BAULKHAM HILLS HIGH SCHOOL

2014
YEAR 12 HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I Pages 2 – 5

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 11

60 marks
- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 The domain of the function $f(x) = \sin^{-1} 3x$ is?

(A) $-\frac{1}{3} \leq x \leq \frac{1}{3}$

(B) $-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$

(C) $-3 \leq x \leq 3$

(D) $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$

2 Given that $0 \leq a \leq \frac{\pi}{2}$ and $\sin a = \frac{3}{5}$, which of the following is an expression for $\sin(x + a)$?

(A) $\sin x + \frac{3}{5}$

(B) $\frac{3}{5}\sin x + \frac{4}{5}\cos x$

(C) $\frac{3}{5}\sin x - \frac{4}{5}\cos x$

(D) $\frac{4}{5}\sin x + \frac{3}{5}\cos x$

3 What is the remainder when $f(x) = x^3 - x^2 + x + 3$ is divided by $(x + 1)$?

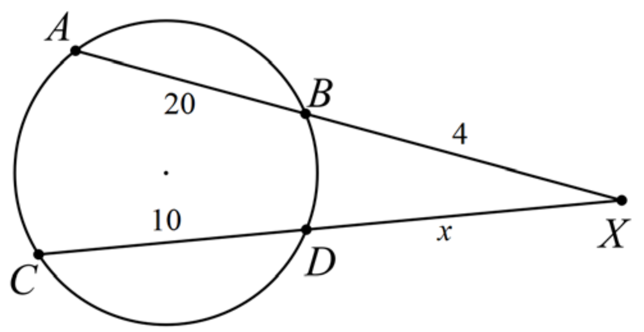
(A) -1

(B) 0

(C) 1

(D) 4

4 The points A, B, C and D lie on the circumference of a circle. The secant passing through AB intersects the secant passing through CD at the point X.



Given that $AB = 20$, $BX = 4$, $CD = 10$ and $DX = x$, find the value of x.

(A) 6

(B) 9.6

(C) 10

(D) 16

5 If $f(x) = e^{x+2}$, what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \ln x - 2$

(D) $f^{-1}(x) = \ln x + 2$

6 If the equation $f(2x) - 2f(x) = 0$ is true for all real values of x, then $f(x)$ could be

(A) $\dfrac{x^2}{2}$

(B) $\sqrt{2x}$

(C) $2x$

(D) $x - 2$

7 How many numbers greater than 5000 can be formed with the digits 4 , 5 , 6 , 7 and 8, if no digit is used more than once in a number?

(A) 96

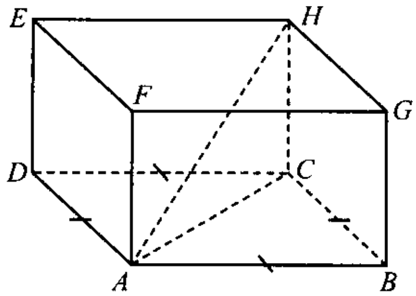
(B) 120

(C) 196

(D) 216

8 A rectangular prism with a square base, *ABCD*, is shown below.

The diagonal of the prism, AH = 8 cm
The height of the prism, HC = 4 cm



The volume of this rectangular prism is

(A) 64 cm^3

(B) 96 cm^3

(C) 128 cm^3

(D) 192 cm^3

9 The cubic function $f(x) = ax^3 + bx^2 + cx$, where *a*, *b* and *c* are positive constants, has no stationary points when

(A) $c > \frac{b^2}{4a}$

(B) $c < \frac{b^2}{4a}$

(C) $c > \frac{b^2}{3a}$

(D) $c < \frac{b^2}{3a}$

10 Which of the following expressions is correct?

(A) $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1 + x^2}}$

(B) $\tan^{-1} x = \cos^{-1} \frac{x}{\sqrt{1 + x^2}}$

(C) $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1 - x^2}}$

(D) $\tan^{-1} x = \cos^{-1} \frac{x}{\sqrt{1 - x^2}}$

END OF SECTION I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Marks

<u>Question 11</u> ***(15 marks)*** Use a *separate* answer sheet

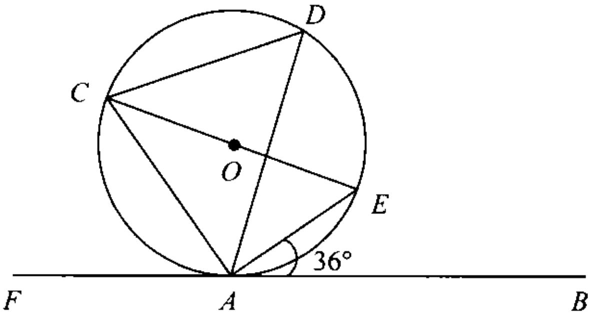
(a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{5x}$ *1*

