CRANBROOK SCHOOL

Year 12 Extension 1 Mathematics

Mini Examination

Tuesday April 8, 2008

Instructions

- There are four (4) questions, each worth 15 marks
- Attempt all questions
- Answer each question in a new booklet
- Show all necessary working
- Calculators are allowed in all sections

Time Allowed: 90 minutes　　　　Total Marks: 60

Question 1 (15 Marks) **START A NEW BOOKLET** **Marked by CJL**

(a) Use one application of Newton's method starting with $x = 1$ to find the next approximation to the root of the equation $\ln x - \frac{1}{x} = 0$ **2**

(b) For what value of p is the expression $4x^3 - x + p$ divisible by $x + 3$ **2**

(c) Let α, β and γ be the roots of $2x^3 - x^2 + 3x - 2 = 0$.

Find the value $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **3**

(d) Consider the function $P(x) = x^3 - \ln(x+1)$. **4**

(i) Show that a root exists between $x = 0.8$ and $x = 0.9$

(ii) By halving the interval twice find a better approximation to the root.

(e) By first factorising $f(x) = x^3 - x^2 - 8x + 12$, draw a neat sketch of the function $y = f(x)$ **4**

EXAMINATION CONTINUES OVER THE PAGE

Question 2 (15 Marks) **START A NEW BOOKLET** **Marked by HRK**

(a) Use the substitution $u = \ln x$ to find $\int \frac{1}{x}(\ln x)^2 \, dx$ **3**

(b) $P(2p, p^2)$ is on the parabola $x^2 = 4y$.

(i) Show that the equation of the normal at P is $x + py = 2p + p^3$ **2**

(ii) The normal at P meets the y-axis at N. M is the midpoint of PN. Find the coordinates of M. **3**

(iii) Show that the locus of M is another parabola with vertex $S(0,1)$ where S is the focus of the original parabola $x^2 = 4y$. **2**

(iv) Prove that SM is parallel to the tangent at P. **2**

(v) Show that there are 2 positions of P for which ΔPNS is equilateral. Find the corresponding coordinates of P. **3**

EXAMINATION CONTINUES OVER THE PAGE

Question 3 (15 Marks) **START A NEW BOOKLET** **Marked by HRK**

(a) **(i)** Show that $\dfrac{u^3}{u+1} = u^2 - u + 1 - \dfrac{1}{u+1}$ **2**

(ii) Using the substitution $u = \sqrt{x}$ and the result shown in part (i),

evaluate $\displaystyle\int_0^2 \frac{x}{1+\sqrt{x}}\,dx$ **4**

(b) For the function $y = x^2 e^{-x}$: **9**

(i) Find any stationary points and establish their nature.

(ii) Find any points of inflexion.

(iii) State any intercepts.

(iv) Find any asymptotes.

(v) Sketch the curve.

EXAMINATION CONTINUES OVER THE PAGE

Question 4 (15 Marks) **START A NEW BOOKLET** **Marked by SKB**

(a) Consider the curve $y = \dfrac{4x}{1+x^2}$.

(i) Prove that the curve represents an odd function. **1**

(ii) Find $\dfrac{dy}{dx}$. **1**

(iii) If $\dfrac{d^2y}{dx^2} = \dfrac{8x(x^2-3)}{(x^2+1)^3}$ find the turning points on the curve and determine their nature. **2**

(iv) Hence find the points of inflexion on the curve by testing concavity changes or otherwise. **2**

(v) By noting any tendencies hence sketch the curve. **1**

(b) **(i)** By noting that $n! = n(n-1)(n-2)(n-3).....(3)(2)(1)$

and that $(n+1)! = (n+1)n!$ whereby $6! = 6(5)(4)(3)(2)(1) = 720$,

use mathematical induction to prove:

$\ln(n!) > n$ for $n \geq 6$ where n is a positive integer. **4**

(ii) Hence show that $\dfrac{1}{n!} < \dfrac{1}{e^n}$ for all positive integers $n \geq 6$. **1**

(iii) Hence show that $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + ... < \dfrac{103}{60} + \dfrac{1}{e^5(e-1)}$ **3**

