CRANBROOK
SCHOOL

Year 12 Extension 1 Mathematics

Mini Examination

Wednesday April 8, 2009

Instructions

- There are four (4) questions, each worth 15 marks
- Attempt all questions
- Answer each question in a new booklet
- Show all necessary working
- Calculators are allowed in all sections
- 5 minutes reading time

Time Allowed: 90 minutes Total Marks: 60

Question 1 (15 Marks) **START A NEW BOOKLET** **Marked by SKB**

(a) Consider the function $P(x) = x - \ln 10x$.

(i) Show that a root exists between $x = 3$ and $x = 4$. **1**

(ii) By choosing $x = 3.6$ as a first approximation and applying Newton's Method once determine a second approximation to this root. **2**

(iii) Comment on the accuracy of your second approximation. **1**

(iv) Why would Newton's Method have failed if $x = 1$ had been chosen as the first approximation? **1**

(b) If α, β and γ are the roots of $x^3 + 4x^2 + 8x + 16 = 0$, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(c) The polynomial $P(x) = x^5 + 3x^4 - 10x^3 + 2x^2 + 9x - 5$ has a triple root at $x = 1$ and two other single roots. Determine the values of these other roots and express $P(x)$ as a product of its factors. **3**

(d) A polynomial $Q(x) = x^4 + px^3 + qx^2 - 5x + 1$ has a zero at $x = 1$. When $Q(x)$ is divided by $x^2 + 2$ it has a remainder of $1 - 7x$. Find p and q. **3**

EXAMINATION CONTINUES OVER THE PAGE

Question 2 (15 Marks) **START A NEW BOOKLET** **Marked by SKB**

(a) **(i)** Use the substitution $u = 1 - x^6$ to find $\int \frac{x^5}{\sqrt{1-x^6}}\,dx$ **3**

(ii) Use the substitution $u = 1 + \log_e x$ to evaluate $\int_1^e \frac{dx}{x(1+\log_e x)^2}$ **3**

(b) $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. Normals to this parabola at P and Q meet at the point R.

(i) Prove that R has coordinates $[-apq(p+q), a(p^2 + pq + q^2 + 2)]$ **4**

(ii) If the normals intersect at right angles prove that the locus of R is the parabola $x^2 = a(y - 3a)$. **4**

(iii) Hence find the coordinates of the focus of the locus of R. **1**

EXAMINATION CONTINUES OVER THE PAGE

Question 3 (15 Marks) **START A NEW BOOKLET** **Marked by HRK**

(a) Differentiate $y = \ln\left(x^3\sqrt{x^2+1}\right)$ **2**

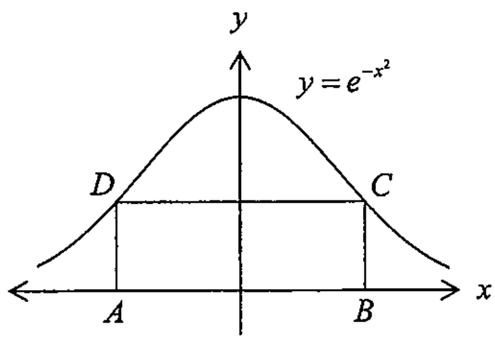
(b) Evaluate $\int_1^3 \left(2x + \frac{3}{x^2}\right)^2 dx$ **3**

(c) Find the exact value of the area enclosed by the curve $y = \frac{e^x}{1+e^x}$, the x-axis,

and the lines $x = 0$ and $x = 1$. **3**

(d) $ABCD$ is a rectangle drawn between the curve $y = e^{-x^2}$ and the x-axis.

4



(i) Show that $ABCD$ has area $2xe^{-x^2}$ $units^2$

(ii) Hence find the maximum area of such a rectangle.

(e) Write down the derivative of $(x-1)e^x$ and use your result to

evaluate $\int_{-1}^{1} xe^x \, dx$ **3**

EXAMINATION CONTINUES OVER THE PAGE

Question 4 (15 Marks) **START A NEW BOOKLET** **Marked by HRK**

(a) Prove by mathematical induction where n is a positive integer,

$3^{3n} + 2^{n+2}$ is divisible by 5. **6**

(b) For the curve $y = xe^{-x}$, **9**

(i) Determine the stationary point and the point of inflexion.

(ii) Sketch the curve.

