GIRRAWEEN HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION

1998

MATHEMATICS

3 UNIT

Time allowed - Two hours

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a new page.
- A table of standard integrals is provided.

Question 1. (18 marks)

(a) Find, correct to 1 decimal place:

(i) $\int_1^3 e^{2x}dx$ (ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x dx$ 6

(b) Find the acute angle between the lines $y=2x-1$ and $x+y=3$. Give your answer correct to the nearest minute. 3

(c) Find an exact value for $\sin 75^0$. 3

(d) Sketch the curve $y=2\cos 3x$ for $0 \le x \le 2\pi$ 3

(e) Solve $\sin 2x = \frac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$ 3

Question 2 (24 marks)

(a) Use the substition $u=x+1$ to evaluate $\int_0^1 x(x+1)^4 dx$. 5

(b) It is known that 90% of students sitting for an examination will pass. In a random sample of 10 students sitting for this examination, what is the probability that exactly 3 students will fail. Give your answer correct to 2 decimal places. 3

(c) Show that $\frac{d}{dx}(x\log_e x) = \log_e x + 1$. Hence find $\int \log_e x dx$. 4

(d) Find the exact value of:

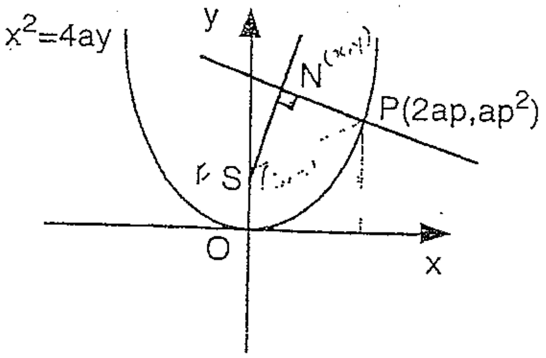
(i) $\cos(\cos^{-1}(\frac{1}{4}))$ (ii) $\sin^{-1}(\cos\frac{\pi}{3})$ (iii) $\tan\left[2\tan^{-1}(\frac{1}{3})\right]$ 8

(e) (i) What is the domain and range of $y=\sin^{-1}\left(\frac{x}{3}\right)$ 2

(ii) Sketch the curve $y=\sin^{-1}\left(\frac{x}{3}\right)$. 2

Question 3 (16 marks)

(a)



$P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. SN is perpendicular to the normal at P, while $S(0,a)$ is the focus of the parabola

(i) Show the equation of the normal is $x + py = 2ap + ap^3$. 3

(ii) Find the equation of SN. 3

(iii) Find the coordinates of N. 3

(iv) Show the locus of N is the parabola $x^2 = a(y-a)$ 3

(b) How many different committees of 3 men and 4 women can be formed from 7 men and 8 women if:

(i) there are no restrictions? 2

(ii) a particular man has to be on the committee? 2

Question 4 (23 marks)

(a) Find $\frac{dy}{dx}$ if:

(i) $y = \frac{\sin x}{x}$ (ii) $y = \ln(\tan(x^2))$ (iii) $y = \sin^{-1}\left(\frac{x}{3}\right)$ 7

(b) Find:

(i) $\int \frac{2x}{2x^2+1}dx$ (ii) $\int \frac{\sin x}{\cos x}dx$ (iii) $\int xe^{x^2}dx$ 9

(c) Find the coefficient of x in the expansion $\left(x^2 - \frac{1}{x^3}\right)^{13}$ 3

(d) Prove $\frac{\sin 2\theta + 1}{\cos 2\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ 4

Question 5 (14 marks)

(a) (i) How many ways can the letters HELLO be arranges in a line? 2

(ii) If one of these arrangements was picked at random, what is the probability that LL would be together? 2

(b) Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of its roots is 4. 4

(c) Find the area enclosed between $y = \sin x$ and the x axis between $x = \frac{\pi}{4}$ and $x = \pi$. Give an exact answer, showing units. 3

(d) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx$ 3

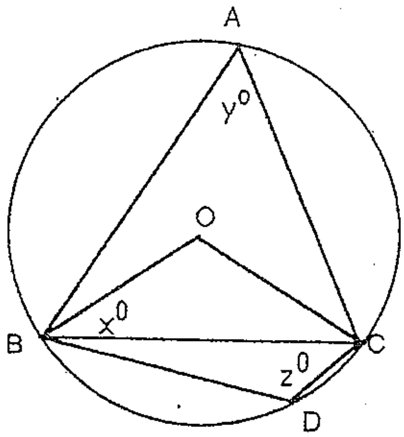
Question 6 (15 marks)

(a) 8

O is the centre of the circle

(i) Prove that $x + y = 90$

(ii) Prove that $z - y = 2x$



(b) If $y = \tan^{-1} x$ prove that $2x\frac{dy}{dx} + (1 + x^2)\frac{d^2x}{dy^2} = 0$ 3

(c) The line $y = mx$ is tangent to the curve $y = e^{3x}$. Find m. 4

180-90-x

Year 12 3unit Half Yearly 1998

Question 1 18 marks

(a) (i) $\int_1^3 e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}\right]_1^3$

