

Name: _____

Class: 12MT3 _____

Teacher: _____

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2000 AP3

YEAR 12 HALF YEARLY HSC

MATHEMATICS
3/4 UNIT (COMMON)

*Time allowed – 1.5 HOURS
(plus 5 minutes' reading time)*

DIRECTIONS TO CANDIDATES:

- * Attempt ALL questions.
- * The value for each question is indicated.
- * All necessary working should be shown in every question.
Full marks may not be awarded for careless or badly arranged work.
- * Standard Integrals are provided. Approved calculators may be used.
- * Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.

***Each page must show your class and your name.**

Marks

Question 1: (12 Marks)

- | | | |
|-------|--|---|
| (a) | Differentiate the following: | 4 |
| (i) | $\log_e(e^{3x} + 2)$ | |
| (ii) | $x^3 \cos 3x$. | |
| (b) | Find the following indefinite integrals: | 4 |
| (i) | $\int \frac{dx}{(7x+4)^2}$ | |
| (ii) | $\int \sin 6x \, dx$ | |
| (iii) | $\int 4xe^{-x^2} \, dx$. | |
| (c) | Solve for x : | 2 |
| | $\log_e 8 + \log_e 16 = x \log_e 2$. | |
| (d) | Find the exact value of $\cos 105^\circ$. | 2 |

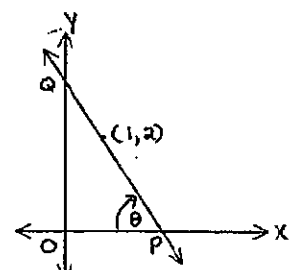
Question 2: (Start a New Page) (12 Marks)

- | | | |
|-----|---|---|
| (a) | Simplify $\frac{\sin x}{\cos x - \sin x} + \frac{\sin x}{\cos x + \sin x}$ | 3 |
| (b) | Simplify $\sec x + \tan x$, in terms of t , where $t = \tan \frac{x}{2}$. | 3 |
| (c) | Use the substitution $u = x^2 - 1$ to find
$\int x^3(x^2 - 1) \, dx$ | 3 |
| (d) | Consider the curve $y = \sin x$, for $0 \leq x \leq 2\pi$.
For what values of x is the gradient equal to $\frac{1}{2}$? | 3 |

- Question 3:** (Start a New Page) (12 Marks) **Marks**
- (a) The quartic expression $x^4 + ax^2 + b$ has factors $(x+1)$ and $(x-2)$. Find the values of a and b . 3
- (b) If $x = c$ is a double root of $P(x)$, show that $x = c$ is a root of $P'(x)$. 3
- (c) p , q and r are the roots of the cubic equation $x^3 + 2x^2 + 3x + 5 = 0$. Evaluate: 4
- (i) $p + q + r$.
- (ii) $p^{-1} + q^{-1} + r^{-1}$.
- (d) The equation $e^x - 4x - 8 = 0$ has a root close to $x = 3$. Using 3 as a first approximation and one application of Newton's Method to find a better approximation for this root. Give your answer correct to three decimal places. 2

- Question 4:** (Start a New Page) (12 Marks)
- (a)
- (i) Find R and α such that $2\cos\theta - \sin\theta = R\cos(\theta + \alpha)$. 4
(Note: $R > 0$ and $0^\circ < \alpha < 90^\circ$.)
- (ii) Hence, solve $2\cos\theta = \sin\theta + 1$, for $0^\circ \leq \theta \leq 360^\circ$
- (b) The curve $y = \cos x$, from $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, is rotated about the x -axis. Find the volume of the solid formed. Leave your answer in exact form. 4
- (c) 4
- (i) Find $\frac{d}{dx}(x \log_e x)$.
- (ii) Prove that $\int_e^{e^2} \frac{1 + \log_e x}{x \log_e x} dx = 1 + \log_e 2$.

