**
The volume of the solid obtained by revolving R about the x –axis is given by



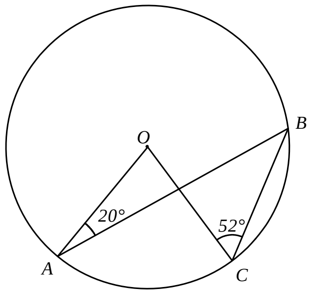
(A) $\pi\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (B) $2\pi\int_0^{\frac{\pi}{2}} x. \sin x \, dx$

(C) $\pi\int_0^{\frac{\pi}{2}} (1-\sin x)^2 \, dx$ (D) $\pi\int_0^{\frac{\pi}{2}} 1 - \sin^2 x \, dx$

7) The inverse function to $y = \frac{6}{2x-3} + 1$ is **1**

(A) $y = \frac{x+1}{2x-2}$ (B) $y = \frac{-x+1}{2x-1}$

(C) $y = \frac{3+3x}{2x-2}$ (D) $y = xy - \frac{3x+3}{2}$

8)	 In the diagram, O is the centre of the circle, $\angle OAB = 20°$ and $\angle OCB = 52°$. The measure of $\angle ABC$, in degrees is (A) 20 (B) 52 (C) 32 (D) 72	**1**
9)	The number of the solutions to the equation $x = 10\cos x$ is (A) 3 (B) 5 (C) 6 (D) 7	**1**
10)	If $2x^3 - 9x^2 + 13x + k$ is divisible by $x - 2$, then it is also divisible by (A) $(x+2)$ (B) $(x+1)$ (C) $(x-1)$ (D) $2x+1$	**1**
	End of Section 1	

Section II – Extended Response

Attempt all questions. Show all necessary working.
Start each question on a new page. Clearly indicate question number.
Write your name and teacher's name at the top of each new page.

Question 11 (15 marks) - Start a new page		**Marks**
a)	Solve for x: $$\frac{2}{x-2} \geq 2x - 1$$	3
b)	Let A be the point $(-8, -3)$ and $B(4,7)$ Find the coordinates of the point P that divides AB externally in the ratio $1:2$	2
c)	State the domain and range of the function $f(x) = 2\sin^{-1}\left(\frac{x}{2}\right)$ and sketch it.	3
d)	Evaluate the integral below in exact form. $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{4x^2+1}\,dx$$	2
e)	Prove that $$\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$$	3
f)	How many five-person committees can be selected from 6 men and 8 women if there must be at least two women included?	2

Question 12 (15 marks) - Start a new page **Marks**

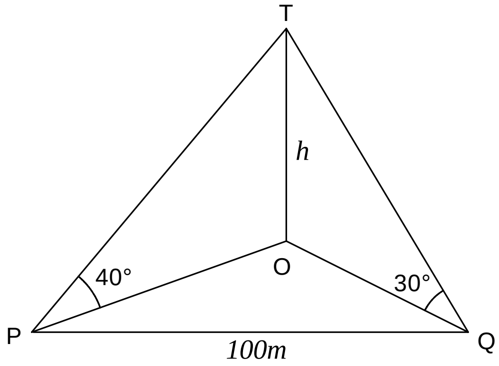
a) i) Write $\sin x - \cos x$ in the form $A\sin(x-\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$ **2**

ii) Hence or otherwise solve the equation **2**

$$\sin x - \cos x = 1 \quad \text{for } 0 \le x \le 2\pi$$

b) When the polynomial $P(x)$ is divided by $x^2 - 5x + 6$, the remainder is $5x + 2$. What is the remainder when $P(x)$ is divided by $x - 2$? **2**

c)



A surveyor stands at the point P due south of a tower OT of height h, and finds the angle of elevation of the top of the tower to be 40°.

Then he walks $100m$ to the point Q due east of the tower. The angle of elevation from Q to the top of the tower is then 30°

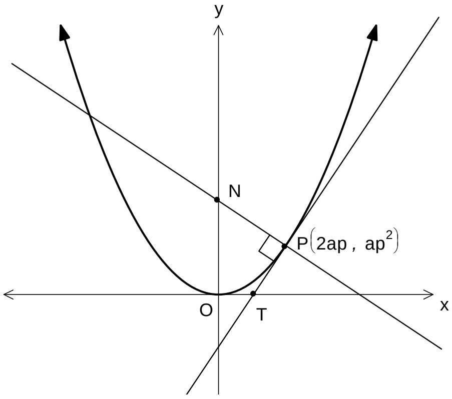
i) Find the expressions for OP and OQ in terms of h **1**

ii) Show that **3**

$$h = \frac{100(\tan 40° \tan 30°)}{\sqrt{\tan^2 40° + \tan^2 30°}}$$

iii) Find h to the nearest metre. **1**

d)



The diagram shows the graph of the parabola $x^2 = 4ay$.