(iii) From your sketch, show that the equation $xe^{-x} = k$ has

(α) Two roots if $0 < k < \frac{1}{e}$

(β) One real root if $k \le 0$

(γ) No real roots if $k > \frac{1}{e}$

END OF EXAMINATION

① (a) $P(x) = x - \ln 10x$

(i) $P(3) = 3 - \ln 30 = -0.40119\ldots$
$P(4) = 4 - \ln 40 \approx 0.31112\ldots$

As $P(3)$ and $P(4)$ have different sign and $P(x)$ is cts for $x > 0$ $\therefore$ at least 1 root exists in interval $3 < x < 4$.

(ii) Let $z_1 = 3.6$

By Newton's Method

$P(x) = x - \ln 10x$
$\therefore P'(x) = 1 - \frac{1}{x}$

$$z_2 = z_1 - \frac{P(z_1)}{P'(z_1)}$$

$$\therefore z_2 = 3.6 - \frac{P(3.6)}{P'(3.6)}$$

$$= 3.6 - \frac{0.01648\ldots}{0.7222\ldots}$$

$$= 3.577180069\ldots$$

(iii) Now $P(z_2) = 0.000020175\ldots$
$\doteqdot 2.0 \times 10^{-5}$

$\therefore$ As its order of accuracy has a magnitude of 10^{-5} its accuracy is very good.

(iv) If $x = 1$ $P'(1) = 0$. This would have meant that z_2 would have been undefined and Newton's Method would have failed. Indeed at $x = 1$ a stationary point exists and any tangent drawn to this point would not have cut the x-axis meaning that Newton's Method would not have applied for finding a closer approximation to the root.

(b) $x^3+4x^2+8x+16=0$ has roots α, β and γ

Now $\alpha+\beta+\gamma = -\frac{b}{a} = -4$

$\alpha\beta+\alpha\beta+\beta\gamma = \frac{c}{a} = 8$

$\alpha\beta\gamma = -\frac{d}{a} = -16$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{8}{-16}$

$= -\frac{1}{2}$

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-4)^2 - 2(8)$

$= 16-16$

$= 0$

(c) $P(x) = x^5+3x^4-10x^3+2x^2+9x-5$

As $P(x)$ has a triple root at $x=1$ $\Rightarrow$ $(x-1)^3$ is a factor.

Now possible zeros of $P(x)$ are $\pm1, \pm5$.

Let $x=-1$ $\therefore P(-1) = -1+3+10+2-9-5 = 0$

$\therefore x+1$ is a further factor

Let $x=-5$ $\therefore P(-5) = -3125+1875+1250+50-45-5 = 0$

$\therefore x+5$ is the other factor

$\Rightarrow$ other roots of $P(x)$ are $x=-1$ and $x=-5$.

$\therefore P(x) = (x-1)^3(x+1)(x+5)$

(d) $Q(x) = x^4 + px^3 + qx^2 - 5x + 1$

As $Q(x)$ has a zero at $x=1$ $\therefore Q(1)=0$

$\therefore 0 = 1 + p + q - 5 + 1$

$\therefore p + q = 3$ —— (1)

$$
\begin{array}{r l}
 & x^2 + px + (q-2) \\
x^2+2 \,\big) & x^4 + px^3 + qx^2 - 5x + 1 \\
 & -(x^4 \qquad + 2x^2) \\
\hline
 & px^3 + x^2(q-2) - 5x + 1 \\
 & -(px^3 \qquad + 2px) \\
\hline
 & x^2(q-2) + x(-5-2p) + 1 \\
 & -(x^2(q-2) \qquad + 2(q-2)) \\
\hline
 & x(-5-2p) + (5-2q)
\end{array}
$$

But the remainder is $-7x+1$.

$\therefore -7 = -5 - 2p$ —— (2)

$1 = 5 - 2q$ —— (3)

from (2) $2p = 2 \therefore p = 1$
and from (3) $2q = 4 \therefore q = 2$ which satisfies (1)

$\therefore (p, q) = (1, 2)$.

2(a) (i) $I = \int \frac{x^5}{\sqrt{1-x^6}} dx$

Let $u = 1 - x^6$

$\therefore \frac{du}{dx} = -6x^5$

$\therefore \frac{du}{-6} = x^5 \, dx$

$$\therefore I = \int \frac{du}{-6} \cdot \frac{1}{\sqrt{u}}$$

$$= -\frac{1}{6} \int u^{-\frac{1}{2}} \, du$$

$$= -\frac{1}{6} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -\frac{1}{3} \sqrt{1-x^6} + c$$

