BAULKHAM HILLS HIGH SCHOOL

Half -Yearly 2013

YEAR 12 TASK 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-14
- Marks may be deducted for careless or badly arranged work

Total marks – 70

Exam consists of 7 pages.

This paper consists of TWO sections.

Section 1 – Pages 2&3 (10 marks)
Questions 1-10

- Attempt Questions 1-10

Section II – Pages 4-7 (60 marks)

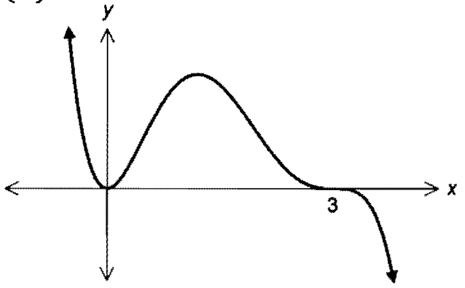
- Attempt questions 11-14

Table of Standard Integrals is on page 8

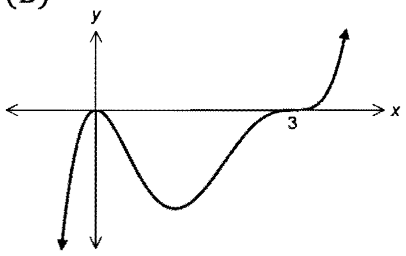
Section I - 10 marks
Use the multiple choice answer sheet for question 1-10

1. The graph of $y = x^2(3 - x)^3$ is best represented by:-

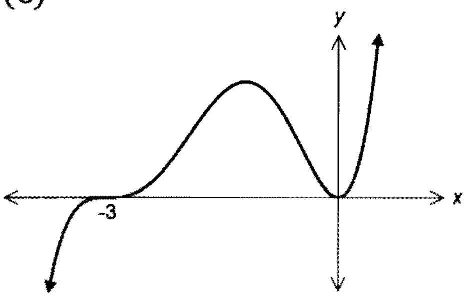
(A)



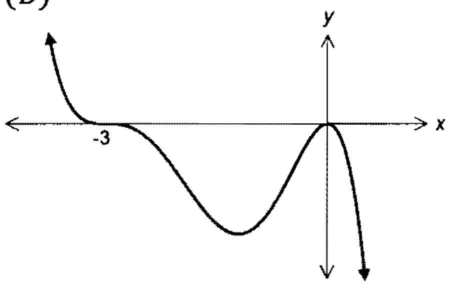
(B)



(C)



(D)



2. The acute angle to the nearest degree between the lines $3x + 2y = 6$ and $x - 4y + 2 = 0$ is:-

(A) 20° (B) 42° (C) 48° (D) 70°

3. The solution to the inequality
$\frac{x}{2 + 3x} \leq 1$ is :-

(A) $x \leq -1$ or $x \geq \frac{-2}{3}$ (B) $x \leq \frac{-3}{2}$ or $x \geq -1$

(C) $x \leq -1$ or $x > \frac{-2}{3}$ (D) $\frac{-2}{3} < x \leq \frac{-1}{2}$

4. The equation of the normal to the curve $y = (e^x + x)^3$ at the point where $x = 0$ is:-

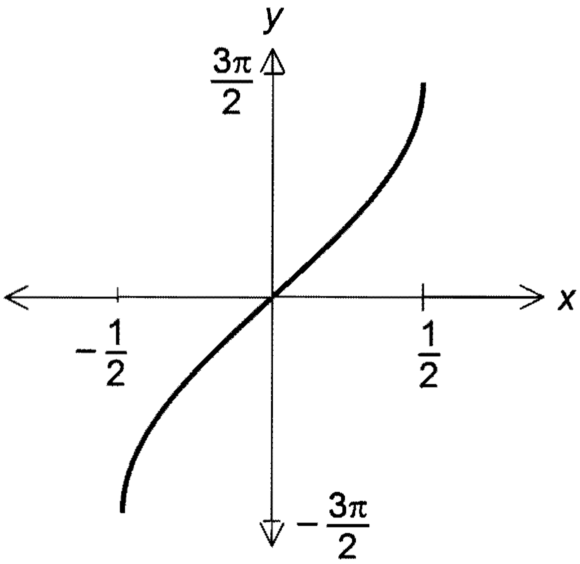
(A) $x + 6y = 0$ (B) $x + 6y - 6 = 0$ (C) $6x - y + 1 = 0$ (D) $x + 3y - 3 = 0$