PT is the tangent and PN is the normal at $P(2ap, ap^2)$.

Given the coordinates for $T(ap, 0)$ and $N\left(0, a(p^2+2)\right)$

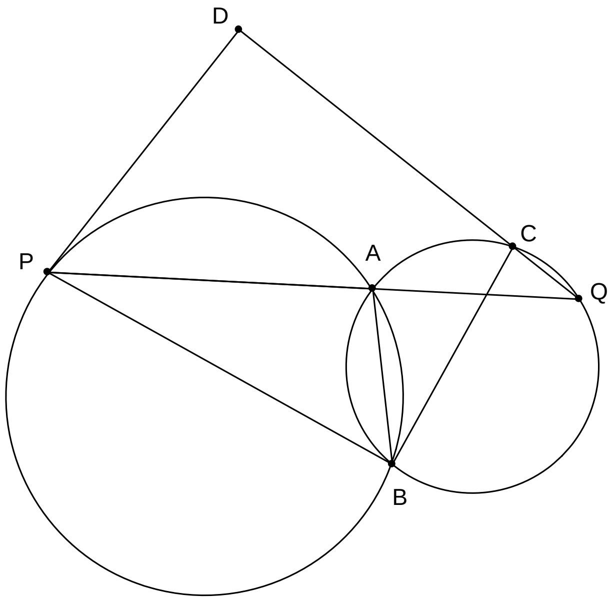
i) Show that the equation of the tangent at P is $y = px - ap^2$ **2**

ii) M is the midpoint of TN, find the cartesian equation for the locus of M **2**

Question 13 (15 marks) - Start a new page **Marks**

a) Use mathematical induction to prove that $16^n + 10n - 1$ is divisible by 25 for all integers $n \geq 1$ **4**

b)



Two circles intersect at A and B.
P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line.
The tangent at P and the line QC produced (where C is on the second circle) meet at D

i) Give a reason why $\angle DPA = \angle PBA$ **1**

ii) Give a reason why $\angle CQA = \angle CBA$ **1**

iii) Hence show that $BCDP$ is a cyclic quadrilateral **2**

c) At time t years the number N of individuals in a population is given by $N = A + Be^{-t}$, where A and B are constants.
After $\log_e 3$ years, there are 60 individuals and after $\log_e 5$ years there are 48 individuals.

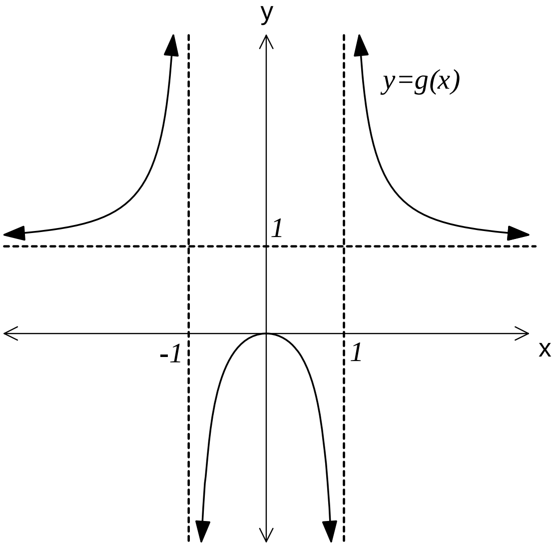
i) Find A and B **3**

ii) Find the limiting population size. **1**

d) Determine how many numbers can be formed using all of the digits 1,2,3, ... 9 without repetition in each of the following cases

