BAULKHAM HILLS HIGH SCHOOL

2015

YEAR 12 HALF-YEARLY

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

Total marks – 70

Exam consists of 8 pages.
This paper consists of TWO sections.

Section 1 – Pages 2-3

Multiple Choice

Question 1-10 (10 marks)

Section 2 – Pages 4-7

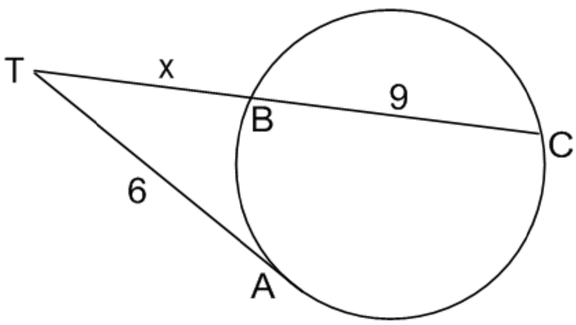
Extended Response

Question 11- 14 (60 marks)

Standard integrals provided on page 8

Section I – 10 marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section.
Use the multiple choice answer sheet provided in the answer booklet.

1 TA is a tangent to the circle at A and TC is a secant, meeting the circle at B.



Given that TA=6, CB=9 and TB=x , what is the value of x?

(A) -12 (B) 2 (C) 3 (D) 4 **1**

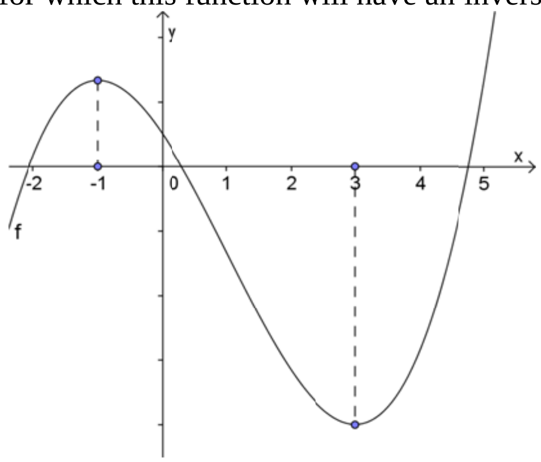
2 Let α, β, γ be the roots of $x^3 - 4x + 1 = 0$. What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) -4 (B) -1 (C) 1 (D) 4 **1**

3 Two of the roots of the equation $x^3 - 2x^2 + kx + 18 = 0$ are equal in magnitude but opposite in sign. What is the value of k?

(A) -9 (B) -6 (C) 6 (D) 9 **1**

4 The diagram shows a polynomial function $y = f(x)$. What is the largest possible domain containing $x = 0$ for which this function will have an inverse function?



(A) $x \leq 1$ (B) $-1 \leq x \leq 3$ (C) $0 \leq x \leq 3$ (D) $x \leq 0$ **1**

5 Given the coordinates of A and B are (-1,1) and (3,-1) respectively, find the coordinates of the point P which divides AB externally in the ratio 1:3.

(A) $(2, -\frac{1}{2})$ (B) $(-\frac{3}{2}, \frac{1}{2})$ (C) $(-3,2)$ (D) $(4,-2)$ **1**

6 The number of different arrangements of the word SIMILES which begin and end with letter S is:

(A) $\frac{7!}{2!}$ (B) $\frac{7!}{2!2!}$ (C) $\frac{5!}{2!}$ (D) $\frac{5!}{2!2!}$ **1**

7 $\int 6\cos^2 3x\, dx =$

(A) $2\sin^3 3x + c$ (B) $2\cos^3 6x + c$

(C) $3x + \frac{1}{2}sin6x + c$ (D) $3x - \frac{3}{2}sin6x + c$ **1**

8 A curve is defined by the parametric equations $x = sin2t$ and $y = cos2t$. Which of the following is equal to $\frac{dy}{dx}$?

