NAME: ______________________________

CLASS: ______________________________

DANEBANK
An Anglican School for Girls

2008
Year 12
Half Yearly Examination

Mathematics
Extension 1

Outcomes Examined: H3, H5, H8, PE2, PE3, PE5, HE2, HE5, HE6, HE7

Weighting of Task: 30%

General Instructions

- Reading time – 5 minutes
- Working time – $1\frac{1}{2}$ hours
- Write using black or blue pen
- All necessary working should be shown in every question otherwise full marks may not be awarded.
- Board-approved calculators may be used
- Start each new question in a new booklet

Total Marks – 63

Attempt All Questions 1 - 5

Questions are NOT of equal value

This paper MUST NOT be removed from the examination room

Total marks – 63
Attempt Questions 1- 5
All questions are not of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $x = 2+\sqrt{3}$, find the exact value of $x+\frac{1}{x}$. Hence, or otherwise find the exact value of $x^2+\frac{1}{x^2}$. **3**

(b) Find the ratio by which the point $C(3,10)$ divides the line segment from $A=(2,8)$ to $B=(6,16)$. **2**

(c) Solve $\left|\frac{x}{x+1}\right| \geq \frac{1}{2}$ **3**

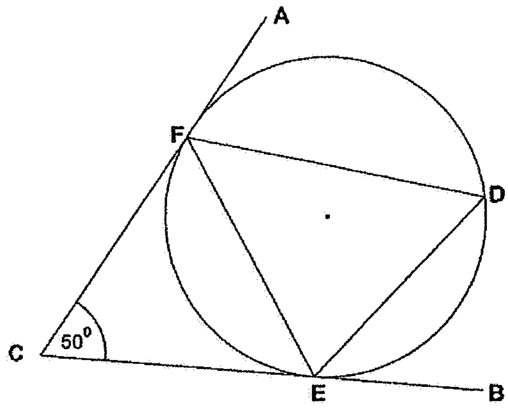
(e) Find the acute angle between the line $y=3x+1$ and $2y=1+x$. **2**

(f) If $\frac{d^2y}{dx^2} = 2e^x - 3e^{-x}$ and when $x=0$, $y=6$ and $\frac{dy}{dx}=5$, find an expression for y in terms of x. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$. 3

(b)



Copy this diagram in your answer booklet

In the diagram, AC and BC are tangents to the circle, touching the circle at F and E respectively. $\angle ACB$ equals 50^0.

(i) Show the $\angle CEF = 65^0$. 2

(ii) Hence find $\angle EDF$. 1

(c) Two of the roots of the equation $x^3 + mx^2 + nx + l = 0$ are equal in value but opposite in sign.

(i) Prove that $x = -m$ is a solution for the above equation. 2

(ii) Hence, show that $l = mn$. 1

(d) Consider the curve $y = 2\sin 2x$

(i) Graph this curve for the domain 2

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) Explain, by the use of this graph, why the area under $y = 2\sin 2x$ from 1

$x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ can be found by $A = 2\int_0^{\frac{\pi}{2}} 2\sin 2x dx$

Question 3 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by Mathematical Induction that $5^{2n}+12^{n-1}$ is divisible by 13 for all positive integers n. **3**

(b) Given $f(x)=\sin^4 3x$

(i) Find $f'(x)$ **1**

(ii) Show that $12\cos 3x\sin^3 3x = 6\sin 6x\sin^2 3x$ **1**

(ii) Hence $\int \sin 6x\sin^2 3x\,dx$ **1**

(c) It is given that $P(x)=2x^4-5x^3+5x^2-4x-4$.

