FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 1
HSC ASSESSMENT TASK 2 2007
ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	HE1	HE2	HE3	HE4	HE5	HE6	HE7
Q1	/15	✓						✓
Q2	/23	✓						✓
Q3	/10	✓						✓
Q4 a,b	/7	✓						✓
Q4c	/5	✓	✓					✓
Total Q4	/12							
Q5	/23	✓						✓
Q6a	/6	✓		✓				✓
Q6b,c	/14	✓						✓
Total Q6	/20							
TOTAL								
	/ 103	/103	/5	/6				/103

GIRRAWEEN HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION

Task 2

2007

MATHEMATICS

Extension 1

Time allowed – Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a new page.
- A table of standard integrals is provided.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (15 marks)

(a) Find $\lim_{x \to 0} \dfrac{\sin \frac{x}{3}}{x}$ (Show working) 2

(b) Solve : $\dfrac{3x-2}{5x-4}+1<0$ 3

(c) Find the acute angle between the lines $3x-4y=3$ and $x-2y=11$. 4

(d) Sketch: (i) $y=3\cos 2x$, $-\pi \le x \le \pi$ 3

(ii) $y=\sin\dfrac{x}{2}$, $-2\pi \le x \le 2\pi$ 3

Question 2 (23 marks)

(a) Differentiate: 17

(i) $y=e^{\ln x}$

(ii) $y=\log\sin(x^3)$

(iii) $y=x^2\tan 2x$

(iv) $y=\dfrac{\cos x}{x}$

(v) $y=e^{\sin 5x}$

(vi) $y=\log_e\left(\dfrac{x-5}{x+5}\right)$

(b) Find the equation of the normal to the curve $y=\log_e x$ at the point where $x=1$. 3

(c) The line $y=mx$ is tangent to the curve $y=e^{4x}$. Find the coordinates of the point of contact. 3

Question 5 (23 marks)

(a) Find: 14

(i) $\int \frac{x+4}{x^2+8x-5}\, dx$ (ii) $\int (\frac{1}{2}\sin 2x - \cos x)\, dx$

(iii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 x\, dx$ (iv) $\int_0^{\frac{\pi}{6}} \sin x \cos x\, dx$

(b) The region under the curve $y = \tan x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ is rotated about the x axis. Find the volume of the solid of revolution. 4

(c) (i) Differentiate $y = \cos^3 x$ 2

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$. 3

Question 6 (20 marks)

(a) It is known that 4% of electric bulbs produced by a certain factory are defective. What is the probability that from a random sample of 20 bulbs

(i) all are good. 2

(ii) exactly two bulbs are defective. 2

(iii) at least one bulb is defective. 2

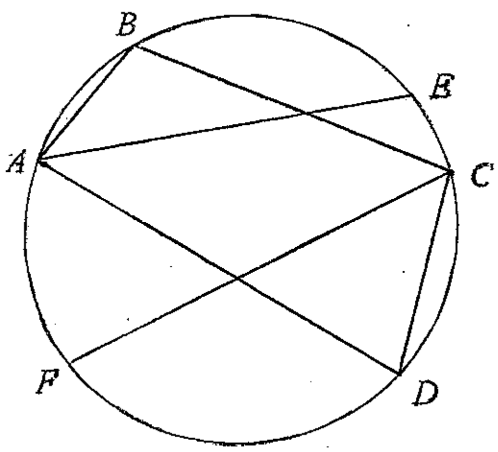
(b) solve:

(i) $\sin 2x = -\frac{\sqrt{3}}{2}, \quad 0 \le x \le 360°$ 4

(ii) $3\cos x + 4\sin x = 4$ using t method. 5

(c) In the diagram given, $ABCD$ is a cyclic quadrilateral where AE bisects $\angle BAD$ and CF bisects $\angle BCD$. Prove that EF is a diameter. 5

(Hint: Draw $\overline{AF}$ and $\overline{EF}$)



Year 12 Half Yearly 2007 – Extension 1 – Solutions

Question 1 (15 marks)

(a) $\lim_{x \to 0} \dfrac{\sin \frac{x}{3}}{x}$

$= \lim_{x \to 0} \dfrac{\sin \frac{x}{3}}{3 \times \frac{x}{3}}$

$= \dfrac{1}{3} \lim_{x \to 0} \dfrac{\sin \frac{x}{3}}{\frac{x}{3}}$ (2)

