PERCENTAGE

GIRRAWEEN HIGH SCHOOL
Mathematics Extension 1

HSC ASSESSMENT
HALF YEARLY
ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	E2	E3	E4	E5	E6	E7
1	/20						✓
Total	/20						
2(a) - (c)	/17						✓
(d)	/13					✓	✓
Total	/30						
3(a)	/3					✓	✓
(b) (c)	/12	✓					✓
(d)	/6		✓				✓
Total	/21						
4(a)	/5	✓					✓
(b)(c)(d)i	/10						✓
(d) ii	/3					✓	✓
(e)	/6		✓				✓
	/24						
5(a) -(d)i	/14						✓
(d) (ii)	/3					✓	✓
(e)	/8						✓
	/25						
TOTAL	/120	/13	/12	/0	/0	/24	/120

GIRRAWEEN HIGH SCHOOL

EXAMINATION

2008

MATHEMATICS

EXTENSION 1

Time allowed - Two hours
(Plus 5 minutes' reading time)

HALF YEARLY

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.

- All necessary working should be shown in every question.
Marks may be deducted for careless or badly arranged work.

- Standard integrals are on the sheets supplied

- Board-approved calculators may be used.

- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.

- You may ask for extra pieces of paper if you need them.

Total marks – 118

Attempt Questions 1 – 5

All questions are NOT of equal value

Answer each question on a separate piece of paper clearly marked Question 1, Question 2 etc.

Each piece of paper must show your name.

Question 1 **(20 Marks)** Use a separate piece of paper **Marks**

(a) Solve for x $2^x = 9$ 2

(b) Solve for x, $\dfrac{2x - 5}{3x + 1} \geq 1$ **2**

(c) Find the acute angle between the lines $2x + 3y - 7 = 0$ and $3x - y + 6 = 0$ **3**

(d) Find $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 3x}$ **2**

(e) Find the coordinates of the point that divides the interval A(2, - 3) and B(-4 , 7) internally in the ratio 3:2. **3**

(f) Sketch the graph of $y = \sin x$ in the domain $0 \leq x \leq 2\pi$. On the same set of axes and in the same domain sketch $y = \operatorname{cosec} x$ **4**

(g) Given that $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

(i) Find an expression for $\cos 2\theta$ in terms of $\cos^2 \theta$ **1**

(ii) Hence or otherwise find the exact value of $cos\ 72^0$ given that $cos\ 36^0 = \dfrac{\sqrt{5} + 1}{4}$ **3**

Question 2 (30 marks) Use a separate piece of paper

(a) Differentiate the following

(i) $y = xe^x$ (ii) $y = \dfrac{e^x}{\tan x}$ **4**

(iii) $y = ln\,(\sin x)$ (iv) $y = ln\left[\dfrac{(x^2 - 7x)}{\sqrt{x}}\right]$ **4**

(b) (i) Differentiate $y = \sin^3 x$ **2**

(ii) Hence or otherwise find $\int 3\cos x \sin^2 x dx$ **2**

(c) Find the equation of the tangent and the normal to the curve $y = tan\, x,$ at the point where $x = \dfrac{\pi}{4}$ **5**

(d) Find

(i) $\displaystyle\int \frac{x^2 - 2x}{x^3 - 3x^2 + 7} dx$ **2**

(ii) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x dx$ (iii) $\displaystyle\int_1^5 \sqrt{2x-1} dx$ **6**

(iv) $\displaystyle\int_2^3 \frac{x^2}{x^3 - 2} dx$ (v) $\displaystyle\int_0^{\pi} \cos x dx$ **5**

Question 3 (19 Marks) Use a separate piece of paper **Marks**

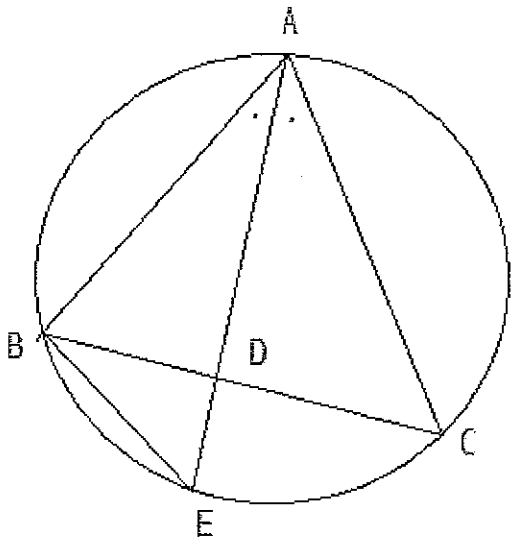
(a) Find the volume when $y = \tan x$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{4}$ **3**

(b) Prove by Mathematical Induction that

$$1 + 8 + 27 + 64 + \dots\dots\dots + n^3 = \frac{1}{4}\{n^2(n+1)^2\}$$

is true for all positive values of the integer n **4**

(c) In the diagram below the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E.