(b) Find $\int \cos^2 5x \, dx$ *2*

(c) Solve $\dfrac{x}{x+2} \geq 3$ *3*

(d) How many eight-letter arrangements can be made using the letters of the word **INFINITE** ? *2*

(e) *FB* is a tangent meeting a circle at *A*. *CE* is the diameter, *O* is the centre and *D* lies on the circumference. $\angle BAE = 36^\circ$.



(i) Find the size of $\angle ACE$, giving reasons. *1*

(ii) Find the size of $\angle ADC$, giving reasons. *2*

(f) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x - \alpha)$, where $0 < \alpha < \dfrac{\pi}{2}$ *2*

(ii) Hence find the general solution to $\sin x - \sqrt{3}\cos x = \sqrt{2}$ *2*

Marks

Question 12 ***(15 marks)*** Use a *separate* answer sheet

(a) Find $\int \frac{dx}{\sqrt{25-4x^2}}$ *2*

(b) Two points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - ap^2$. *1*

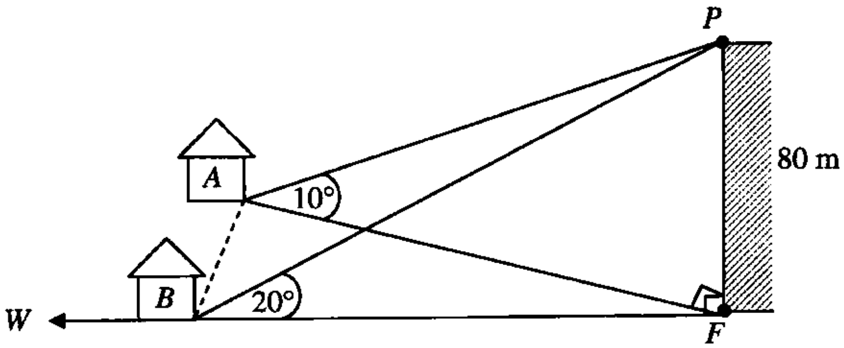
(ii) Show that the tangents at P and Q intersect at the point $R\{a(p+q), apq\}$ *3*

(iii) State the geometric condition for any two tangents to intersect on the directrix. *1*

(c) (i) Find $\frac{d}{dx}(\, x \tan^{-1} x \,)$ *1*

(ii) Hence find the exact value of $\int_0^1 \tan^{-1} x \, dx$ *3*

(d) From a lookout on the top of a vertical cliff, P, which is 80 metres high, the angles of depression of two houses in the valley are observed to be 10° and 20° respectively.



The first farmhouse, A, is northwest and the second farmhouse, B, is due west of the foot of the cliff, F.

(i) Using ΔBPF , show that $BF = 80\tan 70^\circ$, and find a similar expression for AF. *1*

(ii) Show that the distance between the farmhouses is *2*

$$AB = 80\sqrt{\tan^2 80^\circ + \tan^2 70^\circ - 2\tan 80^\circ \tan 70^\circ \cos 45^\circ}$$

(iii) Hence find AB correct to the nearest metre. *1*

Marks

<u>Question 13</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Find the exact value of $\cos\left(\tan^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{4}\right)$ *2*

(b) Two roots of the equation $x^3+px^2+q=0$, where p and q are real, are reciprocals of each other.

(i) Show that the third root is equal to $-q$ *1*

(ii) Show that $p=q-\dfrac{1}{q}$ *2*

(c) After a cricket match, all eleven players must return to school. *2*
One of the players, Rohit, owns a car and takes four passengers with him.
The remaining players must return by bus.
Simon and Vinit, must return to school together.

How many different groups of five players (including Rohit) can return to school by car?