i) Even and odd digits alternate **1**

ii) The digits 1,2,3 are together but not necessarily in that order **2**

Question 14 (15 marks) - Start a new page		
a)	Find $\int 6\cos x\left(e^{3\sin x}\right)dx$	2
b)	The function $g(x)$ is defined by $g(x)=\frac{x^2}{x^2-1}$ and its graph is shown below.  	
	i) What is the largest domain of $g(x)$ including $x=2$ for which an inverse function exist?	1
	ii) Find the equation of the inverse function $g^{-1}(x)$ for this domain	2
	iii) What is the domain of $g^{-1}(x)$?	1
	iv) Find the value of $g^{-1}\left(g(-2)\right)$	1
	v) Find the x-ordinates of all points of intersection of the two curves $y=g(x)$ and $y=g^{-1}(x)$	2
c)	Show that $\tan^{-1}4-\tan^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{4}$ without the use of a calculator.	2
d)	The region bounded by $y=2\cos^{-1}x$ and the x and y $-$axes is rotated around the y-axis. Find the volume of the solid generated.	4
	End of Examination	

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad if \quad n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

BAULKHAM HILLS HIGH SCHOOL
2012 HALF YEARLY
Mathematics Extension 1
MULTIPLE CHOICE ANSWER SHEET

Section 1 - Answer Sheet

1) A [✓ B] C D
2) A B C [✓ D]
3) A [✓ B] C D
4) A [✓ B] C D
5) A B [✓ C] D
6) A B C [✓ D]
7) A B [✓ C] D
8) A B [✓ C] D
9) A B C [✓ D]
10) A B [✓ C] D

Answers

Yr. 12 1/2 yearly 2012

Extension 1

$\boxed{Q.\ 11}$

a) $\frac{2}{x-2} \geq 2x-1 \quad / \times (x-2)^2$

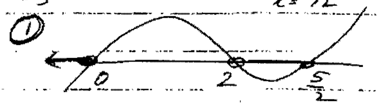
$2(x-2) \geq (2x-1)(x-2)^2$

$0 \geq (x-2)\left[(2x-1)(x-2) - 2\right]$

$0 \geq (x-2)\left[2x^2-5x+2-2\right]$

$0 \geq (x-2)\left[x(2x-5)\right]$ ①

x-int. $x=0$, $x=2$, $x=5/2$



$x \leq 0, \quad 2 < x \leq \frac{5}{2}$ ①

①

b) $A(-8,-3) \quad B(4,7)$

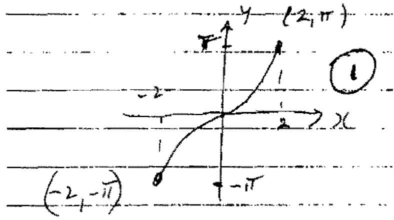
$1 : -2$

$P\left(\frac{-8\times -2+4\times 1}{1-2}, \frac{-3\times -2+7\times 1}{1-2}\right) = (-20, -13)$ ① ①

c) $f(x) = 2\sin^{-1}\left(\frac{x}{2}\right)$ $\quad -1 \leq \frac{x}{2} \leq 1$

$\boxed{D: \ -2 \leq x \leq 2}$ ①

$\boxed{R: -\pi \leq y \leq \pi}$ ①



①

d) $\int_{1/2}^{\sqrt{3}/2} \frac{1}{4x^2+1}\,dx = \int_{1/2}^{\sqrt{3}/2} \frac{1}{4(x^2+\frac{1}{4})}\,dx = \frac{1}{4}\int_{\frac{1}{4}}^{\sqrt{3}/2} \frac{1}{x^2+\frac{1}{4}}\,dx$ $\quad a^2 = \frac{1}{4}$ $\quad a = \frac{1}{2}$

$= \frac{1}{4}\left[\frac{1}{\frac{1}{2}}\tan^{-1}2x\right]_{1/4}^{\sqrt{6}/2}$ ① $= \frac{1}{2}\left[\tan^{-1}2x\right]_{\frac{1}{2}}^{\sqrt{3}/2} = \frac{1}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{24}$ ①

$\boxed{\text{Q.11 cont.}}$

e) Simplify $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \frac{2\cancel{\sin\theta}\cdot\cos\theta}{\cancel{\sin\theta}}$ ① $- \frac{\cos^2\theta - \sin^2\theta}{\cos\theta}$

$= 2\cos\theta - \frac{2\cos^2\theta - 1}{\cos\theta}$ ① $= \cancel{2\cos\theta} - \cancel{2\cos\theta} + \frac{1}{\cos\theta}$ ①