(i) Copy the diagram into your solutions and prove that the $\triangle ABE$ and $\triangle ADC$ are similar. **2**

(ii) Show that $AB \times AC = AD \times AE$ **2**

(iii) Show that $\triangle ABD$ is similar to $\triangle CDE$ **2**

(iv) Prove that $AD^2 = AB \times AC - BD \times DC$ **2**

(d) In a certain country the probability of a child having blue eyes is $\frac{1}{3}$.

(i) If a family has four children what is the probability that at leat 2 children will have blue eyes. **3**

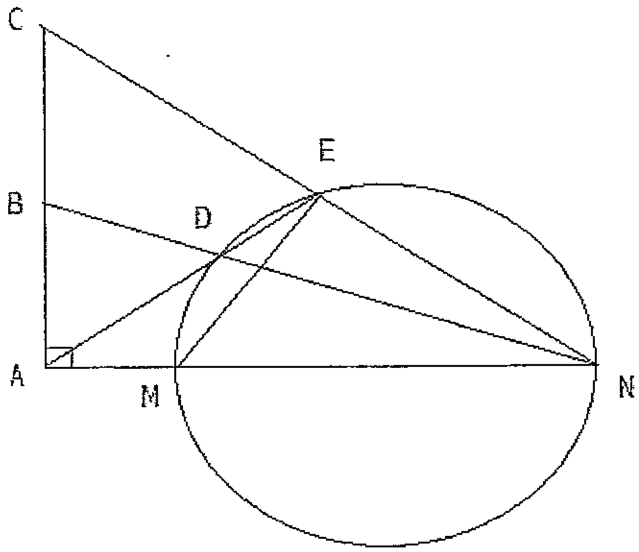
(ii) If 4 such families are surveyed, using your result from part (i) or otherwise, what is the probability that exactly 1 family will have at least 2 children with blue eyes. **3**

Question 4 (**24 Marks**) Use a separate piece of paper **Marks**

(a) In the figure below M,N,E and D are points on a circle with MN as diameter. NE is produced to C and NM is produced to A, such that CA and AN are perpendicular. AE meets the circle at D and ND is produced to meet CA at B.

(i) Show that $ACEN$ are concyclic. **2**

(ii) Show that $BCED$ are concyclic. **3**



(b) Find the term independent of x in the expansion of the following expression

Leave your answer in unexpanded form. **4**

$$(2x^2 - \frac{3}{x})^9$$

(c) Use the t method, where $t = \tan\frac{\theta}{2}$, to solve the following. Give your answer in radians to 2 decimal places $0 \le \theta \le 2\pi$

$$5\sin\theta - 2\cos\theta = 3$$ **4**

(d) (i) Differentiate the function $y = 2^x$ **2**

(ii) Hence or otherwise evaluate $\int_1^3 2^x \, dx$ **3**

(e) (i) How many different 9 letter "words" can be formed by using all 9 letters of the word GIRRAWEEN **2**

(ii) How many different 5 letter selections and how many different arrangements of the 5 letters can be made. **4**

Question 5 (**25** Marks) Use a separate piece of paper **Marks**

(a) Using the formula $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ differentiate $f(x)=\frac{1}{x}$,

from first principles clearly showing all necessary steps. **3**

(b) (i) Show that the first derivative of $f(x)=\sec x$ is $\frac{dy}{dx}=\sec x\tan x$ **2**

(ii) Hence or otherwise find the second derivative of $f(x)=\sec x$ **2**

(ii) For what values of $\theta,\ 0\le\theta\le 2\pi$ is $\sec x$ concave up **2**

(c) Solve for $\theta,\ 0\le\theta\le 2\pi$ the equation $3\tan^5\theta-10\tan^3\theta+3\tan\theta=0$ **4**

