GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

2006
MATHEMATICS
EXTENSION 1

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.

- All necessary working should be shown in every question.
Marks may be deducted for careless or badly arranged work.

- Standard integrals are on the laminated sheets supplied

- Board-approved calculators may be used.

- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.

Total marks – 98
Attempt Questions 1 – 5
All questions are NOT of equal value
Answer each question on a separate piece of paper clearly marked Question 1, Question 2 etc.
Each piece of paper must show your name.

Question 1 (17 Marks) Use a separate piece of paper **Marks**

(a) Solve for x $\dfrac{x+2}{x-4} \leq 3$ **2**

(b) Find the exact value of $\sin 75^0$ with a rational denominator **3**

(c) Sketch the graph of $y = \sin 2x - 1$ $\quad 0 \leq x \leq 2\pi$ **3**

(d) Find the acute angle between the lines $x - 3y + 5 = 0$ and $2x + y - 3 = 0$ **3**

(e) Find the coordinates of the point that divides the interval
A(- 3, - 1) and B(6 , 10) internally in the ratio 2:1. **2**

(f) Solve for x $3^x = 7$ to 2 decimal places **2**

(g) Find $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$ **2**

Question 2 (26 marks) Use a separate piece of paper

(a) Differentiate the following

(i) $y = x \ln x$ (ii) $y = \dfrac{\sin x}{e^{x^2}}$ **4**

(iii) $y = \ln(\tan x)$ (iv) $y = \ln \dfrac{(x-1)}{(x-4)}$ **4**

(v) $y = \cos^3 x$ **2**

(b) Find

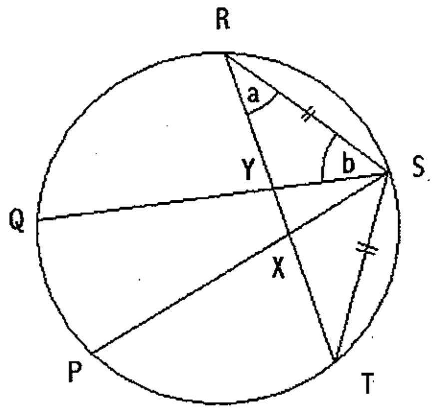
(i) $\displaystyle\int \frac{x^2+2x}{x^3+3x^2-5}dx$ (ii) $\displaystyle\int xe^{2x^2}dx$ **4**

(iii) $\displaystyle\int_0^{\frac{\pi}{12}} \sin^2 3x dx$ (iv) $\displaystyle\int_2^{10} \sqrt{2x+5} dx$ **6**

(v) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx$ (vi) $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin x dx$ **6**

Question 3 **(17 Marks) Use a separate piece of paper** **Marks**

(a) Let PQRST be a circle such that RS = ST, QS meets RT at Y and PS meets RT at X, as in the diagram. Let $\angle YRS = a$ and $\angle QSR = b$



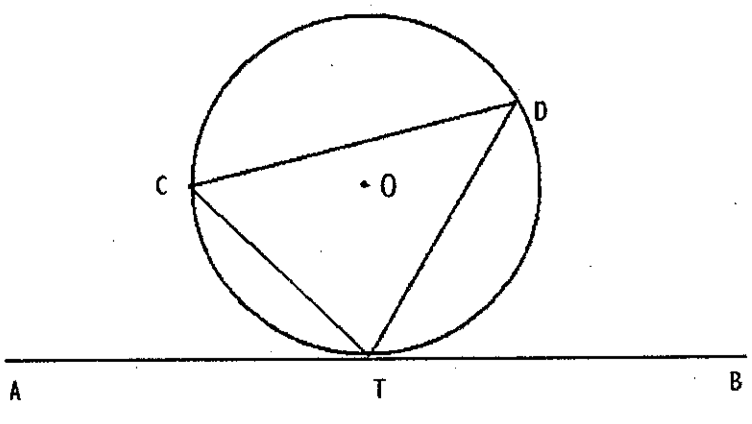
(i) Copy the diagram and state why $\angle SYT = a + b$ **2**

