Name : ____________________
Class : 12 MT___

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

1999 AP3

YEAR 12 HALF YEARLY HSC

MATHEMATICS
3/4 UNIT (COMMON)

Time allowed - 2 HOURS
(plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

* Attempt ALL questions.
* All questions are of equal value
* All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
* Standard Integrals are provided. Approved calculators may be used.
* Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.

***Each page must show your class and your name.**

QUESTION 1 — **Marks**

(a) Expand $(2x-y)^5$ — 2

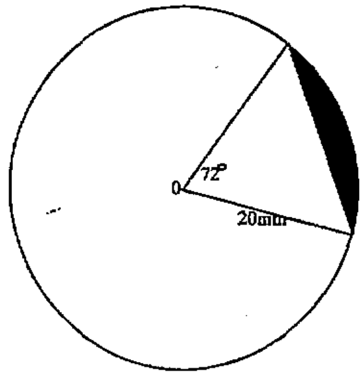
(b) (i) Write down the expansion of $\cos(\alpha-\beta)$. — 3

(ii) Find the exact values of $\cos 45^0$ and $\cos 30^0$.

(iii) Hence find the exact value of $\cos 15^0$.

(c) (i) Convert 72^0 to radians, giving your answer in terms of π. — 3

(ii) Hence or otherwise, find the shaded area below correct to 3 significant figures.



(d) Solve $\sin 2x = \sqrt{3}\cos 2x, 0 \le x \le 2\pi$. — 2

(e) Differentiate with respect to x — 2

(i) $\sqrt[3]{4x-1}$

(ii) $\dfrac{x}{\cot x}$.

QUESTION 2 **BEGIN A NEW PAGE** **Marks**

(a) Given $\int_0^3 f(t)dt = 6$, evaluate **4**

(i) $\int_0^1 f(t)dt + \int_1^3 (f(t)+1)dt$

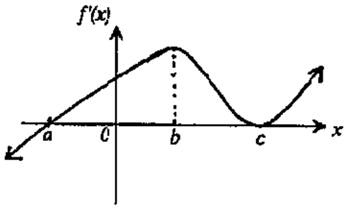
(ii) $\int_3^0 (f(t)+t)dt$

(b) Find **2**

(i) $\int \frac{x^4+2x^3+3}{x^2}dx$

(ii) $\int \frac{dt}{(3-t)^2}$.

(c) The *gradient function* of $y = f(x)$ is graphed below. **6**



(i) Copy this diagram onto your answer sheet.

(ii) On the same diagram, sketch and label a possible graph of $y = f''(x)$.

(iii) State the domain where $y = f(x)$ is concave down.

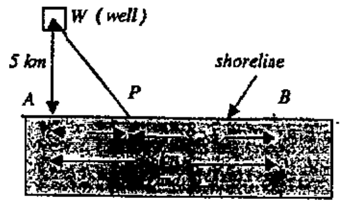
(iv) Find the x values of any points of inflection.

(v) Find any stationary points and determine their nature.

QUESTION 3 **BEGIN A NEW PAGE** **Marks**

(a) Find the equation of any asymptotes of the curve $y = \frac{x^2+x+1}{x}$. **2**

(b)



An offshore oil well is located at a point W, which is 5 *km* from the closest shorepoint A on a straight shoreline.
The oil is to be piped to a shorepoint B that is 8 *km* from A by piping it on a straight line under water from W to some shorepoint P between A and B and then on to B via a pipe along the shoreline.

If the cost of laying the pipe is \$125 000 per *km* under water and \$75 000 per *km* over land.
Let x *km* be the distance between A and P and C (in thousands of dollars) be the cost for the entire pipeline.

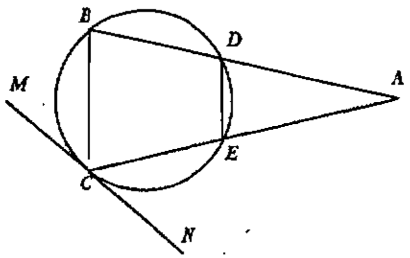
(i) Show that the cost is given by $C = 125\sqrt{x^2+25} + 75(8-x)$ **3**

(ii) Find the domain for x **1**

(iii) Find where the point P should be located to minimise the cost of laying the pipe ? **6**

QUESTION 4 **BEGIN A NEW PAGE** Marks

(a) 6



ABC is a triangle in which $AB = AC$. A circle through B and C cuts AB at D and AC at E. MCN is the tangent at C to the circle through B, C, E, D.