5. The point $P(x, y)$ is on the interval AB such that $3AP = 4PB$. If A has co-ordinates (-1, 2) and B has co-ordinates $(6, -5)$ what are the co-ordinates of P ?

(A) $(2, -1)$ (B) $(3, -2)$ (C) $(-22, 23)$ (D) $(-27, 26)$

6. How many different arrangements of all of the letters of the word $POLYNOMIAL$ are there if the $L's$ are separated?

(A) 907200 (B) 181440 (C) 725760 (D) 1451520

7. 

This inverse trigonometric graph has which of the following equations ?

(A) $y = \frac{1}{3}\sin^{-1}\left(\frac{x}{2}\right)$

(B) $y = 3\sin^{-1}\left(\frac{x}{2}\right)$

(C) $y = \frac{1}{3}\sin^{-1}(2x)$

(D) $y = 3\sin^{-1}(2x)$

8. $\int \frac{dx}{9 + 4x^2}$

(A) $\frac{1}{8}\ln(9 + 4x^2) + c$

(B) $\frac{1}{4}\tan^{-1}\left(\frac{2x}{3}\right) + c$

(C) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$

(D) $\frac{1}{9}\tan^{-1}\left(\frac{4x}{9}\right) + c$

9. The area between the curve $y = 2 + cosx$, the x axis, $x = \frac{\pi}{2}$ and $x = \pi$ is rotated about the x axis. The volume of the solid generated is :-

(A) $\frac{9\pi-16}{4}$ (B) $\frac{9\pi^2}{4}$ (C) $\frac{9\pi^2+16\pi}{4}$ (D) $\frac{9\pi^2-16\pi}{4}$

10. $\int 2^{2x+4}\,dx =$

(A) $\frac{2^{2x+4}}{2} + c$ (B) $\ln 2 \,.\, 2^{2x+4} + c$ (C) $\frac{2^{2x+4}}{ln2} + c$ (D) $\frac{2^{2x+4}}{ln4} + c$

End of Section 1

Section II – Extended Response

Attempt questions 11-14.
Answer each question on the appropriate page in your exam booklet.
All necessary working should be shown in every question.

Question 11 (15 marks) **Marks**

a) If $P(x) = ax^3 - bx^2 + 6$ is a monic polynomial and when divided by $x - 4$ the remainder is -2, find a and b. **3**

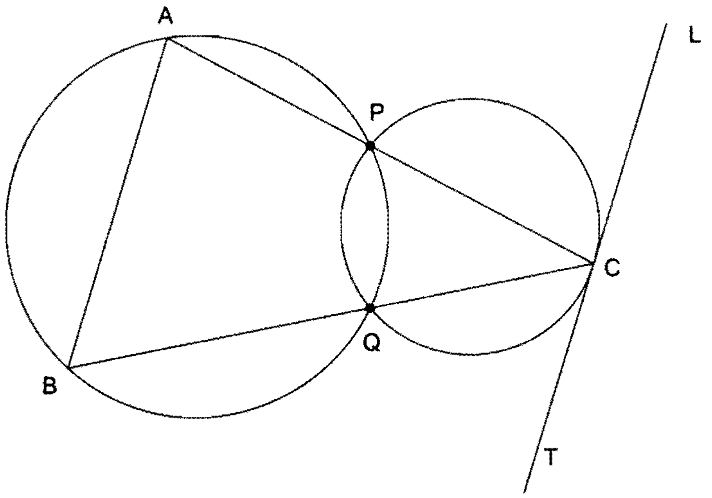
b) (i) Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x - \alpha)$. **2**

(ii) Hence or otherwise solve $\sqrt{3}\cos x + \sin x = 1$ for $0° \leq x \leq 360°$. **2**

c) Find $\int \cos x \sin^4 x \, dx$. **2**

d) If $\sin a = \frac{3}{4}$ and $90° \leq a \leq 180°$ and $\cos b = \frac{1}{5}$ and $0° \leq b \leq 90°$ find the exact value of $\sin(a + b)$. **3**

e) **3**



LT is a tangent to the circle at C.