(ii) Prove that $\angle RPQ = b$ **2**

(iii) Prove that $\angle RPS = a$ **2**

(iv) Prove that PQYX is a cyclic quadrilateral **2**

(b) In the diagram below the tangent ATB meets the circle at T . CT is the chord from the point of contact. Prove the alternate segment theory.
That is $\angle ATC = \angle CDT$ **5**



(c) Prove by mathematical induction that **4**

$$\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+.....+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$$

Question 4 **(18 Marks)** Use a separate piece of paper **Marks**

(a) Find the term independent of x in the following expression.
Leave your answer in unexpanded form. **3**

$$(3x^3 - \frac{2}{x^2})^{10}$$

(b) (i) Give the expansion for tan (a + b) **1**

(ii) Given that a , b and c are the three internal angles of a triangle, by using part (i) or otherwise prove that

tan a +tan b + tan c = tan a tan b tan c **3**

(c) The probability of passing a particular test is $\frac{3}{5}$. If 10 students sit the exam, what is the probability that

(i) all 10 students pass **2**

(ii) at least 2 students pass **3**

(d) Using the letters of the name **JIVAJIRAJAH** find

(i) The number of arrangements **2**

(ii) The number of arrangements with the **J**'s together **2**

(iii) The number of arrangements with the **R** and the **H** at the ends **2**

Question 5 (**20** Marks) Use a separate piece of paper **Marks**

(a) Express $\cos x + \sin x$ in the form $R\cos(x - \alpha)$ clearly stating the values of both R and α. **3**

(b) Hence sketch the the graph of $y = \cos x + \sin x$ in the domain $0 \le x \le 2\pi$ **4**

(c)Use the t method, where $t = \tan\frac{\theta}{2}$, to solve the following. Give your answer in radians to 2 decimal places $0 \le \theta \le 2\pi$ **4**

$$4\sin\theta + 3\cos\theta = 2$$

(d) Find the volume when the curve $y = \cos x + \sec x$ is rotated about the x-axis between $x = 0$ and $\frac{\pi}{3}$ **4**

(e) (i) Differentiate the function $y = x\ln(x) - x$ **2**

(ii) Hence or otherwise find $\int_1^e \ln x\,dx$ **3**

EXTENSION 1 HALF YEARLY SOLUTIONS 2006.

Question 1 17 MARKS

(a) $\frac{x+2}{x-4} \le 3 \qquad x \ne 4$

$(x-4)^2 \frac{x+2}{x-4} \le 3(x-4)^2$

$(x-4)(x+2) \le 3(x-4)^2$

$0 \le (x-4)\{3(x-4) - (x+2)\}$

$0 \le (x-4)(2x-14)$

$\therefore \quad x < 4 \quad x \ge 7.$ (2)

(b) $\sin 75^\circ = \sin(45+30)$

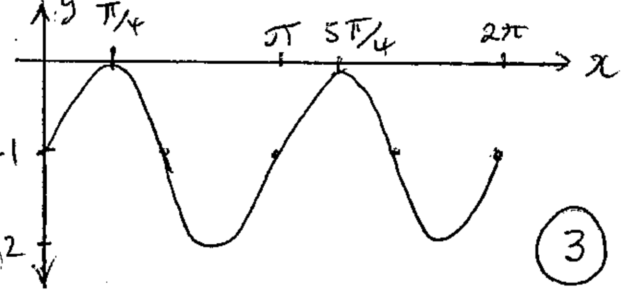
$= \sin 45 \cos 30 + \cos 45 \sin 30$

$= \frac{1}{\sqrt2} \cdot \frac{\sqrt3}{2} + \frac{1}{\sqrt2} \cdot \frac{1}{2}$

$= \frac{\sqrt3 + 1}{2\sqrt2}$

$= \frac{\sqrt6 + \sqrt2}{4}$ (3)

(c)



(3)

(d) $x - 3y + 5 = 0 \Rightarrow 3y = x + 5$

$\therefore \quad m_1 = \frac{1}{3} \qquad y = \frac{1}{3}x + \frac{5}{3}$

$2x + y - 3 = 0 \Rightarrow y = -2x + 3$

$\therefore \quad m_2 = -2$

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= \frac{\frac{1}{3} + 2}{1 - \frac{2}{3}}$