$= \dfrac{1}{3}$

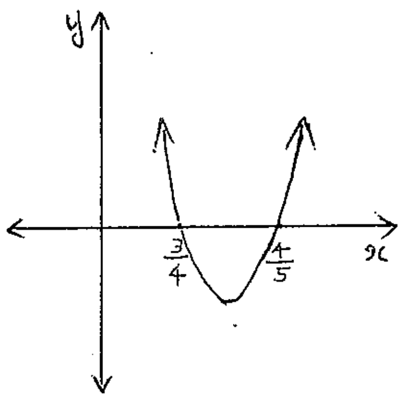
(b) $\dfrac{3x-2}{5x-4} + 1 < 0$

$(5x-4)^2 \times \dfrac{3x-2}{5x-4} + (5x-4)^2 < 0$

$(5x-4)(3x-2) + (5x-4)^2 < 0$

$(5x-4)(3x-2+5x-4) < 0$

$(5x-4)(8x-6) < 0$



(3)

From the graph the solution is

$\dfrac{3}{4} < x < \dfrac{4}{5}$

(c) $\tan \theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$3x - 4y = 3$ $\qquad x - 2y = 11$

$3x - 3 = 4y$ $\qquad x - 11 = 2y$

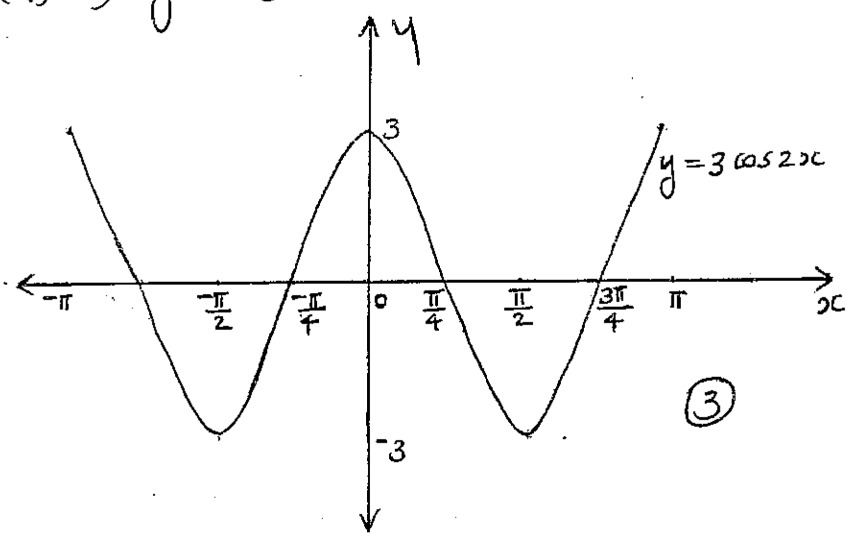
$y = \frac{3}{4}x - \frac{3}{4}$ $\qquad y = \frac{1}{2}x - \frac{11}{2}$

$m_1 = \frac{3}{4}$ $\qquad m_2 = \frac{1}{2}$

$\tan \theta = \left| \dfrac{\frac{3}{4} - \frac{1}{2}}{1 + \frac{3}{4} \times \frac{1}{2}} \right| = \dfrac{2}{11}$ (4)

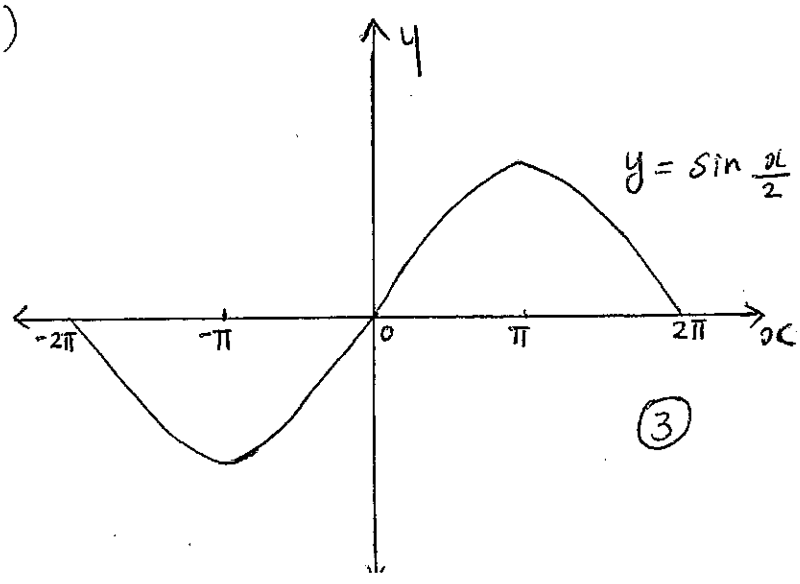
$\theta = 10^\circ 18'$

(d) (i) $y = 3\cos 2x$, $-\pi \le x \le \pi$



(3)