(i) Copy the diagram onto your answer sheet.

(ii) Show that $DE \parallel BC$.

(iii) Show that $\angle ACN = \angle BCD$.

(b) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, whose focus is S. 6
$Q(x,y)$ divides the interval from P to S in the ratio $t^2 : 1$.

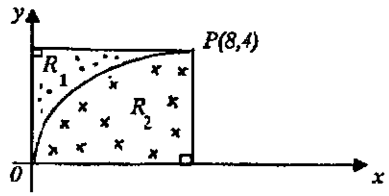
(i) Find x and y in terms of a and t.

(ii) Verify that $\frac{y}{x} = t$.

(iii) Prove that as P moves on the parabola, Q moves on a circle, and state its centre and radius.

QUESTION 5 **BEGIN A NEW PAGE** Marks

(a) 7



OP is an arc of the curve $y^3 = x^2$.
Calculate the volume of the solids generated when

(i) Region R_1 revolves around the y- axis

(ii) Region R_2 revolves around the x- axis

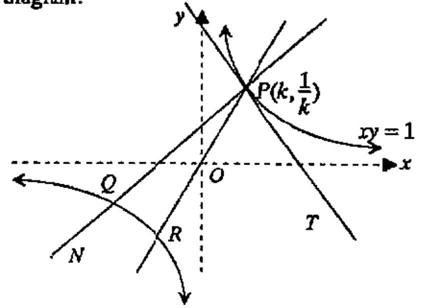
(iii) Region R_2 revolves about the y-axis.

(b) (i) Express $\sin x - \cos x$ in the form $A\sin(x-\alpha)$ with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. 5

(ii) Determine $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$.

QUESTION 6 **BEGIN A NEW PAGE** **Marks**

$P(k, \frac{1}{k})$ is a point on the curve $xy = 1$ where k is a real number, $k \neq 0$.
PT is the tangent to the curve at P and PN is the normal at P.
POR is the line passing through P, the Origin O and R as shown on the diagram.



(a) Find the equation of the line passing through O and P. **1**

(b) The line in part (a) intersects the curve again at R. Find the coordinates of R. **2**

(c) Show that the equation of the tangent at P is given by: **2**

$$x + k^2y = 2k.$$

(d) Find the equation of the normal line at P. **2**

(e) Show that when the normal intersects the curve again at Q, the equation formed to solve is the quadratic equation given by: **3**

$$k^3x^2 - (k^4 - 1)x - k = 0.$$

Hence find the coordinates of point Q

(f) Show that $QR \perp PR$. **2**

QUESTION 7 **BEGIN A NEW PAGE** **Marks**

(a) Given the polynomial function $P(x) = x^3 - 2x^2 - 6x + 4$, when $P(x) = 0$, $P(x)$ has one rational root and two irrational roots.

(i) Find the rational root of $P(x) = 0$. **1**

(ii) Without finding the irrational roots of $P(x)$, show that one of the irrational roots of this equation lies between $x = 3$ and $x = 4$. **1**

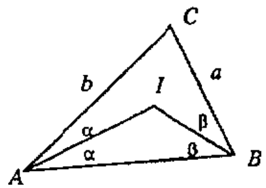
(iii) Using $x = 3 \cdot 5$ as a first approximation, apply Newton's Method once to find a better approximation to the root, to 2 decimal places. **3**

(iv) Sketch $P(x) = x^3 - 2x^2 - 6x + 4$. **1**

(v) Explain why $x = 2$ would not be a good approximation to use when solving $P(x) = 0$ using Newton's Method. **1**

(vi) Find the area bounded by the curve, $x = -3$, $x = -2$ and the x-axis. **2**

(b) IA and IB bisect angles CAB and CBA as shown in the diagram below. **3**



Prove that $\frac{IB}{IA} = \frac{a\cos\beta}{b\cos\alpha}$.