$= 7$

$\theta = 81.87^\circ, \ 81^\circ 52' \, 11.63''$ (3)

(e) $\frac{6\times2 + -3\times1}{2+1}, \ \frac{10\times2 + -1\times1}{2+1}$

$\left(3, \frac{19}{3}\right)$ (2)

(f) $3^x = 7$

$\ln 3^x = \ln 7$

$x \ln 3 = \ln 7$

$x = \frac{\ln 7}{\ln 3}$

$x = 1.77$ (2 D.P.) (2)

(g) $\lim_{x\to0} \frac{\sin 5x}{3x} = \lim_{x\to0} \frac{\sin 5x}{5x} \cdot \frac{5}{3}$

$= 1 \cdot \frac{5}{3}$

$= \frac{5}{3}$ (2)

Question 2. (26 MARKS)

(a) (i) $y = x \ln x$

$\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$ (2)

(ii) $y = \frac{\sin x}{e^{x^2}}$

$\frac{d\frac{u}{v}}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

$= \frac{e^{x^2} \cdot \cos x - \sin x \, (2x\, e^{x^2})}{(e^{x^2})^2}$

$= \frac{\cos x - 2x \sin x}{e^{x^2}}$ (2)

(iii) $y = \ln(\tan x)$

$\frac{dy}{dx} = \frac{\sec^2 x}{\tan x}$ (2)

(iv) $y = \ln \frac{x-1}{x-4}$

$y = \ln(x-1) - \ln(x-4)$

$\frac{dy}{dx} = \frac{1}{x-1} - \frac{1}{x-4}$ (2)

v) $y = \cos^3 x$

$\frac{dy}{dx} = \frac{d\cos^3 x}{d\cos x} \cdot \frac{d\cos x}{dx}$

$= 3\cos^2 x \cdot -\sin x$

$= -3\cos^2 x \sin x$. (2)

) (i) $I = \int \frac{x^2+2x}{x^3+3x^2-5}\, dx$

$= \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-5}\, dx$

$= \frac{1}{3}\ln|x^3+3x^2-5| + C$ (3)

(ii) $I = \int x e^{2x^2}\, dx$

$= \frac{1}{4}\int 4x e^{2x^2}\, dx$

$= \frac{1}{4} e^{2x^2} + C$ (3)

(iii) $I = \int_0^{\pi/12} \sin^2 3x\, dx$

$= \frac{1}{2}\int_0^{\pi/12} 1 - \cos 6x\, dx$

$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\pi/12}$

$= \frac{1}{2}\left[\left(\frac{\pi}{12} - \frac{1}{6}\right) - (0)\right]$

$= \frac{1}{24}(\pi - 2)$ (3)

(iv) $I = \int_2^{10} (2x+5)^{1/2}\, dx$

$= \left[\frac{2}{3}\cdot\frac{1}{2}(2x+5)^{3/2}\right]_2^{10}$

$= \frac{1}{3}\left[(125) - (27)\right]$

$= \frac{98}{3}$ (3)

(v) $I = \int_{\pi/3}^{\pi/2} \sec^2 \frac{x}{2}\, dx$

$= \left[2\tan\frac{x}{2}\right]_{\pi/3}^{\pi/2}$

$= 2\left\{\left(\tan\frac{\pi}{4}\right) - \left(\tan\frac{\pi}{6}\right)\right\}$

$= 2\left(1 - \frac{1}{\sqrt{3}}\right)$ (3)

(vi) $I = \int_{-\pi/2}^{\pi/2} \sin x\, dx$ (3)

$= 0$ ($\sin x$ is an odd fn.)

Question 3. (17 MARKS)

(a) (i) 1 mark for diagram

$\angle SYT$ IS THE EXTERNAL ANGLE TO $\triangle SYR$ ∴

∴ $\angle SYT = \angle SRY + \angle RSY$ (2)

$= a + b$

EXTERNAL $\angle$ = TO THE SUM OF THE INTERIOR OPPOSITE $\angle$'s.