END OF EXAMINATION

12 EXT 1 MINI 2008

a) let $P(x) = \ln x - \frac{1}{x}$

$$P(1) = \ln 1 - \frac{1}{1}$$
$$= -1$$

$$P'(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$P'(1) = 1 + 1 = 2 \quad ✓$$

$$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$$

$$x_2 = 1 - \frac{-1}{2}$$

$$x_2 = 1\tfrac{1}{2} \quad ✓$$

b) divisible by $x+3$ if $P(-3) = 0$

$\therefore 4(-3)^3 - (-3) + p = 0$ ✓

$\therefore p = 105$ ✓

c) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

Now $\Sigma_{\alpha\beta} = \frac{3}{2}$ ✓ $\left(\frac{c}{a}\right)$

$\pi_{\alpha\beta\gamma} = \frac{2}{2} = 1$ ✓ $\left(\frac{-d}{a}\right)$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\frac{3}{2}}{1} \quad ✓$$
$$= \frac{3}{2}$$

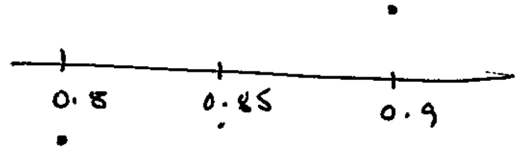
d) (i) $P(0.8) = 0.8^3 - \ln(0.8 + 1)$
$= -0.075 \ldots$
< 0 ✓

$P(0.9) = 0.9^3 - \ln(0.9 + 1)$
$= 0.087 \ldots$
> 0

$\therefore$ as $P(x)$ is continuous and $P(0.8)$ and $P(0.9)$ have opposite signs, a root exists between $x = 0.8$ and $x = 0.9$ ✓

(ii) 1st approx is $x = 0.85$

$P(0.85) = 0.85^3 - \ln(1.85)$
$= -0.001$
< 0 ✓



2nd approx is between $x = 0.85$ and $x = 0.9$ ✓

$\therefore$ 2nd approx is $x = 0.875$

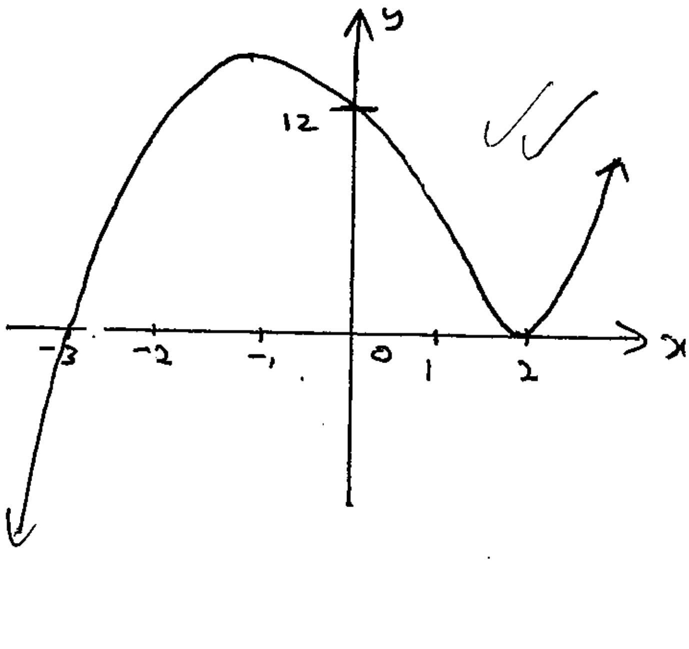
e) $f(x) = x^3 - x^2 - 8x + 12$

$f(2) = 8 - 4 - 16 + 12 = 0$

$\therefore x - 2$ is a factor ✓

$$\begin{array}{r l}
 & x^2 + x - 6 \\
x - 2 \,) & x^3 - x^2 - 8x + 12 \\
 & x^3 - 2x^2 \\
\hline
 & x^2 - 8x \\
 & x^2 - 2x \\
\hline
 & -6x + 12 \\
 & -6x + 12 \\
\hline
 & 0
\end{array}$$
✓

$\therefore f(x) = (x-2)(x^2 + x - 6)$
$= (x-2)(x+3)(x-2)$
$= (x-2)^2(x+3)$



✓✓

Markers Notes 12 Ext 1 2008

Q1 a) Done well. Some struggled to differentiate $\frac{1}{x}$ as x^{-1}, so

$y' = -x^{-2} = -\frac{1}{x^2}$

b) Quickest way is to use $P(-3) = 0$ rather than long division

c) Done well. Remember

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

d) (i) Should really state curve is continuous for all $x > -1$, as well as showing $P(0.8)$ and $P(0.9)$ have opposite signs

(ii) Only asked to halve interval twice, so stop at $x = 0.875$. No need to evaluate $P(0.875)$ and go again as this would be halving the interval 3 times

e) Must answer question as asked ie use factorisation.