SOLUTIONS

Question 1

(a) $(2x-y)^5 = 1(2x)^5(-y)^0 + 5(2x)^4(-y)^1 + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)^1(-y)^4$
$+ 1(2x)^0(-y)^5$ ①
$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$ ①

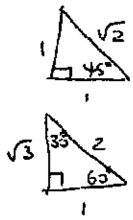
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\boxed{2}$

(b) (i) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ①

(ii) $\cos 45^\circ = \frac{1}{\sqrt{2}}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$



① need both

(iii) $\cos 15^\circ = \cos(45-30)^\circ$
$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$
$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$
$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ ①
$= \frac{\sqrt{6}+\sqrt{2}}{4}$

$\boxed{3}$

(c)(i) $1^\circ = \frac{\pi}{180}$

$72^\circ = \frac{72\pi}{180}$
$= \frac{8\pi}{20}$
$= \frac{2\pi}{5}$ ①

(ii) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2} \cdot 20^2 \cdot \left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)$ ①
$= 61.11610\ldots$
$= 61.1\text{ mm}^2$ (3 sig. fig.) ①

$\boxed{3}$

(d) $\sin 2x = \sqrt{3}\cos 2x$
$\frac{\sin 2x}{\cos 2x} = \sqrt{3}$
$\tan 2x = \sqrt{3}$
$\therefore 2x = \frac{\pi}{3}, \pi+\frac{\pi}{3}, 2\pi+\frac{\pi}{3}, 3\pi+\frac{\pi}{3}, \ldots$ ①
$x = \pi/6, 2\pi/3, 7\pi/6, 5\pi/3$ ①

$\boxed{2}$

(e)(i) $\frac{d}{dx}\sqrt[3]{4x-1} = \frac{d}{dx}(4x-1)^{\frac{1}{3}}$
$= \frac{1}{3}\cdot(4x-1)^{-2/3}\cdot 4$
$= \frac{4}{3}(4x-1)^{-2/3}$ or $\frac{4}{3\sqrt[3]{4x-1}^2}$ ①

(ii) $\frac{d}{dx}\left(\frac{x}{\cot x}\right) = \frac{d}{dx}x\tan x$
$= x\cdot\sec^2 x + \tan x \cdot 1$
$= x\sec^2 x + \tan x$ ①

$\boxed{2}$

Question 2

(a) (i) $= \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt + \int_1^3 1\,dt$

$= \int_0^3 f(t)\,dt + \left\{t\right\}_1^3$ ①

$= 6 + 3 - 1$

$= 8$ ①

(ii) $= \int_3^0 f(t)\,dt + \int_3^0 t\,dt$

$= -\int_0^3 f(t)\,dt + \left(\frac{t^2}{2}\right)_3^0$ ①

$= -6 + 0 - \frac{9}{2}$

$= -10\frac{1}{2}$ ①

$\boxed{4}$

(b) (i) $\int \frac{x^4+2x^3+3}{x^2}\,dx$

$= \int (x^2 + 2x + 3x^{-2})\,dx$

$= \frac{x^3}{3} + x^2 - 3x^{-1} + c$ ①

or $\frac{x^3}{3} + x^2 - \frac{3}{x} + c$

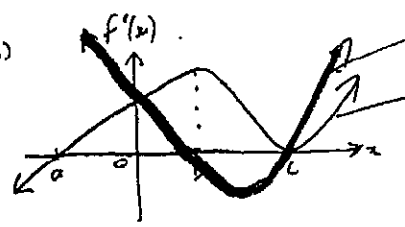
(ii) $\int \frac{dt}{(3-t)^2} = \int (3-t)^{-2}\,dt$

$= \frac{(3-t)^{-1}}{-1 \times -1}$

$= \frac{1}{3-t} + c$ ①

$\boxed{2}$

(c) (i)(ii)



(iii) $b < x < c$ ①

(iv) $x = b$

x	b^-	b	b^+
$f''(x)$	+	0	−

There is a change in concavity at $x=b$

①

$x = c$

x	c^-	c	c^+
$f''(x)$	−	0	+

There is a change in concavity at $x=c$

①

∴ Point of inflection at $x=b$, $x=c$.