(d) After cooking her cheesecake, Christine puts it in the fridge. The fridge is running at a constant temperature of $8^\circ C$. At time t minutes the temperature T of the cheesecake decreases according to the equation:

$$\frac{dT}{dt}=-k(T-8)\text{ , where } k \text{ is a positive constant}$$

Christine puts the cheesecake in the fridge at 9:00 a.m. when the temperature is $85^\circ C$

(i) Show that $T=8+77e^{-kt}$ satisfies both this equation and the initial conditions. *2*

(ii) Christine checks the temperature of the cheesecake at 10:00 a.m. and it is $40^\circ C$. It is best served when it reaches a temperature of $10^\circ C$. *2*

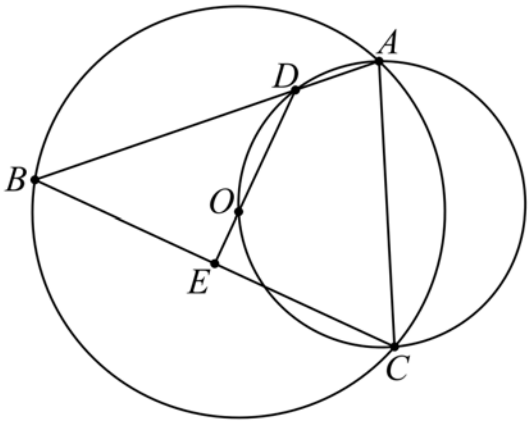
At what time (to the nearest minute) should Christine serve the cheesecake?

Question 13 continues on page 9

Marks

Question 13 (continued)

(e) ABC is a triangle inscribed in a circle with centre O. A second circle through the points A, C, and O cuts AB at D. DO is produced and meets BC at E.



Copy or trace the diagram onto your answer sheet

(i) Prove that $\angle BOE = \angle BAC$ *2*

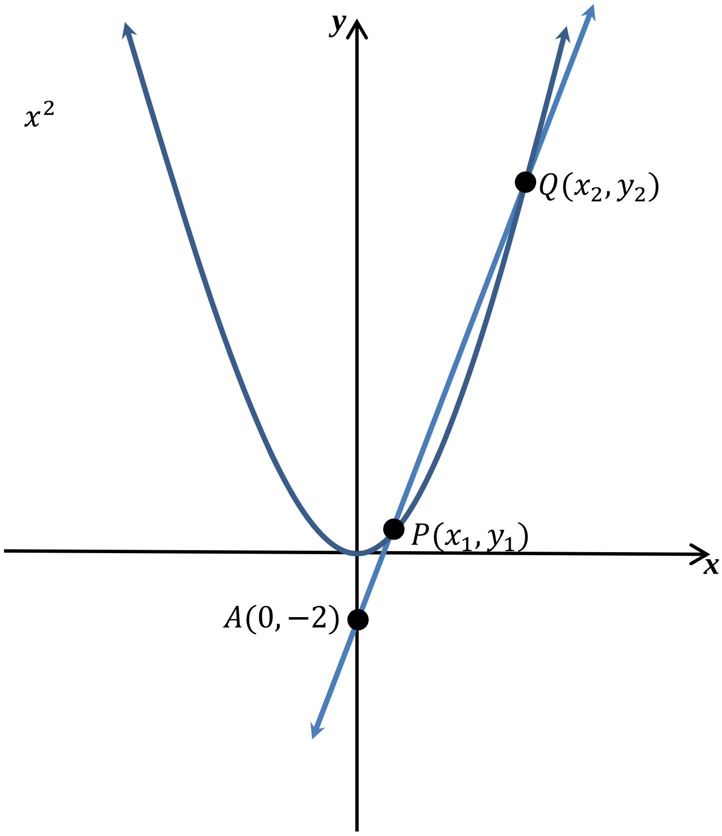
(ii) Show that $BE = CE$ *2*

End of Question 13

Marks

<u>Question 14</u> ***(15 marks)*** Use a *separate* answer sheet

(a) The line l through the point $A(0,-2)$ with slope m meets the parabola $x^2 = 8y$ at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$



(i) The line l has the equation $y = mx - 2$. 1

Show that x_1 and x_2 are the roots of the equation $x^2 - 8mx + 16 = 0$

(ii) Show that $(x_2 - x_1)^2 = 64\left(m^2 - 1\right)$ 2

(iii) Hence show that $PQ^2 = 64\left(1 + m^2\right)\left(m^2 - 1\right)$ 2

(iv) Find the values of m for which the line l is a tangent to the parabola $x^2 = 8y$ 1