(ii) $I = \int_1^e \frac{dx}{x(1+\log_e x)^2}$

Let $u = 1 + \log_e x$

$\therefore \frac{du}{dx} = \frac{1}{x}$

$\therefore du = \frac{dx}{x}$

when $x=1$ $u=1$

$x=e$ $u=2$

$$\therefore I = \int_1^2 \frac{du}{u^2}$$

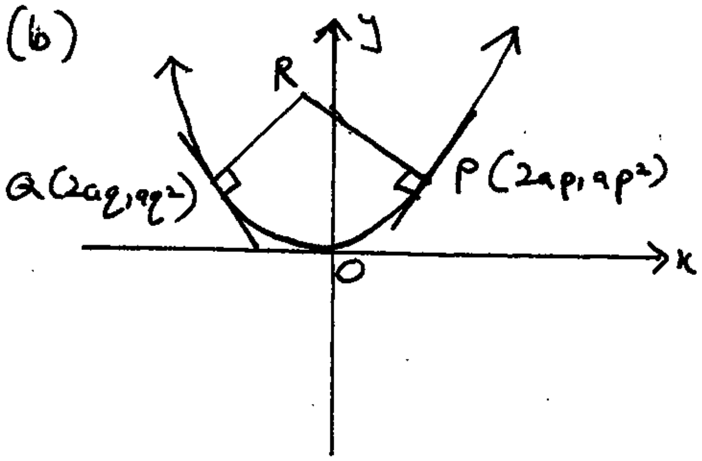
$$= \left[\frac{u^{-1}}{-1}\right]_1^2$$

$$= -\left[\frac{1}{u}\right]_1^2$$

$$= -\left[\frac{1}{2} - 1\right]$$

$$= \frac{1}{2}$$

(b)



(i) $x^2 = 4ay \quad \therefore y = \frac{x^2}{4a}$

$\therefore \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At $P(2ap, ap^2)$ $\frac{dy}{dx} = \frac{2ap}{2a} = p$

$\therefore m_{tan} = p \quad \therefore m_{norm} = \frac{-1}{p}$

$\therefore$ Eq'n of normal at P is:

$y - ap^2 = \frac{-1}{p}(x - 2ap)$

$\therefore py - ap^3 = -x + 2ap$

$\therefore x + py = 2ap + ap^3$ —— (1)

Similarly eq'n of normal at Q is:

$x + qy = 2aq + aq^3$ —— (2)

(1) − (2): $y(p-q) = 2a(p-q) + a(p^3 - q^3)$

$\therefore y = \frac{2a(p-q) + a(p-q)(p^2+pq+q^2)}{p-q}$

$\therefore y = 2a + a(p^2+pq+q^2)$

$\therefore y = a(p^2+pq+q^2+2)$ sub into (1)

$\therefore x + pa(p^2+pq+q^2+2) = 2ap + ap^3$

$\therefore x + \cancel{ap^3} + ap^2q + apq^2 + \cancel{2ap} = \cancel{2ap} + \cancel{ap^3}$

$\therefore x = -apq(p+q)$

$\Rightarrow R = [-apq(p+q), a(p^2+pq+q^2+2)]$

(ii) As normals meet at 90° $\Rightarrow m_{norm\,p} \cdot m_{norm\,q} = -1$

$\therefore \frac{-1}{p} \cdot \frac{-1}{q} = -1 \quad \Rightarrow pq = -1$

$\therefore R = [a(p+q), a(p^2+q^2+1)]$

$\therefore x = a(p+q) \quad \therefore p+q = \frac{x}{a}$ — (1) and $y = a(p^2+q^2+1) \quad \therefore \frac{y}{a} - 1 = p^2+q^2$ — (2

But $p^2+q^2 = (p+q)^2 - 2pq \quad \therefore p^2+q^2 = (p+q)^2 + 2$

$\Rightarrow \frac{y}{a} - 1 = \left(\frac{x}{a}\right)^2 + 2 \quad \therefore \frac{y}{a} - 3 = \frac{x^2}{a^2} \quad \therefore x^2 = a(y - 3a)$

(iii) From (ii) $x^2 = a(y-3a)$ is of the form $X^2 = 4A(Y-k)$

$\therefore$ vertex $= (0, 3a)$

$4A = a \quad \therefore A = \frac{a}{4}$

ie focal length $= \frac{a}{4}$

$\Rightarrow$ focus of the locus of R is: $\left(0, \frac{13a}{4}\right)$.