(i) Show that 2 and $-\frac{1}{2}$ are zeros of $P(x)$. **1**

(ii) Express $P(x)$ as the product of three algebraic factors and hence show that $P(x)=0$ has only two real roots. **2**

(iii) Hence, find the set of values of x for $P(x)>0$. **1**

(d) John was asked to find $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$, using the substitution $u=1+x^2$. **3**

Below is his solution. Clearly identify the mistakes and explain why they are wrong.

$$u=1+x^2 \quad \therefore \quad \frac{du}{dx}=2x$$

$$\frac{du}{2}=xdx$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1+x^2}}dx = \int_0^1 \frac{1}{2\sqrt{u}}du$$

$$= \int 2u^{\frac{1}{2}}du$$

$$= \left[u^{\frac{-1}{2}}\right]_0^1$$

$$= (1)-(0)$$

$$= 1\,units^2$$

Question 4 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove that $\dfrac{1-\cos x}{\sin x} = t$, where $t = \tan\dfrac{x}{2}$. **2**

(b) Find $\int \sin x \cos^3 x \, dx$, using the substitution $u = \cos x$. **2**

(c) (i) Given $y = \sin x + \cos x$, show that $y^2 = 1 + \sin 2x$ **2**

(ii) Find the exact volume of the solid which is formed by rotating $y = \sin x + \cos x$ about the *x-axis*, from $x = 0$ to $x = \dfrac{\pi}{2}$. **2**

(d) (i) Sketch the curves $y = \cos x$ and $y = \cos^2 x$ on the same diagram, for $0 \le x \le \dfrac{\pi}{2}$. **2**

(ii) Find the exact area between these two curves. **3**

Ext.1 Sol^n. 2008 H-Yrly

1\. a) $x = 2+\sqrt{3}$

i) $x + \frac{1}{x} = 2+\sqrt{3} + \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= 2+\sqrt{3} + \frac{2-\sqrt{3}}{1}$

$= 4$

ii) $x^2 + \frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$ $\left\{(x+\frac{1}{x})^2 = x^2+\frac{1}{x^2}+2\right\}$

$= 4^2 - 2$

$= 14$

b) $AC = \sqrt{(3-2)^2+(10-8)^2}$

$= \sqrt{5}$

$CB = \sqrt{(6-3)^2+(16-10)^2}$

$= \sqrt{45} = 3\sqrt{5}$

$AC : CB = \sqrt{5} : 3\sqrt{5}$

$= 1 : 3$

c) $\left|\frac{x}{x+1}\right| \geq \frac{1}{2}$ square both sides

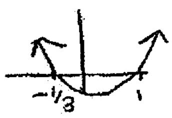
$\frac{x^2}{(x+1)^2} \geq \frac{1}{4}$

$4x^2 \geq (x+1)^2$

$4x^2 - (x^2+2x+1) \geq 0$

$3x^2 - 2x - 1 \geq 0$

$(3x+1)(x-1) \geq 0$



$x \leq -\frac{1}{3}$, $x \geq 1$

but $x \neq -1$

e) $y = 3x+1$ $m_1 = 3$

$2y = 1 + x$ $m_2 = \frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{3 - 1/2}{1 + 3\times\frac{1}{2}}\right|$

$= 1$

$\theta = 45°$

f) $\frac{d^2y}{dx^2} = 2e^{x} - 3e^{-x}$

$\frac{dy}{dx} = \int (2e^{x} - 3e^{-x})\,dx$

$= 2e^{x} + 3e^{-x} + C$ but $\frac{dy}{dx} = 5$ when $x = 0$

$\therefore 5 = 2e^0 + 3e^0 + C$

$C = 0$

$y = \int (2e^{x} + 3e^{-x})\,dx$

$= 2e^{x} - 3e^{-x} + C$ but $y = 6$ when $x = 0$

$\therefore 6 = 2e^0 - 3e^0 + C$

$C = 7$

$\therefore y = 2e^{x} - 3e^{-x} + 7$

Q2.

a) $\sin 2x = \tan x$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x\,(2\cos^2 x - 1) = 0$

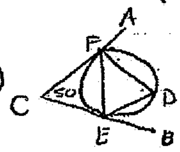
$\therefore \sin x = 0$ $\quad 2\cos^2 x - 1 = 0$

$x = 0, \pi$ $\quad \cos x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}, \pi - \frac{\pi}{4}$

$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

b)



i) CE = CF (tangents from ext. pt are =)

$\therefore \triangle CEF$ is isosceles

$2\angle CEF + 50 = 180$ ($\angle$ sum of $\triangle$)

$\angle CEF = 130 \div 2$

$= 65°$

ii) $\angle CEF = \angle EDF = 65°$ (angle in alternate segment)

2.c. i) $x^3 + mx^2 + nx + l = 0$

roots are $\alpha, -\alpha$ & β

$\therefore$ Sum of roots $= -\frac{b}{a}$

$\alpha + -\alpha + \beta = -m$

$\beta = -m$

$\therefore x = -m$ is a sol^n to the eq^n.