(v) ΔSPQ is formed where S is the focus $(0,2)$ 2

Show that the exact area of ΔSPQ is $16\sqrt{m^2 - 1}$ $units^2$

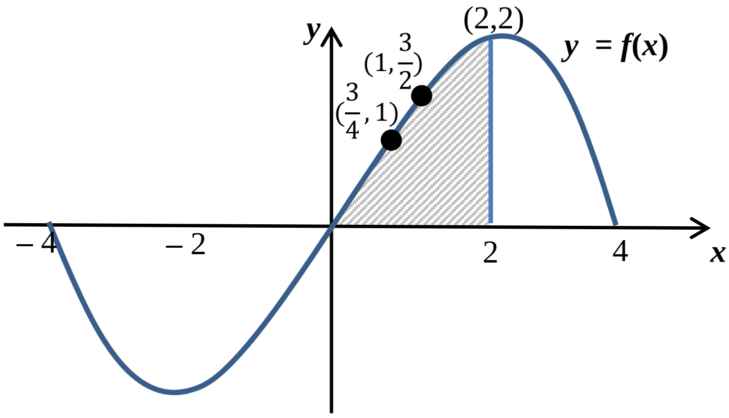
Question 14 continues on page 11

Question 14 (continued)

(b) Using the Principle of Mathematical Induction, prove that for all positive integers n, *3*

$$1 + 27 + 189 + \ldots + \left(2n^2 + 2n - 3\right)3^{n-1} = \left(n^2 - 1\right)3^n + 1$$

(c) The graph of the odd function $y = f(x)$ is shown below for $-4 \leq x \leq 4$



If it is known that the shaded area enclosed by the graph, the x-axis and the line $x = 2$ is 3 units^2 . Determine:

(i) $f^{-1}(-1)$ *1*

(ii) $\int_0^2 f^{-1}(x)\,dx$ *1*

(iii) $\int_{-2}^2 f'(x)\,dx$ *2*

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

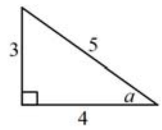
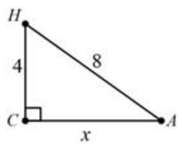
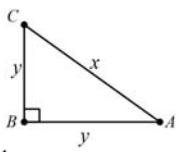
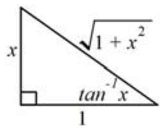
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

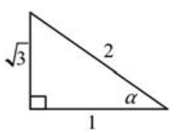
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

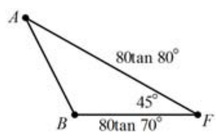
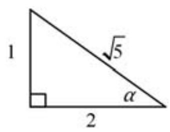
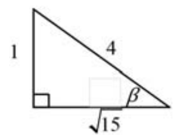
NOTE: $\ln x = \log x, \quad x > 0$

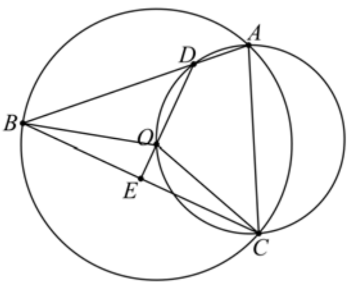
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 1 HALF YEARLY 2014 SOLUTIONS

Solution	Marks	Comments
SECTION I		
1. A – $-1 \le 3x \le 1$ $-\frac{1}{3} \le x \le \frac{1}{3}$	**1**	
2. D – $\cos a = \frac{4}{5}$ $\sin(x+a) = \sin x\cos a + \cos x \sin a$ $= \sin x \times \frac{4}{5} + \cos x \times \frac{3}{5}$ $= \frac{4}{5}\sin x + \frac{3}{5}\cos x$ 	**1**	
3. B - $f(-1) = (-1)^3 - (-1)^2 + (-1) + 3$ $= -1 - 1 - 1 + 3$ $= 0$	**1**	
4. A – $AX \times BX = CX \times DX$ (product of intercepts of intersecting secants) $24 \times 4 = (10 + x) \times x$ $96 = 10x + x^2$ $x^2 + 10x - 96 = 0$ $(x-6)(x+16) = 0$ $x = 6$ or $x = -16$ but $x > 0$ $\therefore x = 6$	**1**	
5. C - $x = e^{y+2}$ $y + 2 = \ln x$ $y = \ln x - 2$ $\therefore f^{-1}(x) = \ln x - 2$	**1**	
6. C – A: $f(2x) - 2f(x)$ $= \frac{(2x)^2}{2} - 2 \times \frac{x^2}{2}$ $= 2x^2 - x^2$ $= x^2$ B: $f(2x) - 2f(x)$ $= \sqrt{2 \times 2x} - 2 \times \sqrt{2x}$ $= \sqrt{4x} - 2\sqrt{2x}$ $= 2\sqrt{x} - 2\sqrt{2x}$ $= 2(\sqrt{x} - \sqrt{2x})$ C: $f(2x) - 2f(x)$ $= 2 \times 2x - 2 \times 2x$ $= 4x - 4x$ $= 0$ D: $f(2x) - 2f(x)$ $= 2x - 2 - 2(x-2)$ $= 2x - 2 - 2x + 4$ $= 2$	**1**	
7. D - <u>5 digit numbers</u> 5 digit $= 5!$ $= 120$ <u>3 digit numbers</u> 4 digits $= 4 \times {}^4\mathbf{P}_3$ $= 96$ $\therefore$ Total $= 120 + 96$ $= 216$	**1**	
8. B –  $x^2 = 8^2 - 4^2$ $= 48$  $y^2 + y^2 = x^2$ $2y^2 = 48$ $y^2 = 24$ Volume $= 4 \times y \times y$ $= 4y^2$ $= 96$	**1**	
9. C - $f(x) = ax^3 + bx^2 + cx$ $f'(x) = 3ax^2 + 2bx + c$ stationary points occur when $\frac{dy}{dx} = 0$ $\therefore f'(x) = 0$ has no solutions $\Delta < 0$ $(2b)^2 - 4(3a)(c) < 0$ $4b^2 - 12ac < 0$ $c > \frac{b^2}{3a}$	**1**	
10. A -  $\theta = \tan^{-1} x$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$ $\theta = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$	**1**	