Prove that this tangent is parallel to AB.

Question 12 (15 marks)

a) The cooling rate of a body is proportional to the difference between the temperature of a body (T) and the temperature of the surrounding medium (M).
ie. $\frac{dT}{dt} = k(T - M)$

(i) Show that $T = M + Ae^{kt}$ is a solution to the above differential equation. **1**

(ii) A hot metal bar of $1000°C$ when immersed in cool water of $15°C$ for 5 minutes cools to $800°C$. Find the temperature of the bar after a further 10 minutes. **2**

b) There are 4 teams of 8 players. The players of each team are numbered 1 to 8. A group of 6 players is to be selected from these 4 teams. **2**
What is the probability that the group of 6 players chosen will have 4 players with the same number?

c) (i) What is the largest domain containing $x = 4$ for $y = x^2 - 4x$ to have an inverse function? **1**

(ii) Sketch $y = x^2 - 4x$ for this domain and its inverse function on the same set of axes. **2**

(iii) What is the domain of the inverse function $y = f^{-1}(x)$? **1**

(iv) Find the equation of the inverse function as a function of x. **3**

(v) Find the point of intersection of the two curves. **2**

(vi) Evaluate $f^{-1}(f(1))$. **1**

Question 13 (15 marks) **Marks**

a) Prove by mathematical induction that $7^n + 3^n$ is a multiple of 10 if n is an odd positive integer. **3**

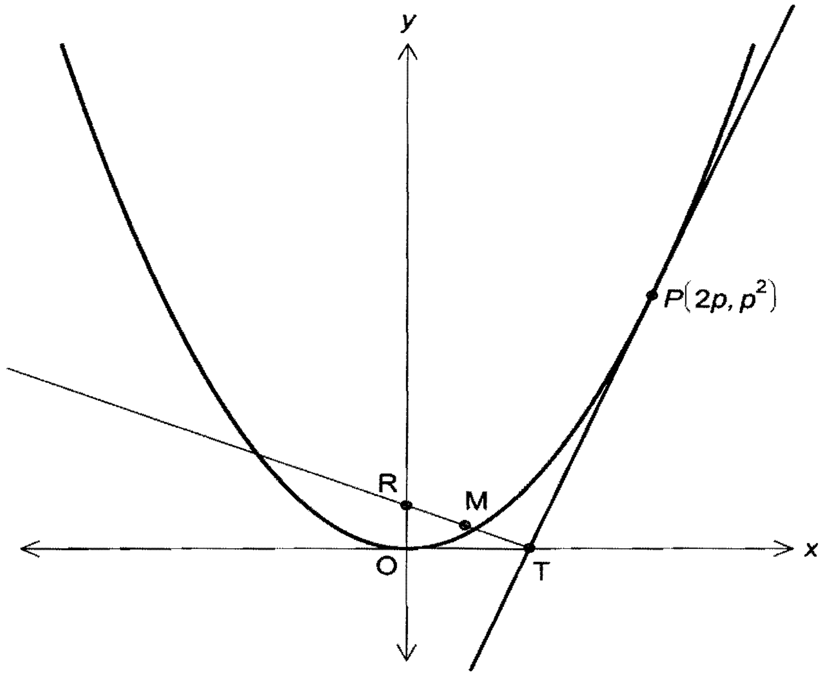
b) If α, β, γ are the roots of $x^3 - 5x^2 + 4x - 2 = 0$, find the values of :-

(i) $\alpha + \beta + \gamma$. **1**

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

c)



(i) Show the Cartesian equation for the parabola above is $x^2 = 4y$. **1**

(ii) Show that the equation of the tangent at P is given by: **2**
$y = px - p^2$.