(A) $cos4t$ (B) $2tan2t$ (C) $2sin4t$ (D) $-tan2t$ **1**

9 If $y = [\ln(4x-1)]^2$ then $\frac{dy}{dx}$ is equal to:

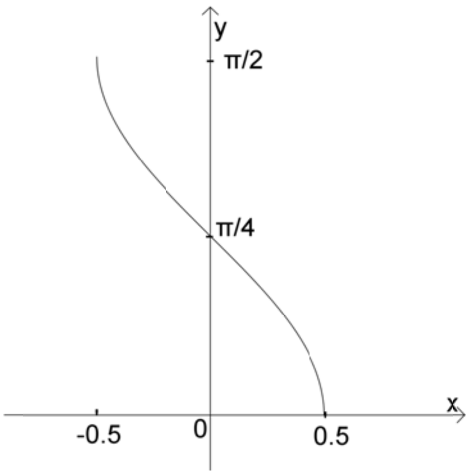
(A) $\frac{8\ln(4x-1)}{4x-1}$

(B) $\frac{2\ln(4x-1)}{4x-1}$

(C) $\frac{8}{4x-1}$ **1**

(D) $\frac{2}{4x-1}$

10



The graph above shows:

(A) $y = 2cos^{-1}2x$ (B) $y = \frac{1}{2}\cos^{-1}(\frac{1}{2}x)$

(C) $y = 2\cos^{-1}(\frac{1}{2}x)$ (D) $y = \frac{1}{2}\cos^{-1} 2x$

Section II – Extended responses
Attempt all questions. Show all necessary working.
Answer each question in the appropriate pages of your answer booklet.

Question 11 (15 marks)

a) (i) Express $\sqrt{3}\,cosx - sinx$ in the form $Acos(x+\alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Without using calculus, find the maximum value of $3 - 3cosx + \sqrt{3}sinx$. **2**

b) Given $P(x) = -x^3 + 3x + 2$:

(i) Show that $x + 1$ is a factor of $P(x)$ **1**
(ii) Fully factorise $P(x)$ **2**
(iii) Sketch the graph of $y = P(x)$ **2**

c) 5 men and 3 women are candidates for election to a committee of 4 people. How many committees are possible if there must be at least one man and at least one woman selected? **2**

d) Given $f(x) = \tan^{-1}\frac{4}{x} + \tan^{-1}\frac{x}{4}$:

(i) find $f'(x)$ **2**
(ii) sketch $y = f(x)$ **2**

Question 12 (15 marks)

a) Without the use of a calculator, find the exact value of: **2**

$$\sin(2\tan^{-1}\sqrt{2})$$

b) For the lines $2x - y + 1 = 0$ and $x - 4y + 4 = 0$

(i) Find the acute angle between the lines, correct to the nearest degree. **2**
(ii) Find the area of the triangle bounded by these two lines and the x-axis. **2**

c) Solve the equation: $\sin 2x = cosx$ for $0 \le x \le 2\pi$ **2**

Question 12 continues on the next page

Question 12 (cont)

d) Solve:

$$\ln(2x^2 + 3x - 9) = \ln(2x^2 - 7x + 6) + 2$$ **3**

expressing your answer in terms of e.

e) Given $sin3\theta = 3sin\theta - 4\sin^3\theta$,

find $\int 2\sin^3\theta \, d\theta$ **2**

f) Simplify :

$$\sin(a+b)\cos\left(\frac{\pi}{2} - a - b\right) - \cos(a+b)\cos(\pi - a - b)$$ **2**

Question 13 (15 marks)

a) (i) Prove by Mathematical Induction that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