(ii)



(3)

Question 2 (23 marks)

(a) (i) $y = e^{\ln x}$

$= x$

$\frac{dy}{dx} = 1$ (2)

(ii) $y = \log \sin(x^3)$

$y' = \frac{1}{\sin(x^3)} \times \cos(x^3) \times 3x^2$

$= \frac{3x^2 \cos(x^3)}{\sin(x^3)}$ (3)

$= 3x^2 \cot(x^3)$

(iii) $y = x^2 \tan 2x$

$y' = x^2 \times (\sec^2 2x) \times 2 + \tan 2x \times 2x$ (3)

$= 2x^2 \sec^2 2x + 2x \tan 2x$

(iv) $y = \frac{\cos x}{x}$

$y' = \frac{x \times -\sin x - \cos x \times 1}{x^2}$ (3)

$= \frac{-x \sin x - \cos x}{x^2}$

(v) $y = e^{\sin 5x}$

$y' = e^{\sin 5x} \times \cos 5x \times 5$ (3)

$= 5 \cos 5x \, e^{\sin 5x}$

(vi) $y = \log_e\left(\frac{x-5}{x+5}\right)$

$= \log_e(x-5) - \log_e(x+5)$

$y' = \frac{1}{x-5} - \frac{1}{x+5}$ (3)

(b) $y = \log_e x$

$\frac{dy}{dx} = \frac{1}{x}$

when $x = 1$, $\frac{dy}{dx} = 1$

gradient of the normal $= -1$

when $x = 1$, $y = \log_e 1 = 0$

gradient -1 and the point is $(1, 0)$

Equation of the normal is

$y - 0 = -1(x - 1)$ (3)

$y = -x + 1$

(c) $y = mx$ —(1) $\quad y = e^{4x}$ —(2)

At the point of contact of (1) and (2) we have $mx = e^{4x}$

$m = \frac{e^{4x}}{x}$

Gradient of the curve $y = e^{4x}$ at the point of contact is equal to the Gradient of $y = mx$

$4e^{4x} = m$

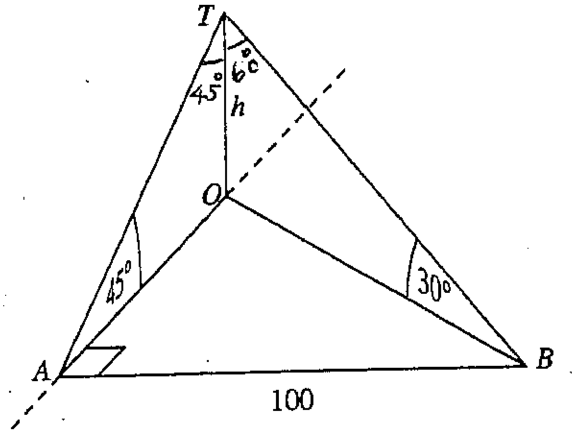
$4e^{4x} = \frac{e^{4x}}{x} \quad \therefore x = \frac{1}{4}$

when $x = \frac{1}{4}$, $y = e^{4 \times \frac{1}{4}}$ (3)

$= e$

The point of contact is $\left(\frac{1}{4}, e\right)$

Question 3 (10 marks)



(i) $\tan 60 = \frac{OB}{h}$ (2)

$\sqrt{3} = \frac{OB}{h}$ $\therefore OB = \sqrt{3}h$

(ii) $\tan 45° = \frac{OA}{h}$

$1 = \frac{OA}{h}$ $\therefore OA = h$

By Pythagoras' theorem in $\triangle OAB$

$OB^2 = OA^2 + 100^2$

$(\sqrt{3}h)^2 = h^2 + 100^2$

$3h^2 = h^2 + 100^2$ (2)

$2h^2 = 100^2$

$h = \frac{100}{\sqrt{2}} = 50\sqrt{2}$

(iii) $\tan \angle AOB = \frac{100}{50\sqrt{2}} = \sqrt{2}$

$\angle AOB = 55°$

The bearing of B from the base of the tower is (2)

$180 - 55° = 125°$

(b) (i) PROBLEM - 7 letters

$\underset{1}{\overset{P}{O}}$ $\underset{6}{O}$ $\underset{5}{O}$ $\underset{4}{O}$

Total number of arrangements

$= 1 \times 6 \times 5 \times 4$ (2)

$= 120$

(ii) O O O O

B can be arranged in 4 ways and the remainder in $5 \times 4 \times 3$ ways

Total number of arrangements

$= 4 \times 5 \times 4 \times 3$ (2)