(ii) $\angle RSQ = \angle RPQ$

ANGLE STANDING ON THE SAME CHORD OR ARC (RQ) ARE EQUAL IN THE SAME SEGMENT

∴ $\angle RPQ = b$. (2)

(iii) $\triangle RST$ IS ISOSCELES

$RS = ST$

∴ $\angle SRT = \angle STR$ (

∴ $\angle STR = a$

∴ $\angle SPR = a$

ANGLES STANDING ON THE SAME ARC (RS) IN THE SAME SEGMENT ARE EQUAL. (2)

(iv) $\angle RPQ = b$

$\angle RPS = a$

$\therefore \angle QPS = a + b$

$(= \angle PPQ + \angle RPS)$

$\angle SYT = a + b$ (PART (i))

$\therefore \angle QPS = \angle SYT$

THAT IS EXTERIOR $\angle SYT$ ⌒ QUADRILATERAL PQYX IS EQUAL TO THE INTERIOR OPPOSITE ANGLE $\angle QPS$ (2)

$\therefore$ PQYX IS A CYCLIC QUAD.

(b) CONSTRUCT THE DIAMETRE FROM T THROUGH O MEETING THE CIRCLE AT F. JOIN C TO F

$\angle ATC + \angle CTF = 90^\circ$
(TANGENT $\perp$ TO RADIUS)

$\angle CTF + \angle CFT = 90^\circ$
($\angle TCF = 90^\circ$, ANGLE IN SEMI-CIRCLE)

$\therefore \angle ATC = \angle CFT$

But $\angle CFT = \angle CDT$

(ANGLES STANDING ON THE SAME ARC IN THE SAME SEGMENT ARE EQUAL)

$\therefore \angle ATC = \angle CDT$ (5)

(c) $\frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} \cdots \frac{1}{(n+1)(n+2)}$

$= \frac{n}{2(n+2)}$

Step 1 Prove true for $n=1$

$LHS = \frac{1}{2\times3} = \frac{1}{6}$

$RHS = \frac{1}{2(1+3)} = \frac{1}{6}$

LHS = RHS TRUE FOR $n=1$

Step 2 Assume true for $n=k$ (1)

$\frac{1}{2\times3} + \frac{1}{3\times4} + \cdots \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$

Step 3 Prove true for $n = k+1$

$RHS = \frac{k+1}{2(k+3)}$

$LHS = \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k(k+3)+2}{2(k+2)(k+3)}$

$= \frac{k^2+3k+2}{2(k+2)(k+3)}$ (2)

$= \frac{(k+2)(k+1)}{2(k+2)(k+3)}$

$= \frac{k+1}{2(k+3)}$

$= RHS$

$\therefore$ TRUE FOR $n = k+1$

Step 4. Proved true for $n=1$
Step 3 implies if true for $n=1$ true for $n = 2, 3, 4, 5 \ldots$
$\therefore$ BY THE PRINCIPAL OF M.I TRUE FOR ALL n. (1)

Question 4 (18 MARKS)

a) $\left(3x^3 - \frac{2}{x^2}\right)^{10} = {}^{10}C_0(3x^3)^{10} + {}^{10}C_1(3x^3)^9\left(\frac{-2}{x^2}\right)$

$\ldots + {}^{10}C_n(3x^3)^{10-n}\left(\frac{-2}{x^2}\right)^n$

${}^{10}C_n(3x^3)^{10-n}\left(\frac{-2}{x^2}\right)^n = {}^{10}C_n 3^{10-n}(-2)^n x^{30-3n} x^{-2n}$

if INDEPENDENT OF x

$x^{30-5n} = x^0$

$30-5n = 0$

$n = 6.$

TERM ${}^{10}C_7\ 3^4\ 2^6$ (3)

b) (i) $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$ (1)

(ii) $a+b+c = \pi^c$

$a+b = \pi - c$

$\tan(a+b) = \tan(\pi - c)$

$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{\tan \pi - \tan c}{1 + \tan \pi \tan c}$

$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{-\tan c}{}$ (3)

$\tan a + \tan b = -\tan c\,(1 - \tan a \tan b)$

$= -\tan c + \tan a \tan b \tan c$

$\tan a + \tan b + \tan c = \tan a \tan b \tan c$

c (i) $\left(\frac{3}{5} + \frac{2}{5}\right)^{10} = {}^{10}C_0 \left(\frac{3}{5}\right)^{10}$ (2)

$= 0.006$ (3 D.P.)