No marks given to those students who use stationary points.

Some sketches were too casual. The turning point is NOT the y intercept.

The scale on the x axis must be reasonable, ie 2 units from origin cannot be same distance as 3 units from origin.

2 a) ✓ = 1 MARK MARKERS NOTES.

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$ ✓

$\int \frac{1}{x}(\ln x)^2\,dx$

$= \int u^2\,du$ ✓

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}(\ln x)^3 + c$ ✓

2a) SPOT THE DIFFERENCE

$\ln x^3 = 3\ln x$

(CLOG LAW ☺)

BUT

$(\ln x)^3 = \ln x \times \ln x \times \ln x$

NOT the clog law

those who made this mistake should differentiate their answer ie $\frac{d}{dx}\ln x = \frac{1}{x}$

NOT what we started with!

b) i) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

$\therefore$ at $x = 2p$ $M_T = \frac{2p}{2}$

$= p$

$\therefore M_N = -\frac{1}{p}$ ✓

Eqn of normal at P is

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = 2p + p^3$ ✓

ii) at N, $x = 0$

ie $py = 2p + p^3$ ✓

$y = 2 + p^2$

$\therefore$ N is $(0, 2 + p^2)$

P is $(2p, p^2)$ ✓

$\therefore M = \left(\frac{2p}{2}, +\frac{2 + 2p^2}{2}\right)$

$= (p, 1 + p^2)$ ✓

b(i)(ii) Very well done those who used a sketch in their working found the rest of the question easier in general

(iii) $x = p$ ①

$y = 1 + p^2$ ②

subst ① into ②

$\therefore y = 1 + x^2$ ✓

$x^2 = (y-1)$

This has vertex $(0,1)$

ie Focus of $x^2 = 4y$ ✓

$(0,1)$ since $4a = 4 \therefore a = 1$

iv) $M_T = p$ $\quad S(0,1)$

$M_{SM} = \frac{p^2}{p}$ ✓ $M(p, 1+p^2)$

$= p$

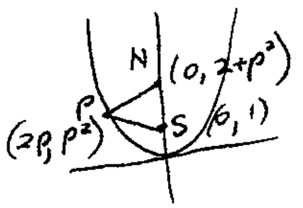
Equal gradients ✓

$\therefore$ SM is parallel to tangent at P

v) If equilateral $PN = NS = SP$

$PN = NS$ ✓

$\left(\sqrt{(2p)^2 + (p^2-(2+p^2))^2}\right)^2 = (1+p^2)^2$



$4p^2 + 4 = p^4 + 2p^2 + 1$

$p^4 - 2p^2 - 3 = 0$ ✓

$(p^2-3)(p^2+1) = 0$

$p = \pm\sqrt{3}$

then $P = (2p, p^2) = (\pm 2\sqrt{3}, 3)$ ✓

also could use

$SP = \sqrt{(2p)^2 + (p^2-1)^2}$

$= \sqrt{(p^2+1)^2}$

$= p^2 + 1$

$= NS.$

NOTES

iii) A simple question but it did, as all questions, need answering

When asked to SHOW YOU MUST SHOW

Vertex could also be shown by calculus or translation of $y = x^2$ up 1.

Mention did need to be made of the focus of the original parabola.

iv) no need here to repeat finding gradient this was done in (i) simply refer back to (i).

v) → A simple concept equilateral = equal sides $\therefore$ use distance formula! & take care!

OR as some began (but none completed ☹) angles would all be $60°$ so can use trig

$\tan 60° = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\sqrt{3} = \ldots$

etc

& this will give the 2 answers for P.

(try this as a correction even if you got it correct using above method.)