$= 240$

Question 4 (12 marks)

(a) $(5 - 2x^3)(1 + 2x)^5$

$= (5 - 2x^3)(1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + {}^5C_3(2x)^3 + {}^5C_4(2x)^4 + {}^5C_5(2x)^5)$

$= (5 - 2x^3)(1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5)$

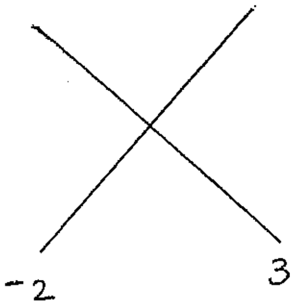
Terms containing x^4 are

$80 \times 5x^4 - 2 \times 10x^4$ (3)

Coefficient of $x^4 = 400 - 20$

$= 380$

(b) $P(2,1)$ $\qquad$ $Q(3,5)$



4

$$x = \frac{-2\times 3 + 3\times 2}{-2+3} = 0 \qquad \therefore \text{ the point is } (0,-7)$$

$$y = \frac{-2\times 5 + 3\times 1}{-2+3} = -7$$

(c) <u>step1</u> Testing $n=1$

when $n=1$, $LHS = 1(1+1)^2$ $\qquad RHS = \frac{1\times 2\times 3\times 8}{12} = 4$

$= 1\times 4 = 4$

$LHS = RHS$ $\therefore$ the result is true for $n=1$

<u>step 2</u> Assume the result is true for $n=k$

ie $1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \cdots + k(k+1)^2 = \frac{k(k+1)(k+2)(3k+5)}{12}$ — ①

To prove that the result is true for $n=k+1$.

ie to prove that

$$1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \cdots + k(k+1)^2 + (k+1)(k+2)^2$$
$$= \frac{(k+1)(k+2)(k+3)(3(k+1)+5)}{12}$$
$$= \frac{(k+1)(k+2)(k+3)(3k+8)}{12}$$

5

Let $S_k = 1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \cdots + k(k+1)^2$

$$S_{k+1} = S_k + (k+1)(k+2)^2$$
$$= \frac{k(k+1)(k+2)(3k+5)}{12} + (k+1)(k+2)^2 \text{ by assumption ①}$$

$$= \frac{k(k+1)(k+2)(3k+5) + 12(k+1)(k+2)^2}{12}$$

$$= \frac{(k+1)(k+2)\left[k(3k+5) + 12(k+2)\right]}{12}$$

$$= \frac{(k+1)(k+2)(3k^2+17k+24)}{12} = \frac{(k+1)(k+2)(k+3)(3k+8)}{12}$$

Thus the result is true for $n = k+1$.

Hence by Mathematical Induction, the result is true for all positive integral values of n.

Question 5 (23 marks)

(a) (i) $\int \frac{x+4}{x^2+8x-5}\,dx$

$$= \int \frac{2(x+4)}{2(x^2+8x-5)}\,dx$$

$$= \frac{1}{2}\int \frac{2x+8}{x^2+8x-5}\,dx \quad (3)$$

$$= \frac{1}{2}\log_e(x^2+8x-5) + C$$

(ii) $\int \left(\frac{1}{2}\sin 2x - \cos x\right) dx$

$$= \frac{1}{2}x - \frac{\cos 2x}{2} - \sin x + C \quad (3)$$

$$= \frac{-1}{4}\cos 2x - \sin x + C$$

(iii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1-\cos 2x}{2}\right) dx$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right) - \left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right\} \quad (4)$$

$$= \frac{\pi - 3\sqrt{3} + 6}{24}$$

(iv) $\int_0^{\frac{\pi}{6}} \sin x \cos x\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin 2x}{2}\,dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin 2x\,dx = \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{-1}{4}\left[\cos 2x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{-1}{4}\left(\cos\frac{\pi}{3} - \cos 0\right)$$

$$= \frac{-1}{4}\left(\frac{1}{2} - 1\right) = \frac{-1}{4} \times \frac{-1}{2}$$

$$= \frac{1}{8} \quad (4)$$

(b) $V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x \, dx$

$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2 x - 1)\, dx$

$= \pi \left[\tan x - x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ (4)

$= \pi \left\{ \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) \right\}$

$= \pi \left(\sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4} \right)$

$= \dfrac{\pi(12\sqrt{3} - 12 - \pi)}{12}$

(c) (i) $y = \cos^3 x$

$\frac{d}{dx}(\cos^3 x) = 3\cos^2 x \times -\sin x$

$= -3\cos^2 x \sin x$ (2)