(v) $f'(a) = 0$
$f''(a) > 0$

tests ①①

∴ $x=a$ is a rel. min. turning pt

$f'(c) = 0$

x	c^-	c	c^+
$f'(x)$	+	0	+
slope	/	—	/

① ∴ $x=c$ is a stationary point of inflection HPOI

(i)
(ii) } 1
(iii) 1
(iv) 1+1
(v) 1+1

$\boxed{6}$

Question 3

(a) $y = \frac{x^2+x+1}{x}$ vertical asymptote $x=0$ ①

$y = x+1+\frac{1}{x}$ diagonal asymptote $y=x+1$ ①

$\boxed{2}$

(b)(i) $WP^2 = 5^2 + x^2$

$WP = \sqrt{25+x^2}$ ①

Cost $= 125\,000 \times WP + 75\,000 \times PB$ ①

$C = 125\sqrt{25+x^2} + 75(8-x)$ ① (C is in thousands of dollars)

$\boxed{3}$

(ii) $x^2+25>0$ and $8-x\geq 0$ and $x\geq 0$

$x^2>-25$ $-x\geq -8$

$\therefore x\in R$ $x\leq 8$

$\therefore \underline{0\leq x\leq 8}$ ①

$\boxed{1}$

(iii) $\frac{dC}{dx} = 125 \cdot \frac{1}{2}(25+x^2)^{-\frac{1}{2}} \times 2x - 75$ ①

$= \frac{125x}{\sqrt{25+x^2}} - 75$

$\frac{dC}{dx} = 0$ for max/min

$\frac{125x}{\sqrt{25+x^2}} - 75 = 0$ ①

$125x = 75\sqrt{25+x^2}$ yes $0\leq x\leq 8$

$5x = 3\sqrt{25+x^2}$ only if $x\geq 0$

Square both sides

$25x^2 = 9(25+x^2)$ ①

$25x^2 = 225 + 9x^2$

$16x^2 = 225$

$x^2 = \frac{225}{16}$

$x = \pm\frac{15}{4}$

Since $0\leq x\leq 8$

$x = \frac{15}{4}$

(this ① allocated at end of question after testing)

x	$\frac{15}{4}^-$	$\frac{15}{4}$	$\frac{15}{4}^+$
$\frac{dc}{dx}$	$-$	0	+
Slope	\	—	/

①

$\therefore$ rel. min when $x = \frac{15}{4}$ and Cost $= \left(125\sqrt{\left(\frac{15}{4}\right)^2+25} + 75\left(8-\frac{15}{4}\right)\right)\times 1000$

$\therefore$ cost $= \$1\,100\,000$

test end points of $0\leq x\leq 8$

$x=0$

Cost $= (5\times 125 + 8\times 75)\times 1000$

$= \$1\,225\,000$

$x=8$

$WP = \sqrt{8^2+5^2}$

$= \sqrt{89}$

$PB = 0$

$\therefore$ Cost $= (\sqrt{89}\times 125)\times 1000$ ①

$= \$1179247.642\ldots$

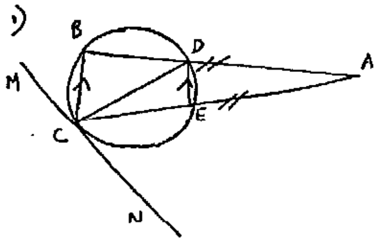
$= \$1179247.64$

$\therefore$ The point P should be located $\frac{15}{4} = 3\frac{3}{4} = 3.75$ km ①

6

Question 4

(a) i)



ii) ΔABC is isosceles ($AB = AC$)