(d) (i) Show that $\frac{1}{x-2}-\frac{1}{x+3}=\frac{5}{(x-2)(x+3)}$ **1**

(ii) Using part (i) or otherwise show that the area under $y=\frac{1}{(x-2)(x+3)}$

between x = 3 and x = 5 is $\frac{2}{5}\ln\frac{3}{2}$ u² **3**

(e) (i) Show that $2\sin\theta+2\sqrt{3}\cos\theta$ can be expressed in the form $R[\sin(\theta+\alpha)]$

clearly showing the values of R *and* α **3**

(ii) Sketch your graph of y = $R[\sin(\theta+\alpha)]$ in the range $0\le\theta\le 2\pi$ **3**

(iii) Solve graphically the equation $R[\sin(\theta+\alpha)]=3$ **2**

YEAR 12 HALF YEARLY 2008

SOLUTIONS

Q1 (a) $2^x = 9$

$x \ln 2 = \ln 9$

$x = \dfrac{\ln 9}{\ln 2}$

$x = 3.17$ (2)

(b) $\dfrac{2x-5}{3x+1} \geq 1$

$(2x-5)(3x+1) \geq (3x+1)^2$

$0 \geq (3x+1)^2 - (3x+1)(2x-5)$

$0 \geq (3x+1)\{3x+1-2x+5\}$

$0 \geq (3x+1)\{x+6\}$

$-6 \leq x < -\frac{1}{3}$ (2)

(c) $2x+3y-7=0$

$3y = -2x+7$

$y = -\frac{2}{3}x + \frac{7}{3}$ $m_1 = -\frac{2}{3}$

$3x - y + 6 = 0$

$y = 3x+6$ $m_2 = 3$.

$\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\dfrac{3+\frac{2}{3}}{1+3(-\frac{2}{3})}\right|$

$\tan\theta = \frac{11}{3}$

$\theta = 74{\cdot}74^\circ$ (3)

(d) $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 3x}$

$= \lim_{x\to 0} \dfrac{\sin 5x}{5x} \cdot \dfrac{3x}{\sin 3x} \cdot \dfrac{5x}{3x}$

$= 1 \cdot 1 \cdot \frac{5}{3} = \frac{5}{3}$ (2)

(e) $(2,-3)$ $(-4,7)$

$3:2$

$\left(\dfrac{2\times 2 + 3\times -4}{3+2}, \dfrac{2\times -3 + 3\times 7}{2+3}\right)$

$= \left(-\frac{8}{5}, 3\right)$ (3)

(f) See grid paper

(g)(i) $\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$

$= 2\cos^2\theta - 1$ (1)