- Question 5:** (Start a New Page) (9 Marks) **Marks**
- (a)
- (i) Sketch $y = \sin 2x$, for $0 \leq x \leq 2\pi$. 4
- (ii) By drawing a suitable straight line, state the number of values of x , in this domain, such that $\sin 2x = \frac{x}{2\pi}$.
- (iii) Can there be further solutions beyond $x = 2\pi$? Briefly justify your answer.
- (b) $A(t, e^t)$ and $B(-t, e^{-t})$ are points on the curve $y = e^x$ and $t > 0$. 5
- The tangents at A and B form an angle of 45° .
- (i) Prove that $e^t - \frac{1}{e^t} = 2$.
- (ii) Solve this equation to show that $e^t = 1 + \sqrt{2}$.

- Question 6:** (Start a New Page) (10 Marks) **Marks**
- 10
- 

A straight line passes through the point $(1, 2)$ and meets the x and y axes at P and Q respectively, as shown. The angle OPQ is θ .

- (a) Show that the equation of the line PQ is given by $y = \tan\theta + 2 - x \tan\theta$.
- (b) Show that the area (A) of $\triangle OPQ$ is given by $A = \frac{\tan\theta}{2} + 2 + \frac{2}{\tan\theta}$.
- (c) Prove that the area is a minimum when $\tan\theta = 2$.
- (d) Hence, find the minimum area.

End of Exam

Question 1

a) $y = \log_e (e^{3x} + 2)$

$$\frac{dy}{dx} = \frac{3e^{3x}}{e^{3x} + 2} \quad (1)$$

(ii) $y = x^3 \cos 3x$
 $y' = \cos 3x \times (3x^2)$
 $+ x^3 \times (-3 \sin 3x)$
 $= 3x^2 (\cos 3x - x \sin 3x)$

b) (i) $\int \frac{dx}{(7x+4)^5}$
 $= \int (7x+4)^{-5} dx$
 $= \frac{(7x+4)^{-4}}{-4 \times 7} + c$
 $= \frac{-1}{28(7x+4)^4} + c$

ii) $\int \sin 6x dx$
 $= -\frac{\cos 6x}{6} + c \quad (1)$

iii) $\int 4x e^{x^2} dx$
 $= 2 \times \int 2x e^{x^2} dx$
 $= 2e^{x^2} + c \quad (1)$

∴ $\log_a 8 + \log_a 16 = x \log_a 2$
 $3 \log_a 2 + 4 \log_a 2 = x \log_a 2 \quad (1)$
 $7 \log_a 2 = x \log_a 2$
 $\therefore x = 7 \quad (1)$

Question 1 (cont)

d) $\cos 105^\circ$
 $= \cos (60^\circ + 45^\circ)$
 $= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \quad (1)$
 $= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$
 $= \frac{1 - \sqrt{3}}{2\sqrt{2}} \quad (1)$
 $\left[= \frac{\sqrt{2}(1 - \sqrt{3})}{4} \right]$

Question 2

a) $\frac{\sin x}{\cos x - \sin x} + \frac{\sin x}{\cos x + \sin x}$
 $= \frac{\sin (\cos x + \sin x + \cos x - \sin x)}{(\cos x - \sin x)(\cos x + \sin x)} \quad (1)$
 $= \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}$
 $= \frac{\sin 2x}{\cos 2x} \quad (1)$
 $= \tan 2x \quad (1)$

b) $\sec x + \tan x$
 $= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \quad (1)$
 $= \frac{1+2t+t^2}{1-t^2}$
 $= \frac{(1+t)^2}{1-t^2} \quad (1)$
 $= \frac{(1+t)^2}{(1+t)(1-t)} = \frac{1+t}{1-t} \quad (1)$

Question 2 (cont)

c) $u = x^2 - 1$
 $\frac{du}{dx} = 2x$
 $\frac{du}{2} = x dx \quad (1)$
 And $x^2 = u + 1$

Now $\int x^3 (x^2 - 1) dx$
 $= \int x^2 (x^2 - 1) x dx$
 $= \int (u+1) \cdot u \cdot \frac{du}{2}$
 $= \frac{1}{2} \int (u^2 + u) du$
 $= \frac{1}{2} \left(\frac{u^3}{3} + \frac{u^2}{2} \right) + c \quad (1)$
 $= \frac{u^2}{2} \left(\frac{u}{3} + \frac{1}{2} \right) + c$
 $= \frac{u^2}{2} \left(\frac{2u+1}{6} \right) + c$
 $= \frac{(x^2-1)^2 (2(x^2-1)+3)}{12} + c$
 $= \frac{(x^2-1)^2 (2x^2+1)}{12} + c \quad (1)$