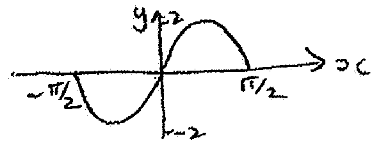
ii) when $x = -m$

$(-m)^3 + m(-m)^2 + n(-m) + l = 0$

$-m^3 + m^3 - mn + l = 0$

$l = mn$

d) i)



Period $= \frac{2\pi}{2}$
$= \pi$

ii) Since $y = 2\sin 2x$ is odd $\therefore$ area from $-\frac{\pi}{2}$ to 0 is = but opp. in sign to area from $\frac{\pi}{2}$ to 0

$\therefore A = 2\int_0^{\pi/2} 2\sin 2x\, dx$

Q3 . a) Test $n=1$

$5^2 + 12^0 = 25 + 1$
$= 26$ which is divisible by 13.

$\therefore$ true for $n=1$

Assume true for $n=k$

$5^{2k} + 12^{k-1} = 13Q$ where Q is an integer

$\therefore 5^{2k} = 13Q - 12^{k-1}$

Prove true for $n=k+1$

Now $5^{2(k+1)} + 12^{k+1-1} = 5^{2k+2} + 12^k$

$= 5^2 . 5^{2k} + 12^k$

$= 25(13Q - 12^{k-1}) + 12^k$

$= 25.13Q - 25.12^{k-1} + 12^k$

$= 25.13Q - 25.12^{k-1} + 12.12^{k-1}$

$= 25.13Q - 13.12^{k-1}$

$= 13(25Q - 12^{k-1})$

which is divisible by 13 sine k is +ve integer.

etc

b) $f(x) = \sin^4 3x$

$= (\sin 3x)^4$

i) $f'(x) = 4\sin^3 3x . 3\cos 3x$

$= 12\cos 3x \sin^3 3x$

ii) $12\cos 3x \sin^3 3x$

$= 12\cos 3x . \sin 3x . \sin^2 3x$

$= 6 \times 2\cos 3x \sin 3x . \sin^2 3x$

$= 6\sin 6x . \sin^2 3x$

iii) $\therefore \int \sin 6x \sin^2 3x\, dx = \int 2\cos 3x \sin^3 3x\, dx$

$= \frac{1}{6}\sin^4 3x + C$

c) $P(x) = 2x^4 - 5x^3 + 5x^2 - 4x - 4$

i) $P(2) = 2\times2^4 - 5\times2^3 + 5\times2^2 - 4\times2 - 4$
$= 0$

$P(-\frac{1}{2}) = 2(-\frac{1}{2})^4 - 5(-\frac{1}{2})^3 + 5(-\frac{1}{2})^2 - 4(-\frac{1}{2}) - 4$
$= 0$

$\therefore$ Since $P(x)$ is 0 in both cases $\therefore$ they are zeros.

ii) $(x-2)(2x+1)$ are factors

$\therefore (2x^2 - 3x - 2)$ is a factor

$$\begin{array}{r} x^2 - x + 2 \\ (2x^2 - 3x - 2)\overline{)2x^4 - 5x^3 + 5x^2 - 4x - 4} \\ \underline{2x^4 - 3x^3 - 2x^2} \\ -2x^3 + 7x^2 - 4x \\ \underline{-2x^3 + 3x^2 + 2x} \\ 4x^2 - 6x - 4 \\ 4x^2 - 6x - 4 \end{array}$$

but $x^2 - x + 2$

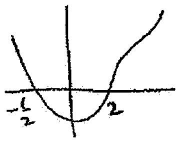
$\Delta = (-1)^2 - 4\times1\times2$
< 0

$\therefore$ +ve def

so no roots.

iii) $(2x+1)(x-2)(x^2-x+2)>0$

↑ +ve def



$\therefore\ (2x+1)(x-2)>0$

$x<-\frac{1}{2}$ or $x>2$

d) $\int_0^1 \frac{1}{2\sqrt{u}}\,du$ - limits have not changed

$\int 2u^{1/2}\,du$ - $\frac{1}{2\sqrt{u}}$ is not $2u^{1/2}$
- no limits of int.