Q3 (a) Let $y = \ln\left(x^3\sqrt{x^2+1}\right)$

$$= \ln x^3 + \ln (x^2+1)^{\frac{1}{2}}$$

$$= 3\ln x + \tfrac{1}{2}\ln(x^2+1)$$

$$\therefore \frac{dy}{dx} = \frac{3}{x} + \frac{2x}{2(x^2+1)}$$

$$= \frac{3}{x} + \frac{x}{x^2+1}$$

$$= \frac{3x^2+3+x^2}{x(x^2+1)}$$

$$= \frac{4x^2+3}{x(x^2+1)}$$

(b) $$I = \int_1^3 \left(2x + \frac{3}{x^2}\right)^2 dx$$

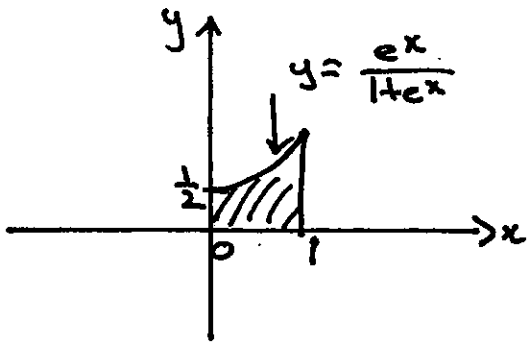
$$= \int_1^3 4x^2 + \frac{12}{x} + 9x^{-4}\, dx$$

$$= \left[\frac{4x^3}{3} + 12\ln x - 3x^{-3}\right]_1^3$$

$$= \left[\left(36 + 12\ln 3 - \tfrac{1}{9}\right) - \left(\tfrac{4}{3} + 0 - 3\right)\right]$$

$$= \frac{338}{9} + 12\ln 3$$

(c)



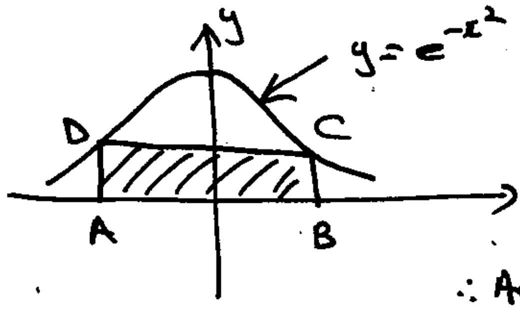
$$\therefore \text{Area} = \int_0^1 \frac{e^x}{1+e^x}\, dx$$

$$= \left[\ln(1+e^x)\right]_0^1$$

$$= \left[\ln(1+e) - \ln(1+1)\right)$$

$$= \ln\left(\frac{1+e}{2}\right) \text{ units}^2$$

(d) (i)



Let $B = (x, 0)$ $\therefore A = (-x, 0)$

$C = (x, e^{-x^2})$, $D = (-x, e^{-x^2})$

$\therefore$ Area of ABCD $= LB$

$= 2x \times e^{-x^2}$

$= 2x\, e^{-x^2}$ units^2

(ii) $A = 2x\, e^{-x^2}$

$\therefore A' = 2x \cdot e^{-x^2} \cdot -2x + e^{-x^2} \cdot 2$

$= 2e^{-x^2}\left[-2x^2 + 1\right]$

For a possible max/min $A' = 0$ $\therefore 1 = 2x^2$ $(e^{-x^2} > 0)$

$\therefore x = \frac{1}{\sqrt{2}}$ $(x > 0)$

x	$\frac{1}{\sqrt{2}}^-$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}^+$
A'	+	0	−

$\Rightarrow$ max. area when $x = \frac{1}{\sqrt{2}}$

$\therefore$ Max. area $= \frac{2}{\sqrt{2}} e^{-\frac{1}{2}}$

$= \sqrt{2}\, e^{-\frac{1}{2}}$ units^2

(e) Let $y = (x-1)e^x$

$\therefore y' = (x-1) \cdot e^x + e^x \cdot 1$

$= e^x(x - 1 + 1)$

$= xe^x$

Now $\int_{-1}^{1} xe^x\, dx = \left[(x-1)e^x\right]_{-1}^{1}$

$= \left[0 - (-2)e^{-1}\right]$

$= \frac{2}{e}$

4(a) <u>TO PROVE:</u> $3^{3n}+2^{n+2}$ is divisible by 5 for $n \in J^+$

<u>PROOF:</u> <u>Step 1:</u> When $n=1$ $\quad 3^{3n}+2^{n+2} = 3^3+2^3$

$$= 27+8$$
$$= 35$$

which is divisible by 5

$\therefore$ it is true for $n=1$.

<u>Step 2:</u> Assume it is true for $n=k$ $(k \le n, k \in J^+)$ and prove it is true for $n=k+1$.