Question 2 (cont)

d) $y = \sin x$
 $\frac{dy}{dx} = \cos x \quad (1)$
 $\cos x = \frac{1}{2} \quad (1)$
 when $x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$
 $= \frac{\pi}{3}, \frac{5\pi}{3} \quad (1)$

Question 3

(a) $P(x) = x^2 + ax + b$
 $P(-1) = 0$
 $\therefore 1 + a + b = 0 \quad (1)$
 $b = -a - 1$
 and $P(2) = 0$
 $4 + 2a + b = 0 \quad (2)$
 Sub. in (1)
 $\therefore 4 + 2a - a - 1 = 0$
 $1 + 3a = 0$
 $a = -\frac{5}{3}$
 $\therefore b = 4 \quad (1)$

b) $P(x) = (x-c)^2 \cdot Q(x) \quad (1)$
 $\therefore P'(x) = Q(x) \cdot 2(x-c)$
 $+ (x-c)^2 \cdot Q'(x) \quad (1)$
 $= (x-c)[2 \cdot Q(x) + (x-c)Q'(x)]$
 $\therefore x=c$ is a root of $P'(x) \quad (1)$

Question 3 (cont)

c) $x^2 + 2x^2 + 3x + 5 = 0$
 $a=1, b=2, c=3, d=5$
 i) $p+q+r = -\frac{b}{a}$
 $= -2$ (1)

ii) $p^{-1} + q^{-1} + r^{-1}$
 $= \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$
 $= \frac{qr + pr + pq}{pqr}$ (1)
 $= \frac{c/a}{-d/a}$
 $= -\frac{c}{d}$ (1)
 $= -\frac{3}{5}$ (1)

d) $f(x) = e^x - 4x - 8$
 $f'(x) = e^x - 4$

$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$ (1)
 $= 3 - \frac{f(3)}{f'(3)}$
 $= 3 - \frac{e^3 - 20}{e^3 - 4}$ (1)
 $= 2.99468...$
 $= 2.995$ (3 dp)

Question 4

a) i) $R = \sqrt{2^2 + 1^2}$
 $= \sqrt{5}$ (1)

$\therefore \frac{2}{\sqrt{5}} \cos \alpha - \frac{1}{\sqrt{5}} \sin \alpha$
 $= \cos \theta \cos \alpha - \sin \theta \sin \alpha$

$\therefore \cos \alpha = \frac{2}{\sqrt{5}}$
 $\alpha = 26^\circ 34'$ (1)

$\therefore R = \sqrt{5}, \alpha = 26^\circ 34'$

ii) $2 \cos \theta - \sin \theta = 1$
 $\sqrt{5} \cos(\theta + \alpha) = 1$
 $\cos(\theta + \alpha) = \frac{1}{\sqrt{5}}$ (1)

$(\theta + \alpha) = 63^\circ 26', 296^\circ 34'$
 $\theta = 63^\circ 26' - 26^\circ 34'$
 $296^\circ 34' - 26^\circ 34'$
 $\theta = 36^\circ 52', 270^\circ$ (1)

b) $V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx$ (1)

$= 2\pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2} \, dx$ (1)

$= \pi \int_0^{\frac{\pi}{2}} (\cos 2x + 1) \, dx$

$= \pi \left[\frac{\sin 2x}{2} + x \right]_0^{\frac{\pi}{2}}$ (1)

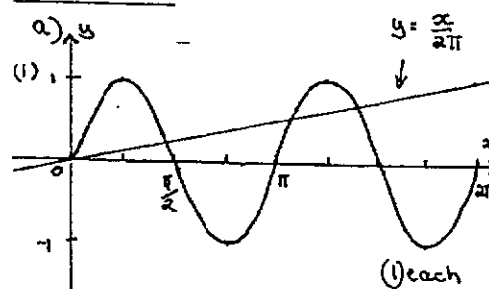
$= \pi \left(\frac{\sin \pi}{2} + \frac{\pi}{2} \right) = \frac{\pi^2}{2}$ (1)