(iii) The tangent cuts the x axis at T and a perpendicular to this tangent is drawn from T to cut the y axis at R.

(I) Show that R is the focus of the parabola. **3**

(II) What is the locus of M, the midpoint of RT? **1**

(III) Find the area of the triangle TPR in terms of p. **2**

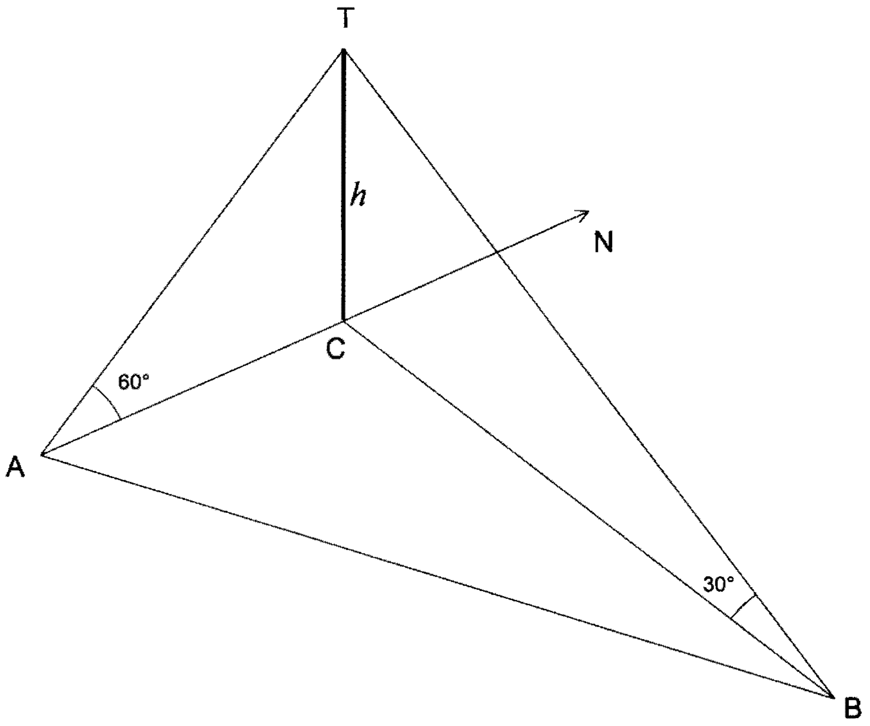
Question 14 (15 marks)

a) (i) Prove $\tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$ for $tan\theta \geq 0$ **2**

(ii) Hence find the exact value of $\tan\frac{\pi}{8}$ in simplified form. **3**

b) Show that $f(x) = x^3 + 2x - 4$ has only 1 root. **2**

c)



From a point A a person notes the elevation to the top of a tower CT due north is 60°.
After walking to B on a bearing of 120° he notes the angle of elevation is then 30°.

(i) If the height of the tower is h, find AC and BC in terms of h in exact form. **1**

(ii) Find the ratio of the distance AB to the height of the tower h.
Give your answer in exact form. **3**

d) Given $\log_b(xy^3) = m$ and $\log_b(x^3y^2) = p$. **4**

Find $\log_b(\sqrt{xy})$ in terms of m and p.

End of Exam

Solutions Yr 12 ½ Yearly 2013

Section 1. M. Choice

1. A 2. D 3. C 4. B 5. B
6. C 7. D 8. C 9. D 10. D

Section 2

11 a) $P(x) = ax^3 - bx^2 + 6$

monic $\Rightarrow a = 1$ (1)

$P(4) = -2 \quad \therefore -2 = (4)^3 - b(4^2) + 6$ (1)

$\therefore -2 = 64 - 16b + 6$

$-72 = -16b$

$b = \frac{-72}{-16}$

$\therefore a = 1 \quad b = \frac{9}{2}$ (1)

b) (i) $\sqrt{3}\cos x + \sin x = A\cos(x - \alpha)$

RHS $A\cos x\cos\alpha + A\sin x\sin\alpha$

$\therefore A\cos\alpha = \sqrt{3} \quad A\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30°$ (1)