$= \underline{\sec\theta}$ $\underline{= \text{RHS} \therefore \text{proven}}$

f) 2W3M or 3W2M or 4W1M or 5W0M

ⓑ 8W ⓑ 6M

$\underset{560}{{}^8C_2 \cdot {}^6C_3}$ ① $+ \underset{840}{{}^8C_3\,{}^6C_2} + \underset{420}{{}^8C_4\,{}^6C_1} + \underset{56}{{}^8C_5 \cdot {}^6C_0}$

$= 1876$ ① (OR) $\underbrace{{}^{14}C_5}_{2002} - \underbrace{({}^6C_5 + {}^6C_4 \cdot {}^8C_1)}_{126} = 1876$

$\boxed{\text{Multiple choice}}$

Question ① B

② D

③ B

④ B

⑤ C

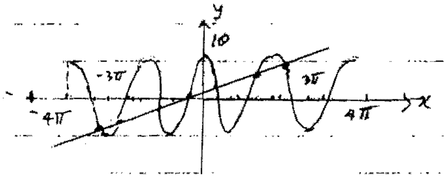
⑥ D

⑦ C

⑧ C

⑨ D

⑩ $k = -6$ $\therefore$ C



Question 12

a) i) $\sin x - \cos x = A\sin(x - \alpha)$

$A = \sqrt{1^2 + 1^2} = \sqrt{2}$ ① $\tan\alpha = \frac{1}{1} \therefore \alpha = 45° = \pi/4$ ①

$\therefore \sin x - \cos x = \sqrt{2}\sin(x - 45°) = \boxed{\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)}$

ii) $\sin x - \cos x = \sqrt{2}\sin(x - 45°) = 1$

$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ ①

$\therefore x - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}$

$x = \frac{\pi}{2}, \pi$ ①

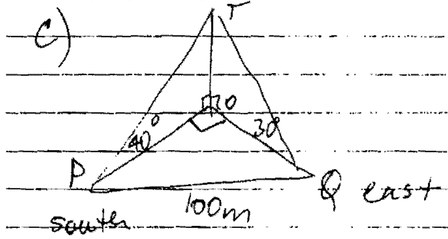
b) $P(x) \div (x^2 - 5x + 6)$ $\therefore R(x) = 5x + 2$

$\therefore P(x) = (x^2 - 5x + 6) \cdot Q(x) + 5x + 2$ ①

$\therefore P(2) = \underbrace{(2^2 - 5\times 2 + 6)}_{0} \cdot Q(x) + 5\times 2 + 2 = \underline{\underline{12}}$

$\therefore \underline{\underline{R(2) = 12}}$ ①

c)



i) $OP = \frac{OT}{\tan 40°} = \frac{h}{\tan 40°}$ $\tan 40° = \frac{OT}{PO}$

$OQ = \frac{OT}{\tan 30°} = \frac{h}{\tan 30°}$ ①

ⓘⓘ) $\triangle POQ$

$100^2 = PO^2 + OQ^2$

$100^2 = \frac{h^2}{\tan^2 40°} + \frac{h^2}{\tan^2 30°}$ ①

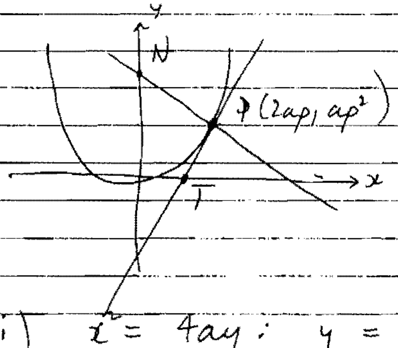
$100^2 = \frac{h^2(\tan^2 30° + \tan^2 40°)}{\tan^2 40° \cdot \tan^2 30°}$ ①

$\therefore h^2 = \frac{100^2(\tan^2 40 \cdot \tan^2 30°)}{\tan^2 30° + \tan^2 40°}$ ① $\therefore h = \frac{100 \cdot \tan 40° \cdot \tan 30°}{\sqrt{\tan^2 30 + \tan^2 40}}$

Q.12 c) cont.

iii) $h = 47.56\ldots = 48\text{ m}$ (nearest metre) ①

d)



i) $x^2 = 4ay$: $y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{2x}{4a}$ ∴ at $x = 2ap$ $m = \dfrac{2\times 2ap}{4a} = p$ ①