SECTION II		
Solution	**Marks**	**Comments**
QUESTION 11		
11(a) $\lim\limits_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}\lim\limits_{x\to 0}\frac{\sin 2x}{2x}$ $=\frac{2}{5}$ ***OR*** $\lim\limits_{x\to 0}\frac{\sin 2x}{5x}=\lim\limits_{x\to 0}\frac{2\cos 2x}{5}$ (L'Hopitals Rule) $=\frac{2}{5}$	1	**1 mark** • Correct solution with correct working • Correct bald answer **0 mark** • Correct answer from incorrect working without $2x$ in the denominator. In particular $\lim\limits_{x\to 0}\frac{2}{5}\times\frac{\sin x}{x}$
11(b) $\int\cos^2 5x\,dx=\frac{1}{2}\int(1+\cos 10x)dx$ $=\frac{1}{2}\left(x+\frac{1}{10}\sin 10x\right)+c$ $=\frac{x}{2}+\frac{1}{20}\sin 10x+c$	2	**2 marks** • Correct solution **1 mark** • Correctly uses $\cos 2\theta$ identity • Integrates correctly from incorrect use of $\cos 2\theta$ identity
11(c) $\frac{x}{x+2}\geq 3$ $x+2\neq 0$ $x\neq -2$ $\frac{x}{x+2}=3$ $x=3x+6$ $2x=-6$ $x=-3$  $-3\leq x<-2$	3	**3 marks** • Correct graphical solution on number line or algebraic solution, with correct working **2 marks** • Bald answer • Identifies the two correct critical points via a correct method • Correct conclusion to their critical points obtained using a correct method **1 mark** • Uses a correct method • Acknowledges a problem with the denominator. **0 marks** • Solves like a normal equation , with no consideration of the denominator.
11(d) Arrangements $=\frac{8!}{3!2!}$ $=3360$	2	**2 marks** • $\frac{8!}{3!2!}$ **1 mark** • Some consideration of repeated letters
11 (e) (i) $\angle ACE=\angle BAE$ (alternate segement theorem) $=36^\circ$	1	**1 mark** • Correct answer with reason
11 (e) (ii) $\angle CAE=90^\circ$ ($\angle$ in a semi-circle) $\angle AEC+\angle CAE+\angle ACE=180^\circ$ ($\angle$ sum of Δ = 180) $\angle AEC+90^\circ+36^\circ=180^\circ$ $\angle AEC=54^\circ$ $\angle ADC=\angle AEC$ $=54^\circ$ ($\angle$'s in same segment are =)	2	**2 marks** • Correct solution **1 mark** • Bald answer • Incorrect answer with correct reasoning
11 (f) (i)  $\alpha=\tan^{-1}\sqrt{3}$ $=\frac{\pi}{3}$ $\sin x-\sqrt{3}\cos x=2\sin\left(x-\frac{\pi}{3}\right)$	2	**2 marks** • Correct solution **1 mark** • Correctly finds A or α