$A = \sqrt{(\sqrt{3})^2 + (1)^2}$

$A = 2$ (1)

$\therefore \sqrt{3}\cos x + \sin x = 2\cos(x - 30°)$

(ii) $\therefore 2\cos(x - 30°) = 1$

$\cos(x - 30°) = \frac{1}{2}$ (1)

$x - 30° = 60°, 300°$

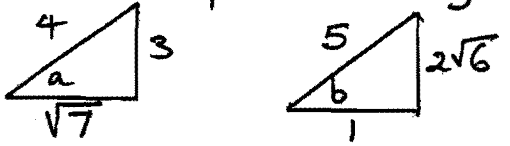
$x = 90°, 330°$ (1)

c) $\int \cos x \sin^4 x \, dx$

$\frac{d}{dx}(\sin^5 x) = 5\cos x \sin^4 x$ (1)

$\therefore \int \cos x \sin^4 x = \frac{1}{5}\sin^5 x + c$ (1)

d) $\sin a = \frac{3}{4} \qquad \cos b = \frac{1}{5}$



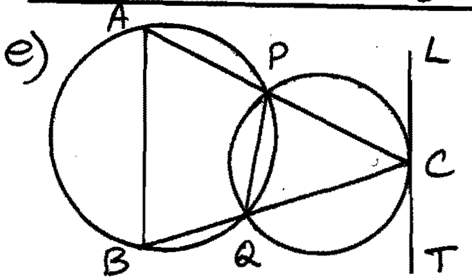
(1)

$\therefore \sin(a+b) = \sin a\cos b + \cos a\sin b$

$= \frac{3}{4} \cdot \frac{1}{5} + \frac{\sqrt{7}}{4} \cdot \frac{2\sqrt{6}}{5}$ (1)

$= \frac{3 + 2\sqrt{42}}{20}$ (or equivalent) (1)

e)



Ext 1. Join PQ.

let $\angle LCP = x°$

$\therefore \angle LCP = \angle PQC = x°$ (1)

(Alternate Segment Theorem)

$\angle PQC = \angle BAP = x°$

(Exterior $\angle$ of a cyclic quad = interior opposite $\angle$). (1)

$\therefore \angle BAP = \angle LCA = x°$

(Since these alternate $\angle$'s are $=$ AB ∥ LT) (1)

12 a) (i) $T = M + Ae^{kt}$

$\Rightarrow Ae^{kt} = T - M$ — (1)

$\frac{dT}{dt} = k(T - M)$

$\frac{dT}{dt} = kAe^{kt}$ from (1)

$= k(T - M)$ (1)

$\therefore T = M + Ae^{kt}$ is a sol'n of $\frac{dT}{dt} = k(T - M)$

(ii) $T = M + Ae^{kt} \quad M = 15°$

when $t = 0 \quad T = 1000$

$\therefore 1000 = 15 + Ae^0$

$\therefore A = 985$.

when $t=5$ $T=800$

$\therefore 800 = 15 + 985e^{5k}$

$e^{5k} = \frac{785}{985}$

$\therefore k = \frac{\ln\left(\frac{785}{985}\right)}{5}$

$k = -0.0453\ldots$ (1)

find T when $t=15$

$T = 15 + 985e^{15(-0.0453\ldots)}$

$= 514°$ (to the nearest deg.)

(accept 514° – 517° allowing for rounding.) (1)

b) Probability 4 of the players have number 1

is $\underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underbrace{\underline{\ }\ \underline{\ }}_{^{28}C_2}$

Nos 1 → 8

$= 8 \times {}^{28}C_2$

$= 3024$ } (1)

$\therefore$ Prob'y $= \frac{3024}{^{32}C_6}$ → (1)