∴ tangent: $y - ap^2 = p(x - 2ap)$ ①

$y = px - 2ap^2 + ap^2$

$\therefore y = px - ap^2$

ii) $M_{TN}\left(\dfrac{ap}{2}, \dfrac{a(p^2+2)}{2}\right)$ ①

$T(ap, 0)$

$N(0, a(p^2+2))$

∴ locus of M $\begin{cases} x = \dfrac{ap}{2} & \therefore p = \dfrac{2x}{a} \\ y = \dfrac{a(p^2+2)}{2} \end{cases}$

$y = \dfrac{a}{2}\left(\left(\dfrac{2x}{a}\right)^2 + 2\right) = \dfrac{a}{2}\left(\dfrac{4x^2}{a^2} + 2\right)$

$\therefore \boxed{y = \dfrac{2x^2}{a} + a}$ ①

Question 13

a) $16^n + 10n - 1$ divisible by 25 $n \geq 1$

STEP 1: prove the expression is div. by 25 for $n = 1$

$\therefore 16^1 + 10\times 1 - 1 = 25$ which is div. by 25 ∴ proven ①

STEP 2: assume that $16^K + 10k - 1$ is div. by 25

$\therefore 16^K + 10K - 1 = 25m$ ① (where m, k are integers)

STEP 3: prove that the expr. is div. by 25 for $n = k+1$

proof: $16^{K+1} + 10(K+1) - 1 = \underline{16^K} \cdot 16 + 10K + 10 - 1$

~~from assumption~~ $16^K = 25m - 10k + 1$

$= (25m - 10K + 1) \cdot 16 + 10K + 9 = 25m\times 16 - \underline{160K} + 16 + \underline{10k} +$ ①

$= 25\times 16m - 150K + 25 = 25(\underbrace{16m - 6K + 1})$

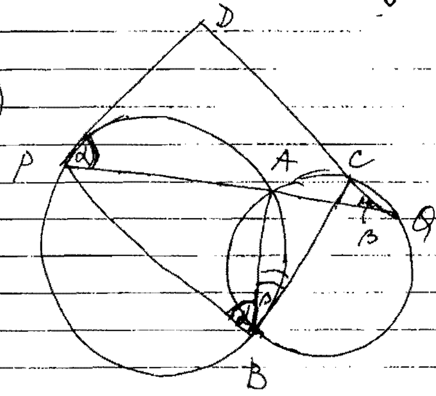
① integer since m, K are integer

∴ divisible by 25.

Conclusion: since the statement is proven to be true for $n = 1$ and if it's true for $n = K$ and since proven for $n = K+1$ ∴ true for $n = 2, 3, \ldots$ ∴ true for all $n \geq 1$ by induction.

(if the conclusion missing) (−1)

b)



i) $\angle DPA = \angle PBA = \alpha$

angle between tangent PD ① and a chord PA at the point of contact is = to the ∠ in alternate segment

ii) $\angle CQA = \angle CBA = \beta$

(angles at the circumference in the same segment are equal)

Q. 13 b - cont.

iii) in $\triangle PDQ$ $\angle PDQ = 180° - \angle DPA - \angle DQP$

$= 180° - \alpha - \beta$ ① (angle sum in triangle)

but $\angle PBC = \angle PBA + \angle CBA = \alpha + \beta$ ① (adjacent angles)

$\therefore \angle PDQ$ is supplementary to $\angle PBC$ (opposite $\angle$'s in cyclic quad are supplementary)

$\therefore$ PBCD is cyclic quadrilateral

c) (i) $N = A + Be^{-t}$

$60 = A + Be^{-\log_e 3}$ $\therefore$ $60 = A + Be^{\log_e 3^{-1}}$ ① $\therefore 60 = A + B \cdot 3^{-1}$

$48 = A + Be^{-\log_e 5}$ $\quad$ $48 = A + Be^{\log_e 5^{-1}}$ $\quad$ $48 = A + B \cdot 5^{-1}$

$\therefore 12 = B(3^{-1} - 5^{-1})$

$12 = B \times \frac{2}{15}$ $\therefore$ $\boxed{90 = B}$ ① $\boxed{A = 30}$ ①

(ii) $N \to \lim_{t \to \infty} (A + Be^{-t}) = \lim_{t \to \infty} (30 + 90\underbrace{e^{-\infty}}_{\to 0}) = 30$ ①

d) 1, 2, 3, ... 9 $\quad$ 4 even $\quad$ 5 odd

i) $5 \times 4 \times 4 \times 3 \times \ldots$ $\therefore$ $5!\,4! = 2880$ ①

odd must be first

ii) (1, 2, 3) 4, 5, 6, 7, 8, 9 $\therefore$ $7! \times 3! = 30240$

① ①

Question 14

a) $\int 6\cos x\,(e^{3\sin x})\,dx = 2\int \underbrace{3\cos x}_{f'(x)} \cdot e^{\overbrace{3\sin x}^{f(x)}}\,dx$ ①