$= \frac{1}{2}\left[e^6 - e^2\right]$

$\therefore = 198.0$ (3)

(ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\,dx = \left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= 1 - \frac{1}{\sqrt{3}}$

$= 0.4$ (3)

(b) $y = 2x - 1 \quad m_1 = 2$

$y = -x + 3 \quad m_2 = -1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2-(-1)}{1+(-2)}\right|$

$= \left|\frac{3}{-1}\right|$

$\tan\theta = +3$

$\therefore \theta = 71^\circ 34'$ (3)

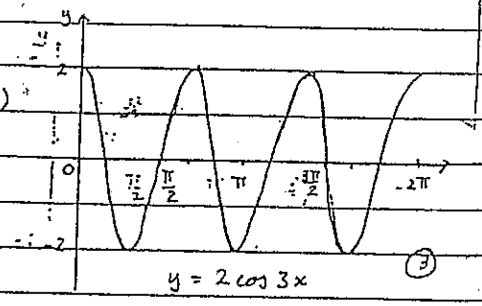
(c) $\sin(30^\circ + 45^\circ)$

$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$

$\sin 75^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}}$ (3)

(d)



$y = 2\cos 3x$ (3)

(e) $\sin 2x = \frac{\sqrt{3}}{2} \qquad 0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$

$\therefore 2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ (3)

Question 2 24 marks

(a) $u = x + 1 \qquad x = 0,\ u = 1$

$\frac{du}{dx} = 1 \qquad x = 1,\ u = 2$

$\therefore du = dx$

$\therefore \int_0^1 x(x+1)^4\,dx = \int_1^2 (u-1)\cdot u^4\,du$

$= \int_1^2 (u^5 - u^4)\,du$

$= \left[\frac{u^6}{6} - \frac{u^5}{5}\right]_1^2$

$= \left(\frac{64}{6} - \frac{32}{5}\right) - \left(\frac{1}{6} - \frac{1}{5}\right)$

$= \frac{64}{15} - \left(-\frac{1}{30}\right)$

$= \frac{43}{10}$ or $4\frac{3}{10}$ (5)

(b) $p(\text{fail}) = 0.1$

q (not fail, but pass) $= 0.9$

$P(X=3) = {}^{10}C_3\,(0.1)^3(0.9)^7$

$= 0.06$ (3)

(c) $y = x\log_e x$

$\frac{dy}{dx} = x \cdot \frac{1}{x} + \log_e x \cdot 1$

$= 1 + \log_e x$

$\therefore \int(1 + \log_e x)\,dx = x\log_e x + c$

$\int 1\,dx + \int \log_e x\,dx = x\log_e x + c$

$x + \log_e x\,dx = x\log_e x + c$

$\therefore \int \log_e x\,dx = x\log_e x - x + c$ (4)

(d) (i) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{4}$ (2)

(ii) $\sin^{-1}\left(\cos\frac{\pi}{3}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$ (2)

(iii) $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]$

Let $\tan^{-1}\left(\frac{1}{3}\right) = \alpha$

$\therefore \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

$\therefore = \frac{2\cdot\tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right)}{1 - \tan^2\left(\tan^{-1}\left(\frac{1}{3}\right)\right)}$

$= \frac{2\cdot\frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2}$

$= \frac{\frac{2}{3}}{1 - \frac{1}{9}}$

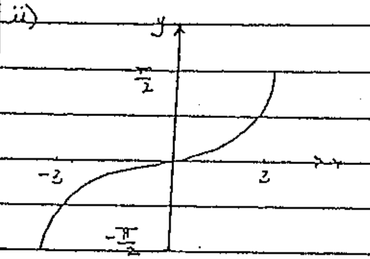
$= \frac{3}{4}$ (4)

(e) (i)

Domain: $-3 \le x \le 3$ (1)

Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (1)

(ii)



(2)

Question 3 (16 marks)

(a) (i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

at $x = 2ap \quad \frac{dy}{dx} = p$

$\therefore$ gradient of normal $= -\frac{1}{p}$

Equation of normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore x + py = 2ap + ap^3$ (3)

(ii) SN: gradient is p

passes through $(0, a)$

$\therefore$ Equation of SN:

$y - a = p(x - 0)$

$px - y + a = 0$ (3)

(iii) coordinates of N

$y = px + a$

$\therefore x + p(px + a) = 2ap + ap^3$

$x + p^2x + ap = 2ap + ap^3$

$x(1 + p^2) = ap + ap^3$

$x(1+p^2) = ap(1+p^2)$

$\therefore x = ap$

$\therefore y = p(ap) + a$

$y = ap^2 + a$

$\therefore N$ is $(ap,\ a(p^2+1))$ (1)

(iv) locus of N:

$x = ap \qquad y = a(p^2+1)$

$\therefore p = \frac{x}{a}$

$\therefore y = a\left(\frac{x^2}{a^2} + 1\right)$

$\therefore y = \frac{x^2}{a} + a$

$ay = x^2 + a^2$

$\therefore x^2 = ay - a^2$

$x^2 = a(y-a)$ (3)