$= \frac{3}{899}$

c) (i) $y' = 0$ $y = x^2 - 4x$

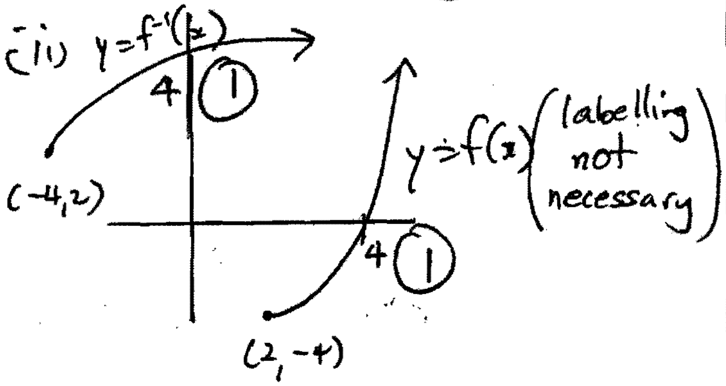
$2x - 4 = 0$

$x > 2$

$\therefore$ largest domain containing $x = 4$

is $x \geq 2$ (1)

(ii)



(iii) D: $f^{-1}(x)$: $x \geq -4$ (1)

(iv) inverse $x = y^2 - 4y$ (1)

$\therefore x + 4 = y^2 - 4y + 4$

$(y-2)^2 = x + 4$ (1)

$y - 2 = \pm\sqrt{x+4}$

$y = 2 \pm \sqrt{x+4}$

but since $y \geq 2$ (because of restriction)

$y = 2 + \sqrt{x+4}$ (1)

(v) Intersect on $y = x$

$y = x$ $\quad y = x^2 - 4x$ } 1

$x = x^2 - 4x$

$x^2 - 5x = 0$

$x(x-5) = 0$ $\quad x = 0, 5$

but $x \geq 2$ $\therefore x = 5$ (1) } award mark at $x = 5$

point is $(5, 5)$

$f^{-1}(f(1)) = f^{-1}(f(3))$ (because of symmetry & $x = 1$ is not in the restricted domain) — no need for reason

$= 3$ (1)

Question 13. (See end of question for marking scale)

a) Prove $7^n + 3^n$ is ÷ by 10 for odd integers n.

Step 1. Prove true for $n = 1$

$7^1 + 3^1 = 10$ which is ÷ by 10

Step 2. Assume true for $n = k$ (k is odd)

$\frac{7^k + 3^k}{10} = M$ (where M is a pos. integer)

$7^k = 10M - 3^k$ — (1)

Step 3 Prove true for $n = k+2$

ie Show → $7^{k+2} + 3^{k+2}$ is ÷ by 10

$7^2 . 7^k + 3^2 . 3^k$

from (1) $= 49(10M - 3^k) + 9(3^k)$

$= 490M - 49(3^k) + 9(3^k)$

$= 490M - 40(3^k)$

$= 10(49M - 4(3^k))$

Since 10 is a factor and M is a positive integer then $7^{k+2}+3^{k+2}$ is ÷ by 10.

Step 4 Proved true for $n=1$ and assumed true for $n=k$ & proven true for $n=k+2$ ∴ true for $n=1$, $n=3$, $n=5$, . . . and for all odd integers n.

Marks – 4 Key components
1. Prove result for $n=1$
2. Clearly stating the assumption what is to be proven – (Must state M is a positive integer)
3. Using the assumption successfully in the proof.
4. Correctly proving the required statement

3 marks – successfully does all 4 components

2 marks – successfully does 3 of the 4 components but must include step 3.

1 mark – does 2 of the 4 components successfully

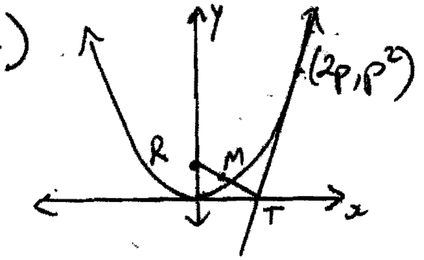
b) $x^3-5x^2+4x-2=0$

(i) $\alpha+\beta+\gamma = 5$ (1)

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma = 4$ (1)

(iii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$= 5^2 - 2(4)$
$= 17$ (1)

c)



(i) $x=2p \quad y=p^2$

$p=\frac{x}{2} \therefore y=\left(\frac{x}{2}\right)^2$ (1)

$\therefore x^2=4y$

or $x^2=4ay$, $x=2at \quad y=at^2$, $x=2p \quad y=p^2$ $\Rightarrow a=1$ $\therefore x^2=4(1)y$, $x^2=4y$.