for all positive integers n **3**

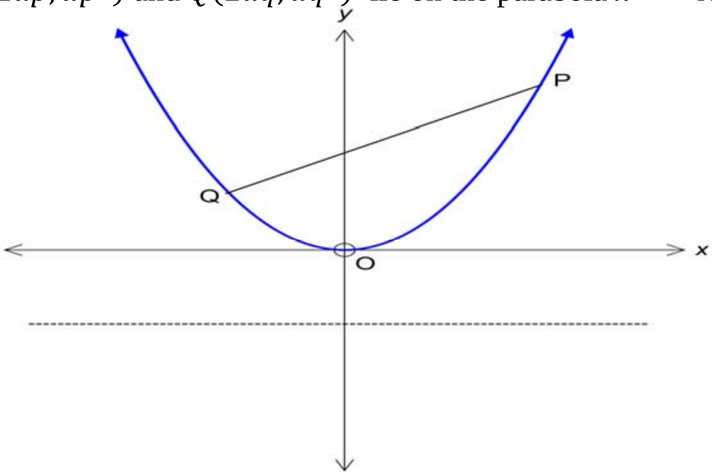
(ii) Hence evaluate **1**

$$11^3 + 12^3 + 13^3 + \ldots + 50^3$$

Question 13 continues on the next page

Question 13 (cont)

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.



The equation of the chord PQ is $y = (\frac{p+q}{2})x - apq$ (Do NOT prove this)

(i) Show that if PQ passes through the focus $S(0, a)$ then $pq = -1$ **1**

(ii) X is the midpoint of the focal chord PQ. T lies on the directrix such that XT is perpendicular to the directrix and W is the midpoint of XT. Find the equation of the locus of W. **3**

c) Newton's Law of Cooling states that the rate of change of the temperature (T) of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium (P)

ie $\frac{dT}{dt} = k(T - P)$

(i) If A and k are constants, show that $T = P + Ae^{kt}$ satisfies Newton's Law of Cooling. **1**

(ii) A cup of coffee is made with temperature $100°C$. Two minutes later, its temperature is 93°C but it is still too hot to drink. If the temperature in the room is 23°C, evaluate A and k (correct to 3 significant figures) **2**

(iii) The coffee is drinkable when it reaches 80°C. How many minutes after the coffee is made will this be? **1**

d) Solve

$$\frac{x^2+x-6}{x} \geq 0$$

3

Question 14 (15 marks)

a) Find $\int \tan^2 2x \, dx$ **2**

b) Simplify **2**

$$\frac{2^{4n} \times 3^{2n}}{8^n \times 6^n} + 3^n$$

c) A function is defined by $f(x) = x - \frac{1}{x}, x > 0$

(i) Show that the inverse function is given by $f^{-1}(x) = \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}$ **2**

(ii) Sketch the curves $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes **2**

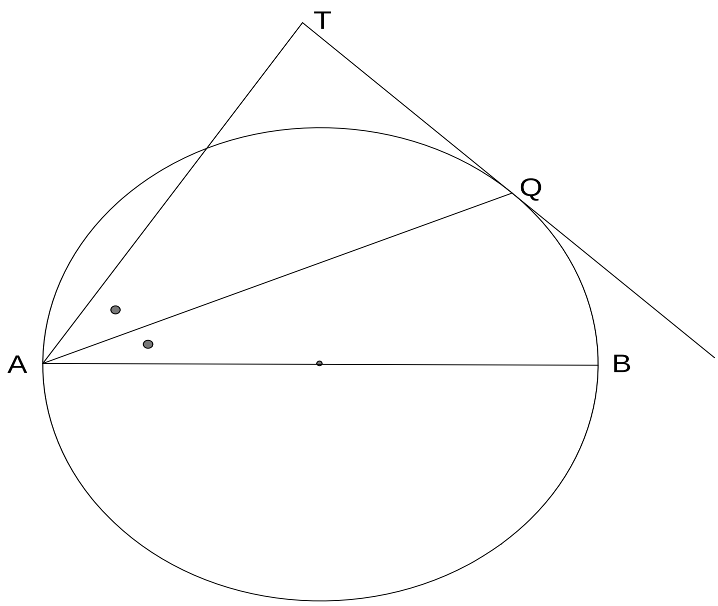
(iii) State the domain and range of $f^{-1}(x)$ **2**

d) Prove that

$$\frac{sin8x}{1 + cos8x} = tan4x$$ **2**

e) In the diagram, AB is a diameter of the circle.
TQ is a tangent and AT is a secant which meets the circle at P.
AQ bisects ∠BAT.

Copy the diagram into your answer booklet.



3

Prove that ∠ATQ=90°.