(b) (i)

Committees $= {}^7C_3 \times {}^8C_4$

$= 2450$ (2)

(ii) If one man is included:

Committees $= {}^6C_2 \times {}^8C_4$

$= 1050$ (2)

Question 4 23 marks

(a)

(i) $\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$ (1)

(ii) $\frac{dy}{dx} = \frac{1}{\tan(x)^2} \cdot \sec^2(x)^2 \cdot 2x$

$= \frac{2x\sec(x)^2}{\tan(x)^2}$ (3)

(iii) $\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$ (2)

(b) (i) $\int \frac{2x}{2x^2+1}\,dx = \frac{1}{2}\int \frac{4x}{2x^2+1}\,dx$

$= \frac{1}{2}\ln(2x^2+1) + C$ (3)

(ii) $\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$

$= -\ln \cos x + c$ (3)

(iii) $\int x e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\cdot 2x\,dx$

$= \frac{1}{2}e^{x^2} + C$ (3)

(c)

Term in x^5

$= {}^{13}C_5\,(x^2)^8\left(-\frac{1}{x^3}\right)^5$

$= {}^{13}C_5 \cdot x^{16} \cdot -\frac{1}{x^{15}}$

$= -{}^{13}C_5\,x$

or $-1287x$ (3)

(d) $\frac{\sin 2\theta + 1}{\cos 2\theta}$

$= \frac{2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$

$= \frac{(\cos\theta + \sin\theta)^{\cancel{2}}}{\cancel{(\cos\theta + \sin\theta)}(\cos\theta - \sin\theta)}$

$= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ (4)

Question 5 (14 marks)

(a) (i)

Arrangements $= \frac{5!}{2!}$

$= 60$ (2)

(ii) consider LL as 1 letter.

Arrangements $= 4!$ (24)

$\therefore$ Probability $= \frac{24}{60}$

$= \frac{2}{5}$ (2)

(b) $3x^3 - 17x^2 - 8x + 12 = 0$

Roots $\alpha + \beta + \gamma = \frac{17}{3}$ (A)

$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{8}{3}$ (B)

$\alpha\beta\gamma = -\frac{12}{3} = -4$ (C)

Let $\alpha\beta = 4$

From (C) $\alpha\beta\gamma = -4$

$4\gamma = -4$

$\gamma = -1$

$\therefore$ one factor is $x+1$

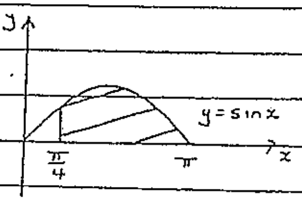
$$
\begin{array}{r|l}
 & 3x^2 - 20x + 12 \\
x+1 & 3x^3 - 17x^2 - 8x + 12 \\
 & 3x^3 + 3x^2 \\
 & -20x^2 - 8x \\
 & -20x^2 - 20x \\
 & 12x + 12 \\
 & 12x + 12 \\
 & 0
\end{array}
$$

$\therefore (x+1)(3x^2 - 20x + 12) = 0$

$(x+1)(3x-2)(x-6) = 0$

Roots are: -1, $\frac{2}{3}$ and 6

(c)



$A = \int_{\frac{\pi}{4}}^{\pi} \sin x\,dx$

$= \left[-\cos x\right]_{\frac{\pi}{4}}^{\pi}$

$= -\left[\cos\pi - \cos\frac{\pi}{4}\right]$

$= -\left[-1 - \frac{1}{\sqrt{2}}\right]$

$= 1 + \frac{1}{\sqrt{2}}$ unit^2

(d) $\cos 2x = 2\cos^2 x - 1$

$\frac{1}{2}(\cos 2x + 1) = \cos^2 x$

$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2x + 1)\,dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\sin\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{1}{8}\left[\pi - 2\right]$

or $\frac{\pi}{8} - \frac{1}{4}$

Q6.

a) i) $OB = OC$ (radii of a circle)

$\therefore \triangle BOC$ is isosceles

$\therefore \angle BCO = x°$

$\angle BOC = 2y$ ($\angle$ at the centre is twice at the circumference)

$\therefore 2x + 2y = 180$ (Area sum of a Δ)

$\therefore x + y = 90°$

ii) $2 + y = 180°$ (opposite $\angle$ of a cyclic quadrilateral are supplementary)

but $2x + 2y = 180°$

$2 + y = 2x + 2y$

$2 - y = 2x$

b) $y = \tan^{-1} x$

$y' = \dfrac{1}{1+x^2}$

$y'' = \dfrac{-2x}{(1+x^2)^2}$

$$2x\left(\frac{1}{1+x^2}\right) + \cancel{(1+x^2)}\left(\frac{-2x}{(1+x^2)^{\cancel{2}}}\right) = 0$$

$$\frac{2x}{1+x^2} - \frac{2x}{1+x^2} = 0$$

$0 = 0$

$\therefore LHS = RHS$

c) $y = mx$

$y = e^{3x}$

$m = 3e^{3x}$

$\therefore m = 3e^{3x}$.