(ii) $y=\frac{x^2}{4}$ (1)

$y'=\frac{2x}{4}$ at $x=2p$ $y'=\frac{4p}{4}=p$

Eq'n of tangent $y-p^2=p(x-2p)$
$y-p^2=px-2p^2$
$y=px-p^2$ (1)

(iii) at T $y=0$ $\therefore 0=px-p^2$ (1)
$x=\frac{p^2}{p} \rightarrow x=p$

(I) $T(p,0)$

RT has eq'n $y-0=-\frac{1}{p}(x-p)$

$y=-\frac{x}{p}+1$ (1)

at $x=0$ $y=1$ $R(0,1)$
now $x^2=4y \Rightarrow a=1$
$\therefore$ focus is $(0,1)$ (1)

$\therefore$ R is the focus.

(II) Midpoint

$M=\left(\frac{0+p}{2}, \frac{1+0}{2}\right)$
$=\left(\frac{p}{2}, \frac{1}{2}\right)$ (not essential)

$\therefore$ locus $y=\frac{1}{2}$ (1) $p\neq 0$

(III) $RT=\sqrt{p^2+1}$

$PT=\sqrt{(2p-p)^2+(p^2-0)^2}$
$=\sqrt{p^2+p^4}$
$=\sqrt{p^2(1+p^2)}$
$=p\sqrt{1+p^2}$ (1)

Area $\Delta = \frac{1}{2}\cdot p\sqrt{1+p^2}\cdot\sqrt{1+p^2}$

or equivalent $\Leftarrow = \frac{p(1+p^2)}{2}$ (1)

14a) i)

Prove $\tan\theta = \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$

$$RHS = \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}} \quad (1)$$

$$= \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \quad (1)$$

$$= \sqrt{\tan^2\theta}$$

$$= \tan\theta$$

(ii)

$$\therefore \tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}$$

$$= \sqrt{\frac{1-\frac{1}{\sqrt2}}{1+\frac{1}{\sqrt2}}} \quad (1)$$

$$= \sqrt{\frac{\sqrt2-1}{\sqrt2}\times\frac{\sqrt2}{\sqrt2+1}} \quad (1)$$

$$\sqrt{\frac{\sqrt2-1}{\sqrt2+1}\times\frac{\sqrt2-1}{\sqrt2-1}}$$

$$= \sqrt{\frac{(\sqrt2-1)^2}{2-1}}$$

$$= \sqrt2-1 \quad (1)$$

or if they have $= \sqrt{3-2\sqrt2}$ ⇓ (2) marks

b) $f(x) = x^3+2x-4$

$f'(x) = 3x^2+2$

now $x^2 \geq 0$ for all x $\therefore 3x^2+2 \geq 0$ (1) for all x

ie $f'(x)$ is increasing for all x.

hence the graph can only cut the x axis at 1 place and hence (1) there is only 1 root

c) (i) In $\Delta ACT \Rightarrow \tan 60° = \dfrac{h}{AC}$

$\therefore AC = \dfrac{h}{\tan 60°} \Rightarrow AC = \dfrac{h}{\sqrt3}$

In ΔCTB similarly $BC = \dfrac{h}{\tan 30°}$

$BC = h\sqrt3$ } (1)

(ii) $\angle CAB = 120°$ let $AB = x$

$\therefore BC^2 = AC^2 + AB^2 - 2AC.AB\cos 120°$

$3h^2 = \dfrac{h^2}{3} + x^2 - 2\dfrac{h}{\sqrt3}\cdot x\cdot\dfrac{-1}{2}$ } (1)