End of Examination

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \; a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \; a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx \qquad = \ln(x + \sqrt{x^2 - a^2}), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx \qquad = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \ \ x > 0$

XI 2015 H.Y. SOLNS

Section I.

1. $x(x+9) = 36$
$x^2 + 9x - 36 = 0$
$(x-3)(x+12) = 0$
$x = 3, \cancel{-12}$ (C)

2. $\dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $[x^3 + 0x^2 - 4x + 1]$
$= \dfrac{-4}{-1} = +4$ (D)

3. Let roots $= \alpha, -\alpha, \beta$.
$\alpha - \alpha + \beta = 2 \quad \therefore \beta = 2$
$\alpha\beta - \alpha\beta - \alpha^2 = k \quad \therefore k = -\alpha^2$
$\alpha . -\alpha . \beta = -18$
$-\alpha^2 . 2 = -18$
$\therefore k = -\alpha^2 = -9$ (A)

4. $-1 \le x \le 3$ (B)

5. $\left(\dfrac{1(3)-3(-1)}{1-3}, \dfrac{1(-1)-3(1)}{1-3} \right)$
$= (-3, 2)$ (C)

6. S _ _ _ _ _ S
5 letters, 2 same.
$\dfrac{5!}{2!}$ (C)

7. $6\int \cos^2 3x \, dx$
$= 6\int \frac{1}{2}(1+\cos 6x) \, dx$
$= 3\int 1 + \cos 6x \, dx$
$= 3\left(x + \frac{1}{6}\sin 6x\right) + c$
$= 3x + \frac{1}{2}\sin 6x + c$ (C)

8. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$
$= \dfrac{-\cancel{2}\sin 2t}{\cancel{2}\cos 2t}$
$= -\tan 2t$ (D)

9. $\dfrac{dy}{dx} = 2\ln(4x-1) . \dfrac{1}{4x-1} . 4$
$= \dfrac{8\ln(4x-1)}{4x-1}$ (A)

10. ~~Ampl. halved~~ (D)

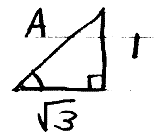
Question 11

a) (i) $\sqrt{3}\cos x - \sin x = A\cos(x+\alpha)$

$= A\cos x\cos\alpha - A\sin x\sin\alpha$

$\cancel{A}\cos\alpha = \sqrt{3}/A$

$\cancel{A}\sin\alpha = 1/A$



$A = 2 \quad \longleftarrow (1)$

$\alpha = \pi/6 \quad \longleftarrow (1)$

$\therefore \quad = 2\cos\left(x + \frac{\pi}{6}\right).$

(ii) $3 - 3\cos x + \sqrt{3}\sin x$

$= 3 - \sqrt{3}(\sqrt{3}\cos x - \sin x)$

$= 3 - \sqrt{3}\left(\underbrace{2\cos(x + \pi/6)}\right) \quad \longleftarrow (1)$

max value 2

min value -2

$\therefore$ Max. value $= 3 - \sqrt{3}(-2)$

$= 3 + 2\sqrt{3} \quad \longleftarrow (1)$

b) $P(x) = -x^3 + 3x + 2$

(i) $P(-1) = -(-1)^3 + 3(-1) + 2$

$= 1 - 3 + 2$

$= 0 \quad \therefore x+1$ is a factor (1)

(ii)

$$
\begin{array}{r l}
 & -x^2 + x + 2 \quad \longleftarrow (1) \\
x+1 \,\big) & -x^3 + 0x^2 + 3x + 2 \\
 & -x^3 - x^2 \\
\hline
 & x^2 + 3x \\
 & x^2 + x \\
\hline
 & 2x + 2 \\
 & 2x + 2 \\
\hline
 & 0
\end{array}
$$