Question 4 (cont)

c) i) $\frac{d}{dx} (x \log_e x)$
 $= (\log_e x) + x \times \frac{1}{x}$
 $= 1 + \log_e x$ (1)

ii) $\int_e^{e^2} \frac{1 + \log_e x}{x \log_e x} \, dx$
 $= \left[\log_e (x \log_e x) \right]_e^{e^2}$ (1)
 $= \log_e (e^2 \cdot 2) - \log_e (e)$
 $= \log_e e^2 + \log_e 2 - 1$ (1)
 $= 2 + \log_e 2 - 1$
 $= 1 + \log_e 2$ (1)

Question 5



- ii) There are 4 values (1)
 iii) No. $\frac{x}{2\pi} > 1$ when $x > 2\pi$ (1)
 \therefore no further solutions because max. value of $\sin 2x$ is 1.

Question 5 (cont)

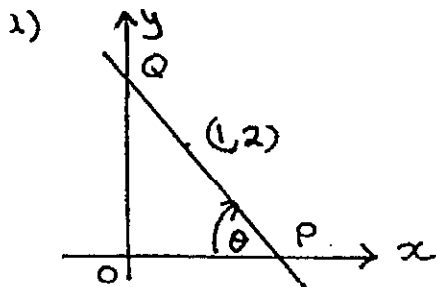
b) i) Let $\theta =$ angle between the tangents.
 At A, $m_1 = e^t$
 B, $m_2 = e^{-t}$ } (1)

$\therefore \tan \theta = \left| \frac{e^t - (e^{-t})}{1 + e^t \cdot e^{-t}} \right|$ (1)
 $1 = \frac{e^t - e^{-t}}{2}$
 i.e. $2 = e^t - e^{-t}$
 or $e^t - \frac{1}{e^t} = 2$ (1)

ii) $e^t - \frac{1}{e^t} - 2 = 0$
 $(e^t)^2 - 2e^t - 1 = 0$ (1)
 $\therefore e^t = \frac{2 \pm \sqrt{4+4}}{2}$ (1/2)
 $= 1 \pm \sqrt{2}$ (1/2)

(Quest 6 on next page)

Question 6



For PQ $m = -\tan\theta$ (1)

Now, $y - y_1 = m(x - x_1)$
 $y - 2 = -\tan\theta(x - 1)$
 $y - 2 = -x\tan\theta + \tan\theta$
 OR $y = 2 + \tan\theta - x\tan\theta$ (1)

b) $A = \frac{1}{2} \times OP \times OQ$

At P $y = 0$
 $\therefore 0 = \tan\theta + 2 - x\tan\theta$
 $x\tan\theta = \tan\theta + 2$
 $x = 1 + \frac{2}{\tan\theta}$ (1)

$\therefore OP = 1 + \frac{2}{\tan\theta}$

At Q, $x = 0$
 $\therefore y = 2 + \tan\theta$
 $\therefore OQ = 2 + \tan\theta$ (1)

$\Rightarrow A = \frac{1}{2} \left(1 + \frac{2}{\tan\theta}\right) (2 + \tan\theta)$
 $= \frac{1}{2} \left(2 + \tan\theta + \frac{4}{\tan\theta} + 2\right)$
 $= \frac{\tan\theta}{2} + 2 + \frac{2}{\tan\theta}$ (1)

Question 6 (cont)

c) Let $t = \tan\theta$
 $\therefore A = \frac{t}{2} + 2 + \frac{2}{t}$

Now $\frac{dA}{dt} = \frac{1}{2} - \frac{2}{t^2}$ (1)

$\frac{d^2A}{dt^2} = \frac{4}{t^3}$ (1)

$\frac{dA}{dt} = 0$ when $\frac{1}{2} = \frac{2}{t^2}$

$t^2 = 4$

$t = \pm 2$

But $t > 0$, since $A > 0$ (1)

\therefore when $t = 2$, $\frac{d^2A}{dt^2} = \frac{4}{2^3} > 0$ (1)

\therefore min. value when $t = 2$

d) When $t = 2$

$A = \frac{2}{2} + 2 + \frac{2}{2}$

$= 4$

\therefore Min area is 4 sq. units.