$\cos^2 x \sin x = \frac{-1}{3} \frac{d}{dx}(\cos^3 x)$

(ii) $\int_0^{\frac{\pi}{2}} \cos^2 x \sin x \, dx$

$= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{d}{dx}(\cos^3 x)\, dx$

$= \frac{-1}{3} \left[\cos^3 x \right]_0^{\frac{\pi}{2}}$ (3)

$= \frac{-1}{3} \left(\cos^3 \frac{\pi}{2} - \cos^3 0 \right)$

$= \frac{-1}{3}(0 - 1)$

$= \frac{1}{3}$

Question 6 (20 marks)

(a) $p = 0.96 \quad q = 0.04$

(i) $P(x = 20) = {}^{20}C_{20}\, p^{20}$ (2)

$= (0.96)^{20} = 0.442$

(ii) Probability that exactly two bulbs are defective

$= P(x = 18)$

$= {}^{20}C_{18}\, p^{18} q^2$ (2)

$= {}^{20}C_{18} (0.96)^{18} (0.04)^2$

$= 0.146$

(iii) Probability that atleast one bulb is defective

$= 1 -$ Probability that all bulbs are not defective (2)

$= 1 - (0.96)^{20} = 0.558$

(b) (i) $\sin 2x = -\frac{\sqrt{3}}{2}, \ 0 \le x \le 360^\circ$

Let $u = 2x$

$\sin u = \frac{-\sqrt{3}}{2}, \ 0 \le u \le 720^\circ$

u is in the 3rd or 4th quadrant.

acute $\angle u = 60^\circ$

$180 + u = 240 \ ; \ 360 - u = 300^\circ$

$u = 240^\circ, 300^\circ, 600^\circ, 660^\circ$

$x = 120^\circ, 150^\circ, 300^\circ, 330^\circ$ (4)

(ii) $3\cos x + 4\sin x = 4$ —— ①

Substitute $\sin x = \frac{2t}{1+t^2}$ and

$\cos x = \frac{1-t^2}{1+t^2}$ in ①

$$3 \times \frac{(1-t^2)}{1+t^2} + \frac{4 \times 2t}{1+t^2} = 4$$

$$\frac{3-3t^2}{1+t^2} + \frac{8t}{1+t^2} = 4$$

$$3 - 3t^2 + 8t = 4 + 4t^2$$

$$7t^2 - 8t + 1 = 0$$

$$(t-1)(7t-1) = 0$$

$$t = 1 \quad \text{or} \quad t = \frac{1}{7}$$

$\tan\frac{x}{2} = 1$ or $\tan\frac{x}{2} = \frac{1}{7}$

$\tan\frac{x}{2} = 1$ (5)

Let $u = \frac{x}{2}$

$\tan u = 1$, $0 \leq u \leq 180^\circ$

$u = 45^\circ$

$\therefore x = 90^\circ$

$\tan\frac{x}{2} = \frac{1}{7}$

Let $u = \frac{x}{2}$,

$\tan u = \frac{1}{7}$, $0 \leq u \leq 180^\circ$

$u = 8^\circ 8'$ $\therefore x = 16^\circ 16'$

Check for $x = 180^\circ$

$3 \times -1 + 4 \times 0 = 4$

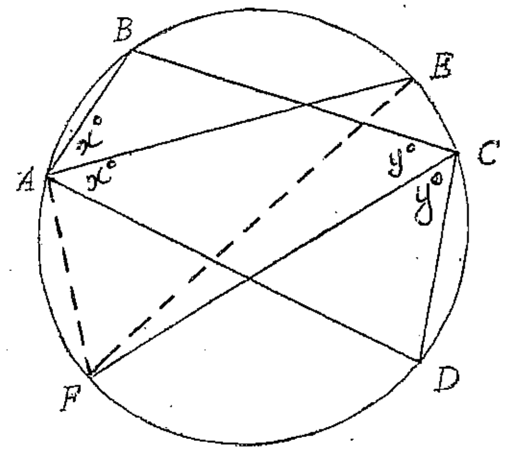
$-3 = 4$ not true

$\therefore x = 180$ is not a solution.

The solutions are

$x = 16^\circ 16', 90^\circ$

(c)



ABCD is a cyclic quadrilateral.

$x + x + y + y = 180$ (opposite angles of a cyclic quadrilateral are supplementary.)

$$2x + 2y = 180$$

$$x + y = 90$$

$\angle FAD = y$ (angles standing on the minor arc FD)

$\angle FAE = \angle FAD + \angle EAD$ (5)

$= y + x$

$= 90^\circ$

$\therefore$ EF is a diameter.