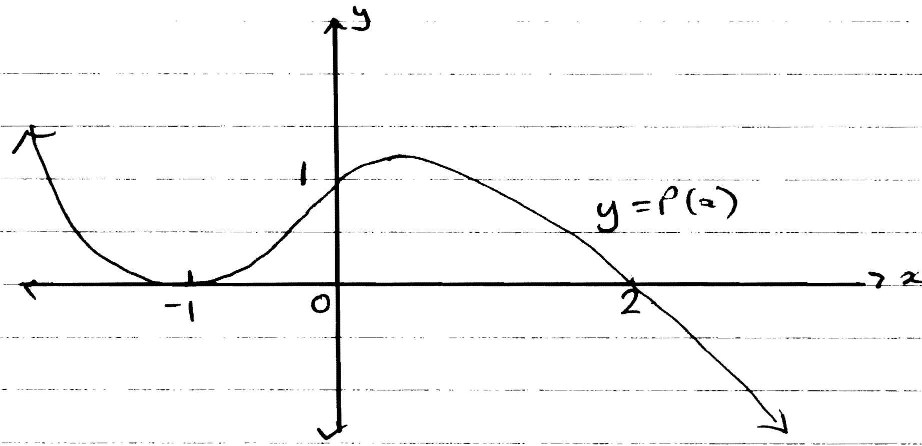
$\therefore P(x) = (x+1) \cdot -(x^2 - x - 2)$

$= -(x+1)(x+1)(x-2)$

$= -(x+1)^2(x-2) \quad \longleftarrow (1)$

$= (2-x)(x+1)^2$

(iii)



(1) Single root + decr. to right

(1) Double root at $x = -1$

c) Without restriction : ${}^8C_4 = 70$ —— (1)

All men : 5

All women ~~(x)~~ : not possible

$\therefore$ No. of ways $= 70 - 5 = 65$ —— (1)

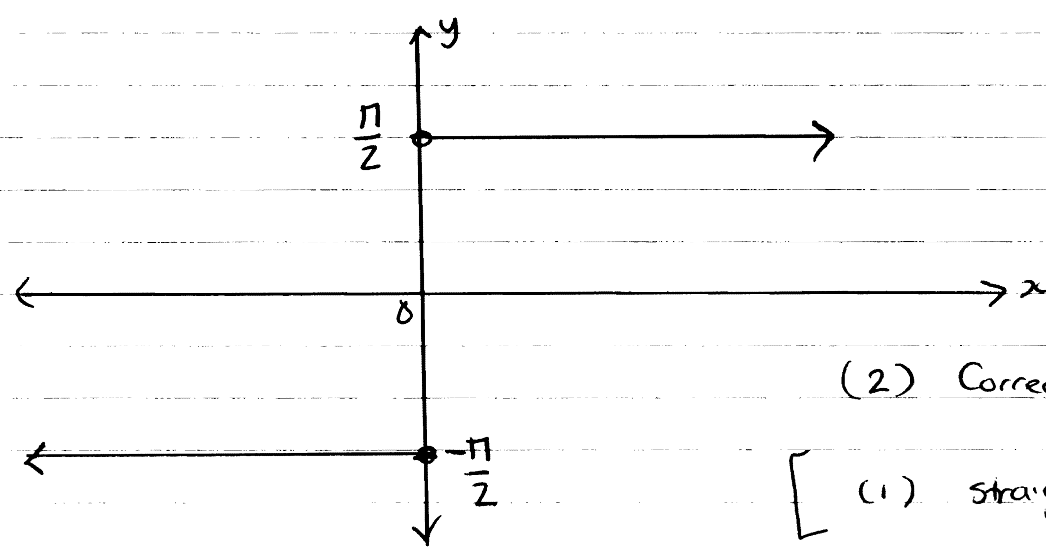
d) $f(x) = \tan^{-1}\frac{4}{x} + \tan^{-1}\frac{x}{4}$

(i) $f'(x) = \frac{1}{1+\left(\frac{4}{x}\right)^2}\cdot\frac{-4}{x^2} + \frac{4}{16+x^2}$ $\leftarrow$ (1) One or both correct

$= \frac{-4}{x^2+16} + \frac{4}{16+x^2}$

$= 0$ $\leftarrow$ (1)

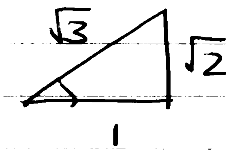
(ii)



(2) Correct 2 rays

[(1) straight line]

Question 12.