$= 2e^{3\sin x} + C$ ①

b) (i) D_g $x \geq 0, x \neq 1$ ①

(ii) $g(x) = \frac{x^2}{x^2 - 1} = y$

$\therefore g^{-1}(x)$: $x = \frac{y^2}{y^2 - 1}$ $\therefore xy^2 - x = y^2$

$y^2(x - 1) = x$

$y^2 = \frac{x}{x - 1}$

$\therefore y = \pm\sqrt{\frac{x}{x - 1}}$ ①

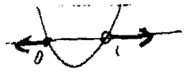
but since D_g $x \geq 0, x \neq 1$ $\therefore$ $R_{g^{-1}}$ is $y \geq 0, y \neq 1$

$\therefore$ $y = +\sqrt{\frac{x}{x-1}}$ ①

(iii) $D_{g^{-1}(x)}$: (Range of the original from graph) $x \leq 0, x > 1$ ①

OR $\frac{x}{x - 1} \geq 0$ $\therefore x(x - 1) \geq 0$

$\therefore x \leq 0, x > 1$



(iv) $g^{-1}(g(-2)) = g^{-1}\left(\frac{4}{3}\right) = \sqrt{\frac{12}{3}} = 2$ ①

(v) $g(x) = \frac{x^2}{x^2 - 1}$ intersect at $y = x$ with $g^{-1}(x)$

$\therefore \frac{x^2}{x^2 - 1} = x$ ①

$x^2 = x(x^2 - 1)$

$0 = x[x^2 - 1 - x]$

$x = 0,$ $x = \frac{1 \pm \sqrt{1 + 4}}{2}$

① $x = \frac{1 \pm \sqrt{5}}{2}$

not in domain

Q. 14

ⓥ cont.

Ⓞⓡ $g(x) = \dfrac{x^2}{x^2-1}$ & $g^{-1}(x) = \sqrt{\dfrac{x}{x-1}}$

$$\therefore \frac{x^2}{x^2-1} = \sqrt{\frac{x}{x-1}} \quad ①$$

$$\frac{x^4}{(x^2-1)^2} = \frac{x}{x-1}$$

$$x^4(x-1) = x(x-1)^2(x+1)^2$$

$$x(x-1)\left[x^3 - (x-1)(x+1)^2\right] = 0$$

$x=0$, $\cancel{x=1}$, $x = \dfrac{1 \pm \sqrt{5}}{2}$ ①

excluded from domain

$\therefore x = 0, \dfrac{1+\sqrt{5}}{2}$

c) Show $\tan^{-1}4 - \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

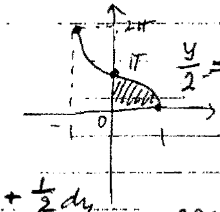
LHS: $\tan\left(\underbrace{\tan^{-1}4}_{\alpha} - \underbrace{\tan^{-1}\left(\frac{3}{5}\right)}_{\beta}\right) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$ ①

$$= \frac{4 - 3/5}{1 + 4 \cdot 3/5} = 1$$ ①

$\text{RHS} = \tan\left(\frac{\pi}{4}\right) = 1 = \text{LHS}$ ∴ proven

d) $y = 2\cos^{-1}x \quad -1 \le x \le 1$

$0 \le 2y \le 2\pi$



$\frac{y}{2} = \cos^{-1}x$

$\boxed{x = \cos y/2}$

$$V = \pi \int_0^{\textcircled{\pi}} \left(\cos^2\frac{y}{2}\right) dy = \pi \int_0^{\pi} \frac{1}{2}\cos y + \frac{1}{2} dy \quad ①$$

$\cos y = 2\cos^2\frac{y}{2} - 1$

$\cos^2\frac{y}{2} = \frac{1}{2}\cos y + \frac{1}{2}$

$$= \frac{\pi}{2}\Big[\underset{①}{\sin y + y}\Big]_0^{\pi} = \frac{\pi}{2}\Big[0 + \pi - 0 - 0\Big] = \frac{\pi^2}{2} \quad ①$$