3(a)

i) $RHS = \frac{u^2}{1} - \frac{u}{1} + \frac{1}{1} - \frac{1}{u+1}$ ✓

$= \frac{(u+1)u^2 - (u+1)u + (u+1) - 1}{u+1}$

$= \frac{u^3 + \cancel{u^2} - \cancel{u^2} - \cancel{u} + \cancel{u} + \cancel{1} - \cancel{1}}{u+1}$

$= \frac{u^3}{u+1}$ ✓

$= LHS$

(ii) $u = x^{\frac{1}{2}}$

$u^2 = x$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

✓ $= \frac{1}{2\sqrt{x}}$

$du = \frac{dx}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

✓ $= 2u\,du$

$x = 2 \quad u = \sqrt{2}$

$x = 0 \quad u = 0$

$\int_0^2 \frac{x}{1+\sqrt{x}}\,dx$

$= \int_0^{\sqrt{2}} \frac{u^2 \times 2u\,du}{1+u}$

$= 2\int_0^{\sqrt{2}} \frac{u^3}{1+u}\,du$ ✓

USING (i)

$= 2\int_0^{\sqrt{2}} u^2 - u + 1 - \frac{1}{u+1}\,du$

$= 2\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_0^{\sqrt{2}}$

$= 2\left[\frac{2\sqrt{2}}{3} - 1 + \sqrt{2} - \ln(\sqrt{2}+1) - (0)\right]$

$= \frac{10\sqrt{2}}{3} - 2 - 2\ln(\sqrt{2}+1)$ ✓ $\doteqdot 0{\cdot}95$

b)(i) $y = x^2 e^{-x} = \frac{x^2}{e^x}$ $\frac{u}{v}$

$\frac{dy}{dx} = \frac{e^x 2x - x^2 e^x}{(e^x)^2}$

$= \frac{\cancel{e^x}(2x - x^2)}{\cancel{e^x}e^x}$ ✓

$2x - x^2 = 0$

$x = 0, 2$

$y = 0, \frac{4}{e^2}$ ✓

$\therefore$ ST. PTS are $(0,0), \left(2, \frac{4}{e^2}\right)$

NOTES

3(a) OR use long division on LHS (Show this works as an extra correction ☺).

Too many students took too many lines to put 4 terms on a common denominator !!

Too many left this unfinished - a good example of a question where those who pushed on and re-read the question gained marks

Please label question parts as in question !! — a pretty basic requirement!!! but one which not done cost some people marks

Untidy work cost others marks

Basic methods were clearly known but many scripts lacked attention to detail.

$\therefore \dfrac{dy}{dx} = \dfrac{2x - x^2}{e^x}$ $\quad u$, v

$$\frac{d^2y}{dx^2} = \frac{\cancel{e^x}(2-2x) - (2x-x^2)\cancel{e^x}}{\cancel{e^{2x}}}$$

$$= \frac{2 - 2x - 2x + x^2}{1}$$

$$= x^2 - 4x + 2$$

$f''(0) = +2 > 0$ $\cup$ $\therefore (0,0)$ is A MIN.

$f''(2) = 4 - 8 + 2 < 0$ $\cap$ $\therefore \left(2, \frac{4}{e^2}\right)$ is A MAX.

(ii) For inflexions $\dfrac{d^2y}{dx^2} = 0$ and concavity changes.

$x^2 - 4x + 2 = 0 \quad x = \dfrac{4 \pm \sqrt{8}}{2}$

$= 2 \pm \sqrt{2}$

x	0	$2-\sqrt{2}$	2	$2+\sqrt{2}$	4
y''	+	0	−	0	+

$x = 2+\sqrt{2} \quad y \doteqdot 0.4$

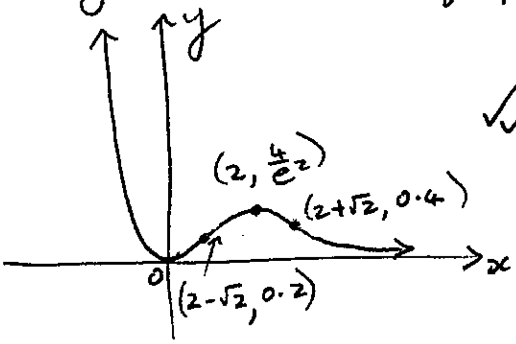
$x = 2-\sqrt{2} \quad y \doteqdot 0.2$

(iii) When $x = 0$ $y = 0$

(iv) $\lim\limits_{x \to +\infty} \dfrac{x^2}{e^x} = 0$

$\lim\limits_{x \to -\infty} x^2 e^{-x} \to \infty$

$\therefore$ x axis is the only asymptote.
ie $y = 0$ is the asymptote



Given you are asked for inflexions in (ii) quickest way of est. nature of st pts is 2nd derivative applied CAREFULLY!! — not always the case

easy questions needing answers to earn marks! DON'T ignore numbering of question parts!!

iv) N.B. The equation of the x-axis is $y=0$

v) Sketches need to be adequately labelled.