Solution	Marks	Comments
11 (f) (ii) $\sin x - \sqrt{3}\cos x = \sqrt{2}$ $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$ $\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ $x - \frac{\pi}{3} = \pi k + (-1)^k \sin^{-1}\frac{1}{\sqrt{2}}$, where k is an integer $x = \frac{\pi(3k+1)}{3} + (-1)^k\left(\frac{\pi}{4}\right)$	**2**	**2 marks** • Correct solution **1 mark** • Obtains correct general solution for $x - \frac{\pi}{3}$ • Leaves in terms of $\sin^{-1}\frac{1}{\sqrt{2}}$
QUESTION 12		
12(a) $\int \frac{dx}{\sqrt{25 - 4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{25}{4} - x^2}}$ $= \frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{5}{2}}\right) + c$ $= \frac{1}{2}\sin^{-1}\frac{2x}{5} + c$	**2**	**2 marks** • Correct solution **1 mark** • Uses correct standard integral
12 (b) (i) $4ay = x^2$ $y = \frac{x^2}{4a}$ $\frac{dy}{dx} = \frac{x}{2a}$ when $x = 2ap$ $\frac{dy}{dx} = p$ $y - ap^2 = p(x - 2ap)$ $y - ap^2 = px - 2ap^2$ $y = px - ap^2$	**1**	**1 mark** • Correct solution
12 (b) (ii) tangent at Q is similarly $y = qx - aq^2$ $y = px - ap^2$ $y = qx - aq^2$ $0 = (p-q)x - a\left(p^2 - q^2\right)$ $x = \frac{a\left(p^2 - q^2\right)}{p - q}$ $x = \frac{a(p-q)(p+q)}{(p-q)}$ $= a(p+q)$ $\Rightarrow y = p(a(p+q)) - ap^2$ $= ap^2 + apq - ap^2$ $= apq$ $\therefore R\{a(p+q), apq\}$	**3**	**3 marks** • Correctly shows that R is the point of intersection **2 marks** • Correctly finds the x or y value of the point R. • Correctly substitutes R into one of the tangents **1 mark** • Attempts to solve simultaneous equations using either the substitution or elimination method
12 (b) (iii) directrix has the equation $y = -a$ $\therefore apq = -a$ $pq = -1$ However p and q are the slopes of the tangents Thus the condition that the tangents intersect on the directix is that **the tangents are perpendicular**	**1**	**1 mark** • Correct condition • Condition could also be that PQ is a focal chord • *Note: simply stating that pq = – 1 is* ***NOT*** *sufficient for the mark.*

Solution	Marks	Comments
12 (c) (i) $\frac{d}{dx}\left(x\tan^{-1}x\right) = (x)\left(\frac{1}{1+x^2}\right) + \left(\tan^{-1}x\right)(1)$ $= \frac{x}{1+x^2} + \tan^{-1}x$	**1**	**1 mark** • Uses product rule correctly.
12 (c) (ii) $\frac{d}{dx}\left(x\tan^{-1}x\right) = \frac{x}{1+x^2} + \tan^{-1}x$ $\therefore \quad x\tan^{-1}x = \int\left(\frac{x}{1+x^2} + \tan^{-1}x\right)dx$ $\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$ $= \tan^{-1}1 - 0 - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	**3**	**3 marks** • Correctly solution **2 marks** • Finds correct indefinite integral • Correctly substitutes into their indefinite integral, (which does not simplify the problem), found via a logical method. **1 mark** • Logically rearranges part (i) in an attempt to find integral. *Note: incorrect answer to part (i) should be treated as correct in part (ii), provided it does not simplify calculations.*
12 (d) (i) $\frac{BF}{PF} = \tan 70^\circ$ Similarly $AF = 80\tan 80^\circ$ $\frac{BF}{80} = \tan 70^\circ$ $BF = 80\tan 70^\circ$	**1**	**1 mark** • Correctly establishing given result **and** writing down a correct expression for *AF*..
12 (d) (ii)  $AB^2 = AF^2 + BF^2 - 2 \times AF \times BF\cos\angle AFB$ $= 80^2\tan^2 80^\circ + 80^2\tan^2 70^\circ - 2(80\tan 80^\circ)(80\tan 70^\circ)\cos 45^\circ$ $= 80^2\left(\tan^2 80^\circ + \tan^2 70^\circ - 2\tan 80^\circ\tan 70^\circ\cos 45^\circ\right)$ $AB = 80\sqrt{\tan^2 80^\circ + \tan^2 70^\circ - 2\tan 80^\circ\tan 70^\circ\cos 45^\circ}$	**2**	**2 marks** • Correct solution **1 mark** • Attempts to use cosine rule with correct data • Provides a diagram with the important information labelled
12 (d) (iii) $AB = 336.3445061\ldots$ $= 336$ metres (to the nearest metre)	**1**	**1 mark** • Correct answer. *Note: no rounding penalty*
QUESTION 13		
13 (a) Let $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \sin^{-1}\frac{1}{4}$   $\cos\left(\tan^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{4}\right) = \cos(\alpha+\beta)$ $= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{\sqrt{15}}{4}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{4}\right)$ $= \frac{2\sqrt{15}-1}{4\sqrt{5}}$	**2**	**2 marks** • Correct solution **1 mark** • Correctly uses $\cos(\alpha+\beta)$ expansion, in an attempt to solve problem.