a) $\sin(2\underbrace{\tan^{-1}\sqrt{2}}_{A})$

$\cancel{=\sin}\left(\qquad \right)$

$= 2\sin A \cos A$ ← (1)

$= 2 \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$

$= \frac{2\sqrt{2}}{3}$ ← (1)

b) (i) $\tan\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$m_1 = 2$, $m_2 = \frac{1}{4}$ ← (1)

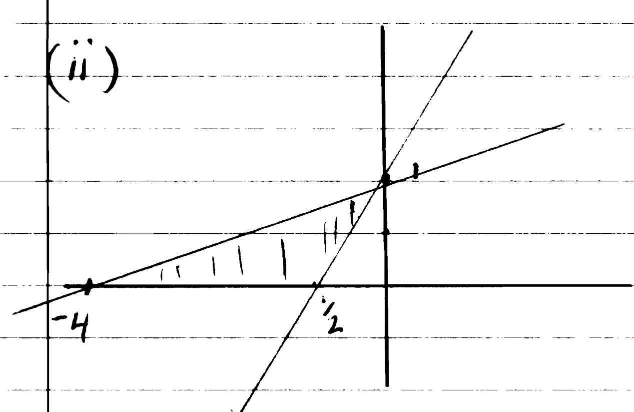
$= \left| \frac{2 - \frac{1}{4}}{1 + 2(\frac{1}{4})} \right|$ ← (1)

$= \frac{1\frac{3}{4}}{1\frac{1}{2}}$

$= \frac{7}{6}$

$\phi = \tan^{-1}\left(\frac{7}{6}\right) \doteqdot 49°$ ← (1)

(ii)



Intersect at $(0, 1)$ ← (1)

$A = \frac{1}{2} \times 3.5 \times 1$

$= 1.75 \text{ units}^2$ ← (1)

c) $2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ (1) $\qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$ (1)

d) $\ln(2x-3)(x+3) - \ln(2x-3)(x-2) = 2$

$$\ln\left(\frac{\cancel{(2x-3)}(x+3)}{\cancel{(2x-3)}(x-2)}\right) = 2 \quad \leftarrow (1)$$

$$\frac{x+3}{x-2} = e^2 \quad \leftarrow (1)$$

$$x+3 = e^2x - 2e^2$$

$$x(1-e^2) = -2e^2 - 3$$

$$x = \frac{-2e^2-3}{1-e^2} \text{ or } \frac{2e^2+3}{e^2-1} \quad (1)$$

e) $4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\therefore \int 2\sin^3\theta \, d\theta = \frac{1}{2}\int 3\sin\theta - \sin 3\theta \, d\theta \quad \leftarrow (1)$$

$$= \frac{1}{2}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right) + c$$

$$= -\frac{3}{2}\cos\theta + \frac{1}{6}\cos 3\theta + c \quad \leftarrow (1)$$

f) $\sin(a+b)\cos\left(\frac{\pi}{2}-(a+b)\right)-\cos(a+b)\cos\left(\pi-(a+b)\right)$

$= \underline{\sin(a+b)\sin(a+b)} - \cos(a+b)\,.\,\underline{-\cos(a+b)}$ (1)

$= \sin^2(a+b) + \cos^2(a+b)$

$= 1.$ —— (1)

Question 13.

a) If $n=1$

$LHS = 1^3 = 1$

$RHS = \frac{1}{4}\cdot 1^2 \cdot 2^2 = \frac{1}{4}(4) = 1$

LHS = RHS

$\therefore$ True for $n=1$

Assume true for $n=k$

ie Assume

$$1^3+2^3+3^3+\dots+k^3 = \frac{1}{4}k^2(k+1)^2$$

Now prove true for $n=k+1$

ie

Prove $\underbrace{1^3+2^3+3^3+\dots+k^3}+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

LHS =

$\frac{1}{4}k^2(k+1)^2 + (k+1)^3$

(3)

$= \frac{1}{4}(k+1)^2\left(k^2+4(k+1)\right)$

$= \frac{1}{4}(k+1)^2(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= RHS$ $\therefore$ If true for $n=k$, then also true for $n=k+1$

Now statement is true for $n=1$

$\therefore$ Also true for $n=2,3,4\dots$ and by induction, true for all positive integers n.