Q4

(a) $y = \dfrac{4x}{1+x^2}$

(i) Let $f(x) = \dfrac{4x}{1+x^2}$

$\therefore f(-x) = \dfrac{4(-x)}{1+(-x)^2}$

$= \dfrac{-4x}{1+x^2}$

$= -f(x)$

$\therefore$ It is an odd function.

(ii) $\dfrac{dy}{dx} = \dfrac{(1+x^2).4 - 4x(2x)}{(1+x^2)^2}$

$= \dfrac{4+4x^2-8x^2}{(1+x^2)^2}$

$= \dfrac{4-4x^2}{(1+x^2)^2}$

(iii) For a stat. pt $\dfrac{dy}{dx} = 0$

$\therefore 4 - 4x^2 = 0$

$\therefore x = \pm 1$

when $x = 1$ $\dfrac{d^2y}{dx^2} < 0 \Rightarrow$ max. turn. pt at $(1, 2)$

When $x = -1$ $\dfrac{d^2y}{dx^2} > 0 \Rightarrow$ min turn. pt at $(-1, -2)$.

(iv) For a possible pt of inflexion $\dfrac{d^2y}{dx^2} = 0$

$\therefore x = 0$ or $x = \pm\sqrt{3}$

$x=0$

x	0^-	0	0^+
y''	$+$	0	$-$

concavity change $\Rightarrow$ pt. of inflexion at $(0, 0)$

At $x = \sqrt{3}$

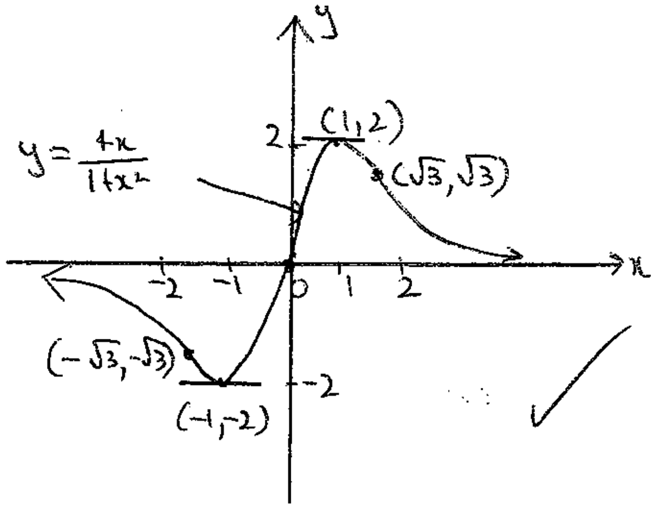
x	$\sqrt{3}^-$	$\sqrt{3}$	$\sqrt{3}^+$
y''	$-$	0	$+$

concavity change

$\Rightarrow$ pt. of inflexion at $(\sqrt{3}, \sqrt{3})$

Now as function is odd $\Rightarrow$ further pt of inflexion at $(-\sqrt{3}, -\sqrt{3})$.

(v) As $x \to \pm\infty$ $y \to 0$.



(b) (i) TO PROVE: $\ln(n!) > n$ for $n \geq 6$ $(n \in J^{+})$

PROOF: Step 1: When $n=6$ LHS $= \ln(6!) = \ln(720) = 6.579\ldots$

RHS $= 6$ $\therefore$ LHS $>$ RHS $\therefore$ true for $n=6$

Step 2: Assume it is true for $n=k$ $(k \leq n)$ and prove it is true for $n=k+1$

ie $\ln(k!) > k$ ——— (1)

If $n=k+1$
$$\begin{aligned}\ln(n!) &= \ln([k+1]!)\\ &= \ln[(k+1)k!]\\ &= \ln(k+1) + \ln(k!)\\ &> \ln(k+1) + k \quad (\text{using } (1))\\ &> 1+k \quad [\text{As } \ln(k+1) > 1 \text{ for } k \geq 6]\end{aligned}$$

ie $\ln[(k+1)!] > k+1$

$\therefore$ if it is true for $n=k$ $(k \geq 6)$ so it is true for $n=k+1$.