Solution	Marks	Comments
13 (b) (i) Let the roots be $\alpha, \frac{1}{\alpha}$ and β $(\alpha)\left(\frac{1}{\alpha}\right)(\beta) = -\frac{q}{1}$ $\beta = -q$	1	**1 mark** • Correct solution
13 (b) (ii) $\alpha + \frac{1}{\alpha} + \beta = -p$ $\alpha + \frac{1}{\alpha} = q - p$ $(\alpha)\left(\frac{1}{\alpha}\right) + \alpha\beta + \frac{\beta}{\alpha} = 0$ $1 - q\alpha - \frac{q}{\alpha} = 0$ $1 - q\left(\alpha + \frac{1}{\alpha}\right) = 0$ $1 - q(q-p) = 0$ $1 - q^2 + pq = 0$ $pq = q^2 - 1$ $p = q - \frac{1}{q}$	2	**2 marks** • Correct solution **1 mark** • Correctly establishes sum of roots one at a time or two at a time or equivalent merit
13 (c) Case 1: twins return by car (R, S , V and two others go by car) Ways = ${}^8\mathbf{C}_2$ = 28 Case 2: twins return by bus (S , V and four others go by bus R does not go by bus) Ways = ${}^8\mathbf{C}_4$ = 70 Total number of ways = 98	2	**2 marks** • Correct solution ${}^8\mathbf{C}_2 + {}^8\mathbf{C}_4$ is sufficient **1 mark** • Identifies two logical cases that must be considered • Successfully finds one of the required cases
13 (d) (i) $T = 8 + 77e^{-kt}$ $\frac{dT}{dt} = -77ke^{-kt}$ $= -k\left(77e^{-kt} + 8 - 8\right)$ $= -k(T-8)$ when $t = 0$, $T = 8 + 77e^0$ $= 8 + 77$ $= 85$	2	**2 marks** • Correct solution **1 mark** • Establishes initial temperature is 85° • Verifies given equation is solution to the differential equation
13 (d) (ii) when $t = 1$ $T = 40$ $40 = 8 + 77e^{-k}$ $e^{-k} = \frac{32}{77}$ $-k = \ln\frac{64}{91}$ $k = -\ln\frac{32}{77}$ $= 0.8780695191\ldots$ when $T = 10$, $10 = 8 + 77e^{-kt}$ $77e^{-kt} = 2$ $e^{-kt} = \frac{2}{77}$ $-kt = \ln\frac{2}{77}$ $t = -\frac{1}{k}\ln\frac{2}{77}$ $t = 4.15759591\ldots$ The cheesecake should be served at 1:09 p.m.	2	**2 marks** • Correct solution **1 mark** • Finds correct value of k • Uses their value of k correctly to find a solution *Note: no rounding penalty*
13 (e) (i) $\angle COE = \angle BAC$ (exterior $\angle$ cyclic qudrilateral = opposite interior $\angle$) $\angle BOC = 2\angle BAC$ ($\angle$ at centre twice $\angle$ at circumference on same arc) $\angle BOC = \angle COE + \angle BOE$ (common $\angle$) $2\angle BAC = \angle BAC + \angle BOE$ $\angle BOE = \angle BAC$  	2	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution
13 (e) (ii) In ΔBOE and ΔCOE $OB = OC$ (= radii) (S) $\angle BOE = \angle COE$ (proven in (i)) (A) OE is common (S) $\therefore \Delta BOE \equiv \Delta COE$ (SAS) $BE = CE$ (corresponding sides in $\equiv$ Δ 's)	2	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution

Solution	Marks	Comments
QUESTION 14		
14 (a) (i) x_1 and x_2 would be the solutions (i.e. roots) to the parabola and the line solved simultaneously by eliminating y $x^2 = 8(mx - 2)$ $x^2 = 8mx - 16$ $x^2 - 8mx + 16 = 0$	1	**1 mark** • Correct explanation
14 (a) (ii) sum of roots: $x_1 + x_2 = 8m$ product of roots: $x_1 x_2 = 16$ $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1 x_2$ $= (8m)^2 - 4 \times 16$ $= 64m^2 - 64$ $= 64\left(m^2 - 1\right)$	2	**2 marks** • Correct solution **1 mark** • Finds the sum and product of the roots • Finds both roots
14 (a) (iii) $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ $= (x_2 - x_1)^2 + (mx_2 - 2 - mx_1 + 2)^2$ $= (x_2 - x_1)^2 + m^2(x_2 - x_1)^2$ $= (x_2 - x_1)^2\left(1 + m^2\right)$ $= 64\left(m^2 - 1\right)\left(1 + m^2\right)$	2	**2 marks** • Correct solution **1 mark** • Eliminates y from distance formula, using equation of l, or equivalent merit
14 (a) (iv) If l is a tangent then $PQ = 0$ $64\left(m^2 - 1\right)\left(1 + m^2\right) = 0$ $m^2 - 1 = 0 \qquad 1 + m^2 = 0$ $m^2 = 1 \qquad m^2 = -1$ $m = \pm 1 \qquad$ no real solutions $\therefore m = \pm 1$	1	**1 mark** • Correct solution
14 (a) (v) perpendicular distance from S to PQ $d = \dfrac{\lvert m(0) - (2) - 2 \rvert}{\sqrt{m^2 + 1}}$ $= \dfrac{4}{\sqrt{m^2 + 1}}$ $\Rightarrow \; Area = \dfrac{1}{2} \times \sqrt{64\left(1 + m^2\right)\left(m^2 - 1\right)} \times \dfrac{4}{\sqrt{m^2 + 1}}$ $= 2 \times 8\sqrt{m^2 - 1}$ $= 16\sqrt{m^2 - 1}$	2	**2 marks** • Correct solution **1 mark** • Finds the perpendicular height of triangle, or equivalent merit
14 (b) When $n = 1$; LHS $= \left[2\left(1^2\right) + 2 \times 1 - 3\right] \times 3^{1-1}$ $= (2 + 2 - 3) \times 3^0$ $= 1$ RHS $= \left(1^2 - 1\right) \times 3^1 + 1$ $= 0 \times 3 + 1$ $= 1$ *LHS = RHS* Hence the result is true for $n = 1$ Assume the result is true for $n = k$ i.e. $1 + 27 + 189 + \ldots + \left(2k^2 + 2k - 3\right)3^{k-1} = \left(k^2 - 1\right)3^k + 1$ Prove the result is true for $n = k + 1$ i.e. Prove $1 + 27 + 189 + \ldots + \left(2(k+1)^2 + 2k - 1\right)3^k = \left((k+1)^2 - 1\right)3^{k+1} + 1$ ***PROOF:*** $1 + 27 + 189 + \ldots + \left(2k^2 + 2k - 3\right)^{3^{k-1}} + \left(2(k+1)^2 + 2k - 1\right)3^k$ $= \left(k^2 - 1\right)3^k + 1 + \left(2(k+1)^2 + 2k - 1\right)3^k$ $= \left(k^2 - 1 + 2(k+1)^2 + 2k - 1\right)3^k + 1$ $= \left(3k^2 + 6k\right)3^k + 1$ $= 3\left(k^2 + 2k\right)3^k + 1$ $= \left(k^2 + 2k + 1 - 1\right)3^k + 1$ $= \left((k+1)^2 - 1\right)3^k + 1$ Hence the result is true for $n = k + 1$,if it is true for $n = k$ Since the result is true for $n = 1$, then it is true for all positive integers by induction.	3	There are 4 key parts of the induction; 1. Proving the result true for $n = 1$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement **3 marks** • Successfully does all of the 4 key parts **2 marks** • Successfully does 3 of the 4 key parts **1 mark** • Successfully does 2 of the 4 key parts

Solution	Marks	Comments
14 (c) (i) If $\left(\frac{3}{4},1\right)$ is a point on $f(x)$, then $\left(1,\frac{3}{4}\right)$ is a point on $f^{-1}(x)$ $f^{-1}(-1)= -f^{-1}(1)$ $= -\frac{3}{4}$	1	**1 mark** • Correct answer
14 (c) (ii) $\int_0^2 f^{-1}(x)\,dx = 2\times 2-\int_0^2 f(x)\,dx$ $=4-3$ $=1$	1	**1 mark** • Correct answer
14 (c) (iii) $\int_{-2}^{2} f'(x)dx = [f(x)]_{-2}^{2}$ $=f(2)-f(-2)$ $=2+2$ $=4$	2	**2 marks** • Correct solution **1 mark** • Realises that the integral of the derivative is the original function **0 marks** • Bald answer