<u>i.e.</u> $\quad 3^{3k}+2^{k+2} = 5M \quad$ (where $M \in J$)

$$\therefore 3^{3k} = 5M - 2^{k+2} \quad \text{———} \ (1)$$

When $n=k+1$

$$3^{3n}+2^{n+2} = 3^{3(k+1)} + 2^{k+1+2}$$
$$= 3^{3k}.3^3 + 2^{k+3}$$
$$= (5M-2^{k+2})27 + 2^{k+3} \quad \text{(sub (1))}$$
$$= 135M - 27.2^{k+2} + 2^{k+2}.2$$
$$= 135M - 25.2^{k+2}$$
$$= 5[27M - 5.2^{k+2}]$$

which is divisible by 5

$\therefore$ if it is true for $n=k$ so it is true for $n=k+1$.

<u>Step 3:</u> It is true for $n=1$ and so it is true for $n=1+1=2$.
It is true for $n=2$ and so it is true for $n=2+1=3$
and so on for all positive integral values of n.

(b) $y = xe^{-x}$

(i) $y' = x.-e^{-x} + e^{-x}.1$

$= -e^{-x}[x-1]$

$y'' = -e^{-x}.1 + (x-1).+e^{-x}$

$= -e^{-x}[1-(x-1)]$

$= -e^{-x}[2-x]$

For a stat. pt $y'=0$ $\therefore x=1$ $(e^{-x}>0)$

When $x=1$ $y''<0$ $\Rightarrow$ max. turn. pt at $(1, \frac{1}{e})$

For a possible pt of inflexion $y''=0$ $\therefore x=2$ $(e^{-x}>0)$

x	2^-	2	2^+
y''	$-$	0	$+$

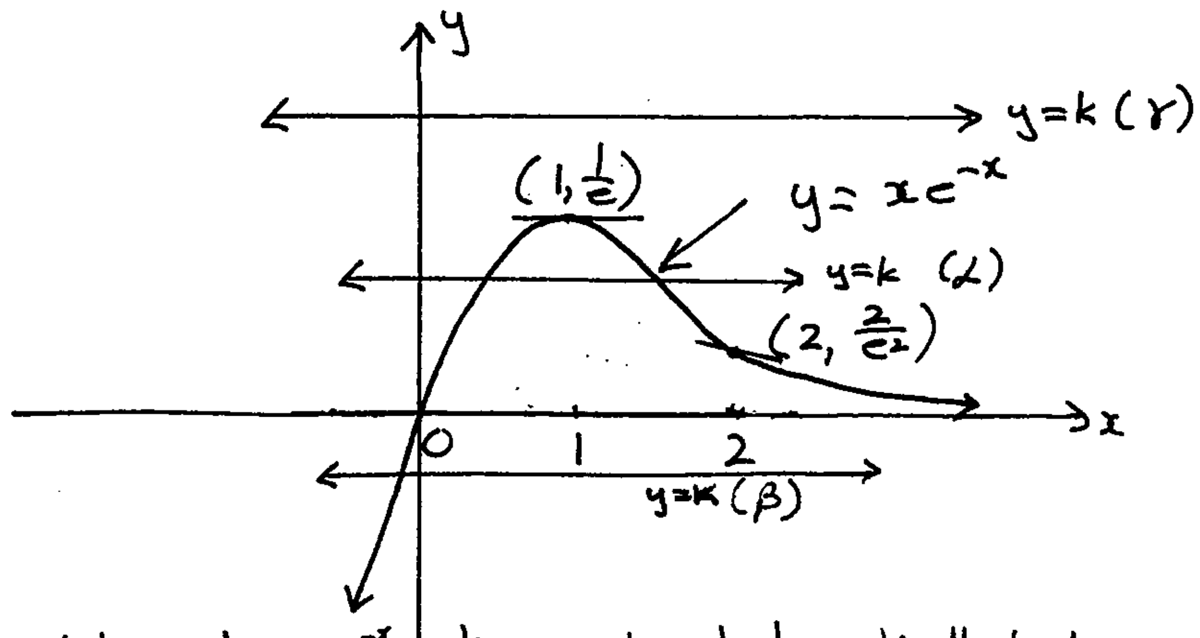
concavity change $\Rightarrow$ pt. of inflexion at $(2, \frac{2}{e^2})$

As $x \to \infty$ $y \to 0$

As $x \to -\infty$ $y \to -\infty$

Intercept at $(0,0)$

(ii)



The solution to $xe^{-x}=k$ can be solved graphically by drawing $y=k$ on the graph of $y=xe^{-x}$.

(α) If $0<k<\frac{1}{e}$ $y=k$ will cut twice $y=xe^{-x}$ $\therefore$ 2 roots

(β) If $k \leq 0$ $y=k$ will only cut once $y=xe^{-x}$ $\therefore$ 1 real root

(γ) If $k>\frac{1}{e}$ $y=k$ will lie above the curve $y=xe^{-x}$ $\therefore$ No real roots