Let $\angle ABC = \alpha$

$\therefore \angle ABC = \angle BCA = \alpha$ (equal base $\angle$'s, isos Δ) ①

$\angle BDE = 180° - \angle BCE$ (opp $\angle$'s cyclic quad are supp) ①

$= 180° - \alpha$

$\angle CBD + \angle BDE = 180°$ ①

and these are cointerior angles ①

$\therefore BC \parallel DE$

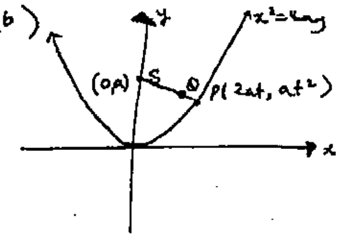
iii) $\angle ACN = \angle CDE$ (angle in the alternate segment thm) ①

$\angle CDE = \angle BCD$ (alt $\angle$'s $=$, $BC \parallel DE$) ①

$\therefore \angle ACN = \angle BCD$

$\boxed{6}$

(b)



(i) $Q = \left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$

$= \left(\frac{1\times 2at + t^2\times 0}{t^2+1}, \frac{1\times at^2+t^2\times a}{t^2+1}\right)$ ①

$= \left(\frac{2at}{t^2+1}, \frac{2at^2}{t^2+1}\right)$ ①

(ii) $\frac{y}{x} = \frac{\frac{2at^2}{t^2+1}}{\frac{2at}{t^2+1}}$

$= \frac{2at^2}{2at}$

$\therefore \frac{y}{x} = t$ ①

(iii) $x = \frac{2at}{t^2+1}$

$= \frac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1}$

$= \frac{\frac{2ay}{x}}{\frac{y^2+x^2}{x^2}}$

$\therefore x = \frac{2axy}{x^2+y^2}$ ①

$x^2+y^2 = 2ay$

$x^2+y^2-2ay = 0$

$x^2+(y-a)^2 = a^2$ ①

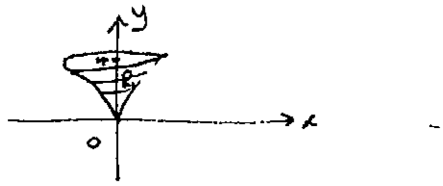
circle centre $(0,a)$
radius a units ①

$\boxed{6}$

Question 5

(a)

(i) Volume $= \pi \int_0^4 y^3 \, dx$



$= \pi \left[\frac{y^4}{4} \right]_0^4$ ①

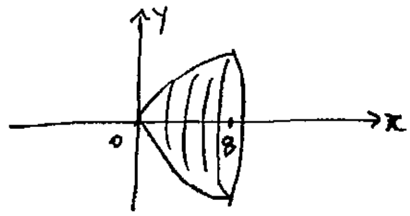
$= \pi \left[\frac{4^4}{4} - 0 \right]$

$= 64\pi$ cubic units ①

(ii) Volume $= \pi \int_0^8 x^{4/3} \, dx$ ①

$y^3 = x^2$
$y = x^{2/3}$
$y^2 = x^{4/3}$

$= \pi \left[\frac{3}{7} x^{7/3} \right]_0^8$ ①



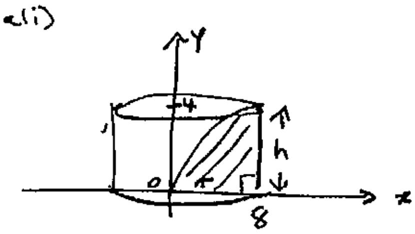
$= \pi \left[\frac{3}{7} 8^{7/3} - 0 \right]$

$= \pi \cdot \frac{3}{7} \cdot 2^7$

$= \frac{384}{7}\pi$ cubic units ①

(iii) Volume $= \pi \times 8^2 \times 4 - 64\pi$ ① ← Q5(a)(i)

$= 192\pi$ cubic units ①



$\boxed{7}$

(b) (i) $\sin x - \cos x \equiv A \sin(x - \alpha)$

$= A \sin x \cos\alpha - A \cos x \sin\alpha$ ①

$\therefore A\cos\alpha = 1$ - - - - - - - ①

$A \sin\alpha = 1$ . . . . ②

①^2 + ②^2:

$A^2 \sin^2\alpha + A^2\cos^2\alpha = 1 + 1$

$= 2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 2$

$A^2 = 2$

$A = \sqrt{2}$ ①

from ①, ② $\left.\begin{array}{l} \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = \frac{1}{\sqrt{2}} \end{array}\right\}$ quad ①, $\sqrt{2}$, 1, $\frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$ ①

$\sin x - \cos x = \sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$

(ii) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$ ①

$= \sqrt{2}$ ①

$\boxed{5}$

QUESTION 6

(a) $m_{OP} = \dfrac{\frac{1}{k} - 0}{k - 0}$

$= \dfrac{1}{k^2}$

eqn OP is $y = \dfrac{1}{k^2}x$ $\quad x - k^2 y = 0$ ① $\boxed{1}$

(b) Solve simultaneously $y = \dfrac{1}{k^2}x$

and $y = \dfrac{1}{x}$

$\therefore \dfrac{1}{k^2}x = \dfrac{1}{x}$ ①

$x^2 = k^2$

$x = k$ or $x = -k$

Since P is $(k, \frac{1}{k})$

R is $(-k, -\frac{1}{k})$ ① $\boxed{2}$

(c) $y = x^{-1}$

$\dfrac{dy}{dx} = -x^{-2}$

$= -\dfrac{1}{x^2}$

at $x = k$, $\dfrac{dy}{dx} = -\dfrac{1}{k^2}$

$\therefore$ gradient of tangent $= -\dfrac{1}{k^2}$ ①

equation of tangent is $y - \dfrac{1}{k} = -\dfrac{1}{k^2}(x - k)$

$k^2 y - k = -x + k$

$x + k^2 y = 2k$ as required ① $\boxed{2}$

(d) Slope of tangent $= -1/k^2$

$\therefore$ slope of normal $= k^2$ $\quad (m_1 m_2 = -1)$ ①

Equation of normal $y - \dfrac{1}{k} = k^2(x - k)$

$ky - 1 = k^3 x - k^4$ $\quad y = k^2 x + \frac{1}{k} - k^3$

$k^3 x - ky = k^4 - 1$ ① $\boxed{2}$

(e) Solve simultaneously

$y = \dfrac{1}{x}$ (eqn ①) and $k^3 x - ky = k^4 - 1$ (eqn ②)

Sub eqn ① into eqn ②

$$\left.\begin{aligned} k^3 x - k\left(\tfrac{1}{x}\right) &= k^4 - 1 \\ k^3 x^2 - k &= (k^4 - 1)x \\ k^3 x^2 - (k^4 - 1)x - k &= 0 \end{aligned}\right\} \text{①}$$

Since $P(k, \frac{1}{k})$ lies on the normal

$x = k$ is a root of the equation.

or use $\alpha + \beta = \frac{k^4}{k^3}$

method ①

$$
\begin{array}{r}
k^3x + 1 \\
x-k \overline{\smash{)}\, k^3x^2 - (k^4-1)x - k} \\
\underline{k^3x^2 - k^4x} \\
x - k \\
\underline{x - k} \\
0
\end{array}
$$

$\therefore (x-k)(k^3x+1) = 0$

$x = k$ or $x = -\frac{1}{k^3}$

Q is $\left(-\frac{1}{k^3}, -k^3\right)$ ② or

Q: formula

method ②

product of roots

$\alpha\beta = -\frac{k}{k^3}$

$= -\frac{1}{k^2}$

since $\alpha = k$ is one root,

$k\beta = -\frac{1}{k^2}$

$\beta = -\frac{1}{k^3}$

$\therefore Q$ is $\left(-\frac{1}{k^3}, -k^3\right)$ ②

1 metho
1 answ

$\boxed{3}$

(f) Slope $QR = \dfrac{-\frac{1}{k} - -k^3}{-k - -\frac{1}{k^3}}$

$= \dfrac{\frac{(k^4-1)}{\cancel{k}}}{\frac{-(k^4-1)}{\cancel{k^3}\,k^2}}$ ①

$= -k^2$

Slope $PR = \frac{1}{k^2}$

Slope $QR \times$ slope $PR = -k^2 \times \frac{1}{k^2}$ ①

$= -1$

$\therefore QR \perp PR$

$\boxed{2}$