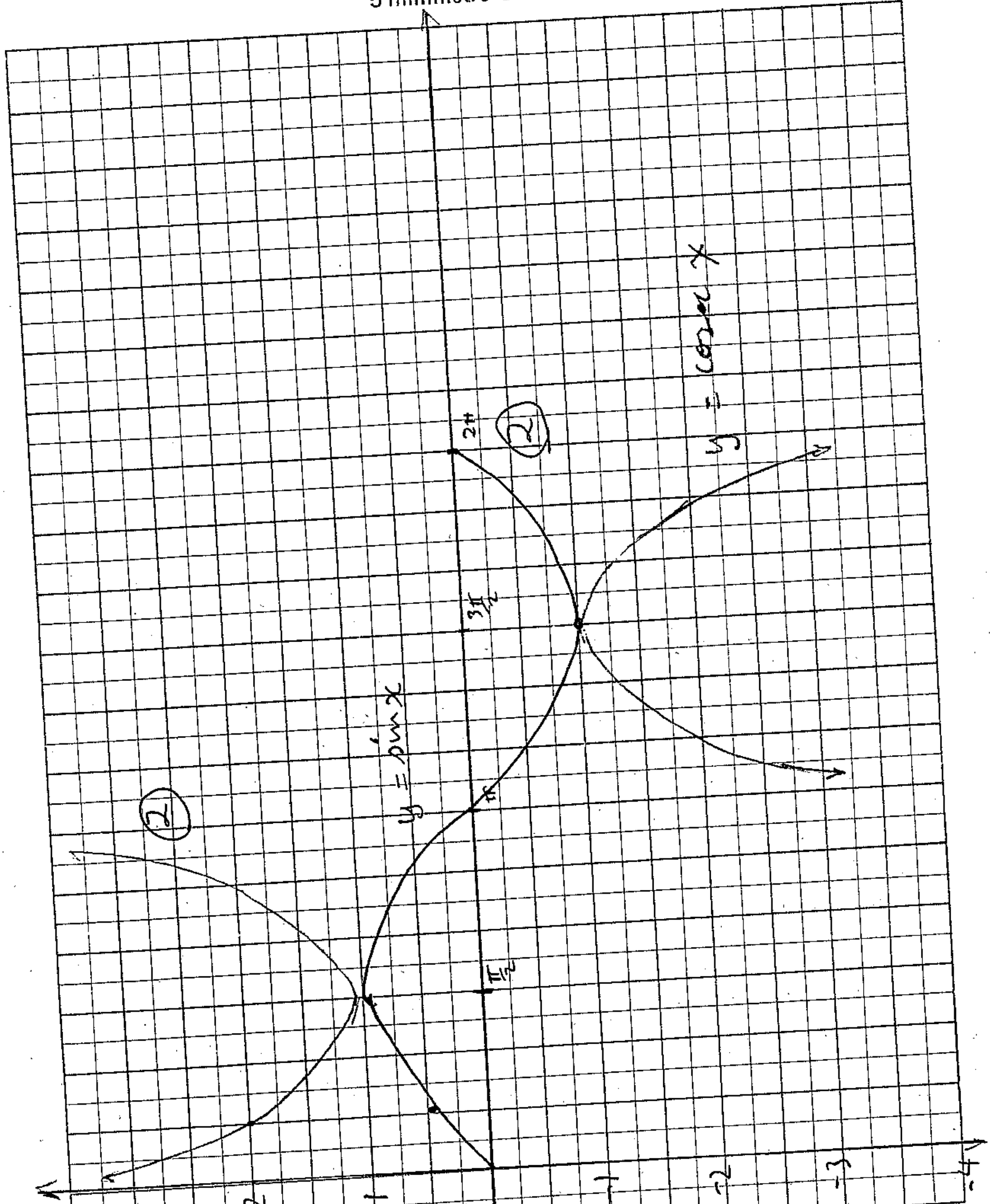
(ii) $\cos 72^\circ = 2\cos^2 36^\circ - 1$

$= 2\left(\dfrac{\sqrt{5}+1}{4}\right)^2 - 1$

$= 2\left(\dfrac{6+2\sqrt{5}}{16}\right) - 1$

$= \dfrac{3+\sqrt{5}}{4} - \dfrac{4}{4} = \dfrac{\sqrt{5}-1}{4}$ (3)

5 millimetre Grid



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2008 EXT 1 ½ YR 2008

Q2.(a) (i) $y = xe^x$

$\frac{dy}{dx} = e^x + xe^x$ ②

(ii) $y = \frac{e^x}{\tan x}$

$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

$= \frac{\tan x (e^x) - e^x \sec^2 x}{\tan^2 x}$ ②

(iii) $y = \ln (\sin x)$

$\frac{dy}{dx} = \frac{\cos x}{\sin x}$

$= \cot x$ ②

(iv) $y = \ln \left[\frac{x^2 - 7x}{\sqrt{x}}\right]$

$y = \ln(x^2 - 7x) - \frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{2x - 7}{x^2 - 7x} - \frac{1}{2x}$ ②

(b) (i) $y = \sin^3 x$

$\frac{dy}{dx} = 3\sin^2 x \cos x$ ②

(ii) $\int 3\sin^2 x \cos x \, dx$

$= \sin^3 x$ ②

(c) $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$ ①

$\frac{dy}{dx}_{(x = \pi/4)} = 2$ ①

EQN TANGENT

$y - 1 = 2(x - \pi/4)$

$y = 2x - \frac{\pi}{2} + 1$ ①

EQN NORMAL

$y - 1 = -\frac{1}{2}(x - \pi/4)$

$y = -\frac{x}{2} + \frac{\pi}{8} + 1$ ②

(d) (i) $\int \frac{x^2 - 2x}{x^3 - 3x^2 + 7} \, dx$

$= \frac{1}{3}\int \frac{3x^2 - 6x}{x^3 - 3x^2 + 7} \, dx \quad \left(\int \frac{f'(x)}{f(x)}\right)$