(ii) $\frac{1}{4} \cdot 50^2 \cdot 51^2 - \frac{1}{4} \cdot 10^2 \cdot 11^2$

$= 1622600$ $\longleftarrow$ (1)

b) (i) ~~#~~ Sub. $S(0, a)$ into eqn:

$a = \frac{p+q}{2} \cdot 0 - apq$ $\longleftarrow$ (1)

$a = -apq$ $\therefore$ $pq = -1$

(ii) $X = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2} \right) = \left(a(p+q), \frac{a(p^2+q^2)}{2} \right)$

$T = (a(p+q), -a)$ (1) for T

$W = \left(a(p+q), \frac{\frac{a}{2}(p^2+q^2) - a}{2} \right)$ (1) for W

$= \left(a(p+q), \frac{a(p^2+q^2-2a)}{4} \right)$

At W: $x = a(p+q)$ $p+q = \frac{x}{a}$

$y = \frac{a(p^2+q^2-2a)}{4}$ $p^2+q^2 = \frac{4y}{a} + 2$

$(p+q)^2 - 2pq = \frac{4y}{a} + 2$

$\left(\frac{x}{a}\right)^2 - 2(-1) = \frac{4y}{a} + 2$

$\frac{x^2}{a^2} + \cancel{2} = \frac{4y}{a} + \cancel{2}$

Locus: $\underline{x^2 = 4ay}$ (1) for locus

c) $T = P + Ae^{kt}$

(i) $\dfrac{dT}{dt} = \dfrac{d}{dt}\left(P + Ae^{kt}\right)$

$= 0 + Ae^{kt} . k$ but $Ae^{kt} = T - P$ (1)

$= k(T - P)$ as required.

(ii) $P = 23$: $T = 23 + Ae^{kt}$

$t = 0, T = 100$: $100 = 23 + Ae^{k(0)}$ $\therefore \underline{A = 77}$ — (1)

$t = 2, T = 93$ $93 = 23 + 77e^{2k}$

$70 = 77e^{2k}$

$\dfrac{70}{77} = e^{2k}$

$k = \dfrac{1}{2} \ln \dfrac{70}{77} \doteqdot \underline{-0.0477}$ — (1)

(iii) $T = 23 + 77e^{-0.0477t}$

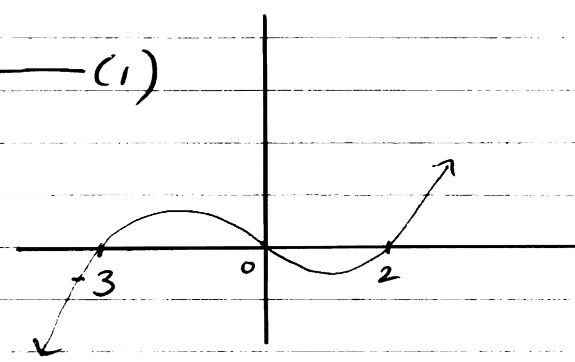
$80 = 23 + 77e^{-0.0477t}$

$\dfrac{57}{77} = e^{-0.0477t}$

$t = \dfrac{-1}{0.0477} \ln \dfrac{57}{77} \doteqdot \underline{6.3 \text{ min}}$ ← (1)

d) $\left(\dfrac{x^2 + x - 6}{x}\right)^{\times x^2} \geq 0^{\times x^2}$ (note : $x \neq 0$)

$x(x+3)(x-2) \geq 0$ —— (1)



Soln : $\underline{-3 \leq x < 0}$, $\underline{x \geq 2}$

(1) (1)

Q14

a) $\int \tan^2 2x \, dx = \int \sec^2 2x - 1 \, dx$ (1)

$= \frac{1}{2} \tan 2x - x + c$ (1) must have $+c$.

b) $\dfrac{2^{4n} \times 3^{2n}}{8^n \times 6^n} + 3^n$

$= \dfrac{2^{4n} \times 3^{2n}}{2^{3n} \times 2^n \times 3^n} + 3^n$ (1)