Step 3: It is true for $n=6$ and so it is true for $n=6+1=7$. It is true for $n=7$ and so it is true for $n=7+1=8$ and so on for all positive integral values of n.

(ii) As $\ln(n!) > n$, for $n \geq 6$
$$\therefore e^{\ln(n!)} > e^{n}$$
$$\therefore n! > e^{n}$$
$$\therefore \frac{1}{n!} < \frac{1}{e^{n}}, \text{ for } n \geq 6.$$

(iii) Now $\displaystyle\sum_{n=1}^{5} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = \frac{103}{60}$.

and
$$\sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots$$
$$< \underbrace{\frac{1}{e^6} + \frac{1}{e^7} + \frac{1}{e^8} + \ldots}_{\text{Limiting sum with } a = \frac{1}{e^6},\ r = \frac{1}{e},\ |r|<1} \quad [\text{using (ii)}]$$

$$\therefore \text{Limiting Sum} = \frac{\frac{1}{e^6}}{1-\frac{1}{e}} = \frac{1}{e^6} \div \frac{e-1}{e} = \frac{1}{e^5(e-1)}$$

$$\Rightarrow \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < \frac{103}{60} + \frac{1}{e^5(e-1)}$$

Q4 Comments

(a) (i) Most students proved that the curve was an odd function

(ii) Most found $\frac{dy}{dx}$ successfully.

(iii) Some students failed to determine the nature of the stationary points and lost marks accordingly. Follow on marks were awarded from (ii) errors.

(iv) Many students did not bother testing concavity changes to confirm the existence of the points of inflexion. Some students compared the x-value for the point of inflexion with 'y'' and not 'y''' to substantiate that at the value of x there was indeed a point of inflexion.

(v) Mostly well done. However, as the curve was proven to be odd from (i) then it must exhibit odd function properties ie point symmetry about the origin needed to be shown.

(b) (i) Not generally well done. Many students did not use the log law property of $\log_e[(k+1)k!] = \log_e(k+1) + \log_e(k!)$ to justify their proof nor did they explain why $\ln(k+1) > 1$.

(ii) Many students did not raise both sides of the result from (i) ie $\ln(n!) > n$, to the power of e to achieve the required result.

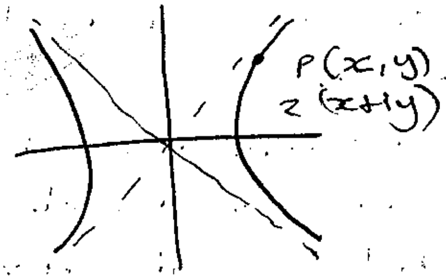
(iii) By not splitting the summation into two sections ie $\sum_{n=1}^{5} \frac{1}{n!}$ and $\sum_{n=6}^{\infty} \frac{1}{n!}$ very few students were able to obtain the correct result.

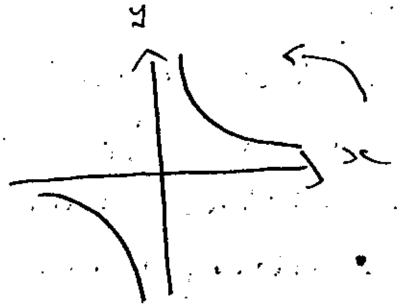
Rectangular hyperbola

corollary

wi

$e = \sqrt{2}$





$\frac{\pi}{4}$ $\times \operatorname{cis}\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$x^2 - y^2 = a^2$

$$z' = (x+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{x}{\sqrt{2}} + i\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} + i^2\frac{y}{\sqrt{2}}$$

$$= \frac{x-y}{\sqrt{2}} + i\left(\frac{x+y}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(x-y + i(x+y)\right)$$

$\frac{dy}{dx}$ $(\ln x)^3$ $(\ln x^3)$ $\frac{3}{x}$

$\frac{dy}{dx}$ $3\ln x$

$(\ln x)^3$

$f'(x)$ $\ln x \cdot \ln x \cdot \ln x$

Let $u = \ln x$.

$u' = \frac{1}{x}$

$\ln x$

$f'(x)$ u^3

$\frac{dy}{du} \times \frac{du}{dx}$

$3u^2 \times \frac{1}{x}$

$= \frac{3u^2}{x}$

$= \frac{3(\ln x)^2}{x}$

$\ln x^3$

$(\ln x)^3$