$= \left[u^{-1/2}\right]_0^1$ limits still wrong
this has been diff. not integrated.

$= 1 - 0$. $0^{-1/2}$ is undefined not 0.

units^2 - not an area

4(a) $\frac{1-\cos x}{\sin x} = t$

$\therefore \text{LHS} = \dfrac{1 - \left(\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}}$

$= \dfrac{1+t^2-(1-t^2)}{1+t^2} \times \dfrac{1+t^2}{2t}$

$= \dfrac{2t^2}{2t}$

$= t$

$= \text{RHS}$

b) $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

$\int \sin x \cos^3 x\,dx = \int -u^3\,du$

$= -\frac{u^4}{4} + c$

$= -\frac{\cos^4 x}{4} + c$

c) i) $y = \sin x + \cos x$

$y^2 = (\sin x + \cos x)^2$

$= \sin^2 x + \cos^2 x + 2\sin x \cos x$

$= 1 + \sin 2x$

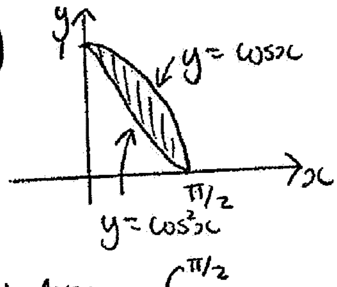
ii) $\text{Vol} = \pi\int_0^{\pi/2} y^2\,dx$

$= \pi\int_0^{\pi/2} (1 + \sin 2x)\,dx$

$= \pi\left[x - \frac{\cos 2x}{2}\right]_0^{\pi/2}$

$= \pi\left\{\left(\frac{\pi}{2} - \frac{\cos\pi}{2}\right) - \left(0 - \frac{\cos 0}{2}\right)\right\}$

$= \pi\left(\frac{\pi}{2}+1\right)\ \text{units}^3$

d) 

ii) $\text{Area} = \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx$

$= \int_0^{\pi/2} \left(\cos x - \frac{1}{2}\left[\cos 2x + 1\right]\right)dx$

$= \left[\sin x - \frac{1}{4}\sin 2x - \frac{1}{2}x\right]_0^{\pi/2}$

$= \left(\sin\frac{\pi}{2} - \frac{1}{4}\sin\pi - \frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(\sin 0 - \frac{1}{4}\sin\right.$

$= 1 - \pi/4\ \text{units}^2$

Q5 a) Period $= \frac{2\pi}{n}$ amp $= 4$

$= 24$

$\frac{2\pi}{n} = 24$

$n = \frac{2\pi}{24}$

$= \pi/12$

$\therefore$ eq^n $y = 4\cos\left(\frac{\pi x}{12}\right)$

ii) $y = 4\sin\left(\frac{\pi}{2} - \frac{\pi x}{12}\right)$

iii) α) $6 - x = 24\cos\left(\frac{\pi x}{12}\right)$

$\frac{6-x}{6} = 4\cos\left(\frac{\pi x}{12}\right)$

$1 - \frac{x}{6} = 4\cos\left(\frac{\pi x}{12}\right)$

Draw $y = 1 - \frac{x}{6}$ on the same graph and read off x-values of pts of intersection.

β) 3 sol^{n}s

γ) $x \doteqdot -4.3$

b) i) $\triangle BCD$ is isosceles

$\therefore \angle BDC = a$ (equal base $\angle$'s)

$\angle DCR = a + a$ (ext. $\angle$ of $\triangle BCD$)

$= 2a$

ii) $\angle DCR = \angle DAB$ (ext. $\angle$ of cyclic quad = opp int. $\angle$)

$\therefore \angle BAD = 2a$

but OA bisects $\angle BAD$

$\therefore \angle OAD = 2a \div 2$

$= a$

iii) $\angle TAD = 90 - a$ ($OA \perp AT$)

$\angle ABD = 90 - a$ ($\angle$ in alt. seg)

$\angle ABC = 90 - a + a$ (adj. $\angle$'s)

$= 90°$