$= \dfrac{2^{4n} \times 3^{2n}}{2^{4n} \times 3^n} + 3^n$

$= 3^n + 3^n$

$= 2 \times 3^n$ (1)

c) i) $f: y = x - \frac{1}{x}$ $(x > 0)$

$f^{-1}: x = y - \frac{1}{y}$ $(y > 0)$

$xy = y^2 - 1$

$y^2 - xy - 1 = 0$ (1)

$y = \dfrac{x \pm \sqrt{(-x)^2 - 4.1.-1}}{2} = \dfrac{x \pm \sqrt{x^2+4}}{\cancel{4}\ 2}$

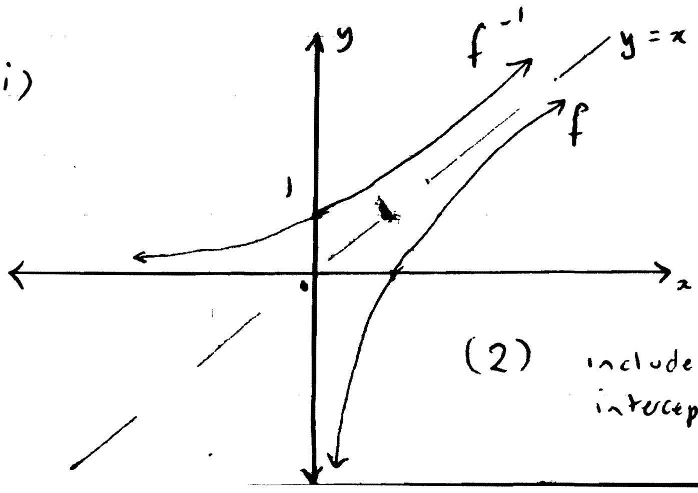
$= \frac{x}{2} \pm \sqrt{\dfrac{x^2+4}{4}}$

$\therefore \quad y = \frac{x}{2} + \sqrt{\dfrac{x^2}{4} + 1}$ (1) Some explanation needed

because ~~since~~ $x > 0$ in $f(x)$, then $y > 0$ in $f^{-1}(x)$

$\therefore +$ (not $\pm$).

ii)



(2) include intercepts.

(iii) D: all real x (1)

R: $y > 0$ (1)

1) $LHS = \dfrac{\sin 8x}{1+\cos 8x} = \dfrac{2\sin 4x\cos 4x}{\cancel{1}+2\cos^2 4x-\cancel{1}}$ (1)

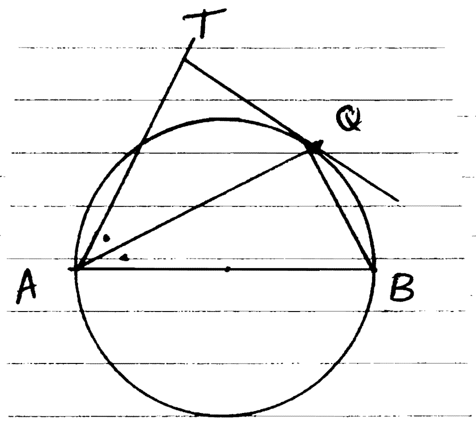
$= \dfrac{\cancel{2}\sin 4x\,\cancel{\cos 4x}}{\cancel{2}\cos^{\cancel{2}} 4x}$

$= \dfrac{\sin 4x}{\cos 4x}$ (1)

$= \tan 4x$

$= RHS$

2)



Join BQ.

In $\triangle TQA$, $\triangle QBA$

$\angle TQA = \angle QBA$ (alternate segment theorem) (1)

$\angle TAQ = \angle QAB$ (QA bisects $\angle TAB$)

$\therefore \triangle TQA \parallel\!| \triangle QBA$ (matching $\angle$s equal) or (AA)

$\therefore \angle QTA = \angle BQA$ (matching $\angle$s of similar $\triangle$s are equal) (1)

but $\angle BQA = 90^\circ$ ($\angle$ in semicircle) (1)

$\therefore \angle QTA = 90^\circ$