$3h^2 = \dfrac{h^2}{3} + x^2 + \dfrac{hx}{\sqrt3}$

$\therefore x^2 + \dfrac{hx}{\sqrt3} - \dfrac{8h^2}{3} = 0$

$$\therefore x = \frac{-\frac{h}{\sqrt3} \pm\sqrt{\frac{h^2}{3}+4.1.\frac{8h^2}{3}}}{2.1} \quad (1)$$

$$= \left(-\frac{h}{\sqrt3} \pm \sqrt{\frac{33h^2}{3}}\right)\div 2$$

$$x = \frac{-\frac{h}{\sqrt3} \pm h\sqrt{11}}{2} \quad \text{but } x>0$$

$$\therefore x = -\frac{h}{2\sqrt3} + \frac{h\sqrt{11}}{2}$$

$$= h\left(\frac{\sqrt{11}}{2} - \frac{1}{2\sqrt3}\right)$$

$$= h\left(\frac{\sqrt{33}-1}{2\sqrt3}\right)$$

$\therefore$ Ratio is $\dfrac{\sqrt{33}-1}{2\sqrt3} : 1$

(accept $1\cdot37 : 1$) } (1)

d) <u>Method 1</u>

$\log_b xy^3 = m \qquad \log_b x^3y^2 = p$

$\therefore \log_b x + 3\log_b y = m$ and

$3\log_b x + 2\log_b y = p$ } (1)

let $\log_b x = c$ and $\log_b y = d$.

$\therefore c + 3d = m$ — (1)

$3c + 2d = p$ — (2)

(1) × 3 $\quad 3c + 9d = 3m$ — (3)

(3) − (2) $\quad 7d = 3m - p$

$\therefore d = \dfrac{3m-p}{7}$ sub in (2)

$3c + 2\left(\dfrac{3m-p}{7}\right) = p$

$\therefore c = \dfrac{9p-6m}{21} = \dfrac{3p-2m}{7}$

$\therefore \log_b y = \frac{3m-p}{7}$ and
$\log_b x = \frac{3p-2m}{7}$ } (1)

Require $\log_b \sqrt{xy}$

$= \frac{1}{2}(\log_b x + \log_b y)$

$= \frac{1}{2}\left(\frac{3p-2m}{7} + \frac{3m-p}{7}\right)$ (1)

$= \frac{1}{2}\left(\frac{2p+m}{7}\right)$

$\therefore \log_b \sqrt{xy} = \frac{2p+m}{14}$ (1)

Method 2.

$\log_b (xy^3) = m$

$\therefore b^m = xy^3$

$x = \frac{b^m}{y^3}$ — (1)

$\log_b x^3y^2 = p$

$\therefore b^p = x^3y^2$ -- (2) } (1)

sub (2) into (1)

$b^p = \left(\frac{b^m}{y^3}\right)^3 \cdot y^2$

$b^p = \frac{b^{3m}}{y^7}$

$\therefore y^7 = \frac{b^{3m}}{b^p} \Rightarrow y^7 = b^{3m-p}$

$y = \sqrt[7]{b^{3m-p}}$

$y = b^{\frac{3m}{7} - \frac{p}{7}}$ (1) -- (3)

sub into (1)

$x = \frac{b^m}{\left(b^{\frac{3m}{7} - \frac{p}{7}}\right)^3}$

$= \frac{b^m}{b^{\frac{9m}{7} - \frac{3p}{7}}}$

$= b^{m - \frac{9m}{7} + \frac{3p}{7}}$

$x = b^{\frac{3p}{7} - \frac{2m}{7}}$ (4) (1)

$\therefore \sqrt{xy} = \sqrt{b^{\frac{3m}{7} - \frac{p}{7}} \cdot b^{\frac{3p}{7} - \frac{2m}{7}}}$

$= \sqrt{b^{\frac{m}{7} + \frac{2p}{7}}}$

$= \left(b^{\frac{m}{7} + \frac{2p}{7}}\right)^{\frac{1}{2}}$

$= b^{\frac{m}{14} + \frac{2p}{14}}$

$\therefore \log_b \sqrt{xy} = \frac{m+2p}{14}$ (1)