(ii) $P(\text{at least } 2) = 1 - \{P(1\text{ pass}) + P(0\text{ pass})\}$

$= 1 - \left({}^{10}C_9\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^9 + {}^{10}C_{10}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^{10}\right)$

$= 1 - (1.57 \times 10^{-3} + 1.05 \times 10^{-4})$

$= 0.998$ (3 D.P.) (3)

(d) (i) # arrangements $= \frac{11!}{3!\,3!\,2!}$

(3 J's, 3 A's, 2 I's) (2)

$= 554400$

(ii) # arrangement $= \frac{9!}{3!\,2!}$ (2)

REGARD J's AS ONE LETTER.

$= 5040$

(iii) # arrangement $= \frac{2 \times 9!}{3!\,3!\,2!}$ (2)

$= 1680$

QUESTION 5 (22 MARKS)

(a) $R\cos(x+\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$ (1)

$\cos x = R\cos\alpha\cos x$ (A)

$1 = R\cos\alpha$ (A)

$\sin x = R\sin x\sin\alpha$

$1 = R\sin\alpha$ (B)

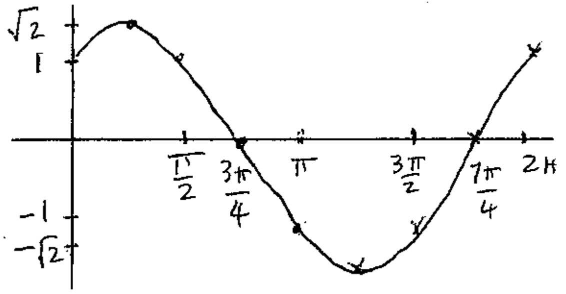
B/A $\frac{R\sin\alpha}{R\cos\alpha} = 1$ (1) $\tan\alpha = 1$ $\boxed{\alpha = 45^\circ}$

$B^2 + A^2$ $R^2\sin^2\alpha + R^2\cos^2\alpha = 2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 2$

$\boxed{R = \sqrt{2}}$ (1)

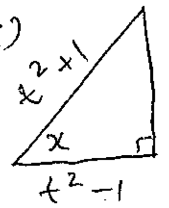
b)



4

$y = \cos x + \sin x.$

(c)



$\cos x = \dfrac{t^2-1}{t^2+1}$

$\sin x = \dfrac{2t}{t^2+1}$

$4\sin\theta + 3\cos\theta = 2$

$4\dfrac{(2t)}{t^2+1} + 3\dfrac{(t^2-1)}{t^2+1} = 2$

$8t + 3t^2 - 3 = 2t^2 + 2$

$t^2 + 8t - 5 = 0$

$t = \dfrac{-8 \pm \sqrt{84}}{2}$

$\tan\frac{x}{2} = 0.5826, \ -8.5826$

$\frac{x}{2} = .5275, \ 3.6691,$
$1.6870, \ 4.8284$

$x = 1.055^c, \ 3.374^c$ 4

(d) $y = \cos x + \sec x$

$V = \pi \int_0^{\pi/3} y^2 \, dx$

$= \pi \int_0^{\pi/3} \cos^2 x + \sec^2 x + 2 \, dx$

$= \pi \int_0^{\pi/3} \frac{1}{2}\cos 2x + \sec^2 x + \frac{5}{2} \, dx$

$= \pi \left[\frac{\sin 2x}{4} + \tan x + \frac{5x}{2} \right]_0^{\pi/3}$

$= \pi \left[\left(\frac{\sqrt{3}}{8} - \sqrt{3} + \frac{5\pi}{6} \right) - (0) \right]$

$= \pi \left[\frac{5\pi}{6} - \frac{7\sqrt{3}}{8} \right] u^3$ 4

(e) (i) $y = x\ln x - x$

$\dfrac{dy}{dx} = x \cdot \frac{1}{x} + 1 \ln x - 1$

$= \ln x$ 2

(ii) $I = \int_1^e \ln x \, dx$

$= \left[x\ln x - x \right]_1^e$

$= \left[(e - e) - (-1) \right]$

$= 1 \ u^2.$ 3