$= \frac{1}{3}\ln |x^3 - 3x^2 + 7| + C$ ②

(ii) $\int_0^{\pi/8} \sec^2 2x \, dx$

$= \left[\frac{1}{2}\tan 2x\right]_0^{\pi/8}$

$= \frac{1}{2}$ ③

(iii) $\int_1^5 (2x - 1)^{1/2} \, dx$

$= \left[\frac{1}{2} \cdot \frac{2}{3}(2x - 1)^{3/2}\right]_1^5$

$= \frac{1}{3}[27 - 1] = 26/3$ ③

Q2 cont $\frac{1}{2}$ YR EXT 1 '08

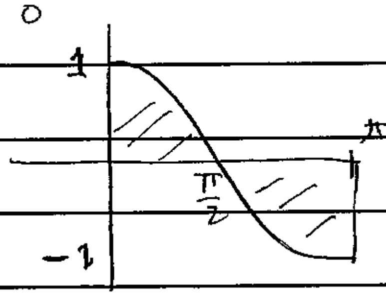
(iv) $\int_2^3 \frac{x^2}{x^3-2}\,dx$

$= \frac{1}{3}\int_2^3 \frac{3x^2}{x^3-2}\,dx$

$= \frac{1}{3}\left[\ln\left|x^3-2\right|\right]_2^3$

$= \frac{1}{3}\ln\frac{25}{6}$ (3)

(v) $\int_0^{\pi} \cos x\,dx$



Areas are equal $\therefore A = 0$

OR $= \left[\sin x\right]_0^{\pi} = 0 - 0$

$= 0.$ (2)

Q3 (a) $V = \pi\int_0^{\pi/4} \tan^2 x\,dx$

$V = \pi\int_0^{\pi/4} \sec^2 x - 1\,dx$

$= \pi\left[\tan x - x\right]_0^{\pi/4}$

$= \pi\left[(1-\pi/4) - 0\right]$

$= \pi(1-\pi/4)\ u^3$ (3)

(b) $1+8+27+\ldots+n^3$

$= \frac{1}{4}\left\{n^2(n+1)^2\right\}$

Step 1 Prove true for $n=1$.

LHS $= 1$ RHS $= \frac{1}{4}\left\{1^2(2^2)\right\}$

$= 1$

LHS = RHS $\therefore$ TRUE FOR $n=1$.

Step 2. Assume true for $n=k$

Prove true for $n=k+1$

VIZ $1+8+27+\ldots+k^3 = \frac{1}{4}\left\{k^2(k+1)^2\right\}$

R.H.S. $= \frac{1}{4}\left\{(k+1)^2(k+2)^2\right\}$

L.H.S. $= 1+8+27+\ldots+k^3+(k+1)^3$

$= \frac{1}{4}\left\{k^2(k+1)^2\right\} + (k+1)^3$

$= \frac{1}{4}\left\{(k+1)^2(k^2+4k+4)\right\}$

$= \frac{1}{4}\left\{(k+1)^2(k+2)^2\right\}$ = RHS.

$\therefore$ True for $n = k+1$

Step 3. Proved true for $n=1$ Step 2 implies if true for $n=1$ true for $n=2,3\ldots$

$\therefore$ By the principle of Math Ind true for all n

(c)(i) IN $\triangle ABE$ and $\triangle ADC$

$\angle BAE = \angle DAC$ (DATA)

$\angle BEA = \angle ACD$ ($=\angle ACB$) SAME ARC

$\therefore \triangle ABE \,|||\, \triangle ADC$ AAA (2)

(ii) $\therefore \frac{AB}{AD} = \frac{AE}{AC}$ RATIO SIDES SIMILAR $\triangle$'s

$\therefore AB \cdot AC = AD \cdot AE$ (2)

(iii) IN $\triangle ABD$ and $\triangle CED$

$\angle BDA = \angle EDC$ (VERTICALLY OPP.)

$\angle ABC = \angle AEC$ (STAND SAME ARC)

$\therefore \triangle ABD \,|||\, \triangle CED$ (AAA) (2)

Now $\frac{BD}{DE} = \frac{DA}{DC}$

$BD \cdot DC = DA \cdot DE$

Q3 cont using result (ii)

$AD\cdot AE + BD\cdot DC = AB\cdot AC + DA\cdot DE$

$AD\cdot AE - AD\cdot DE = AB\cdot AC - BD\cdot DC$

$AD(AE - DE) = AB\cdot AC - BD\cdot DC$

$AD^2 = AB\cdot AC - BD\cdot DC$ (2)

(d) (i) $\left(\frac{1}{3}+\frac{2}{3}\right)^4 = {}^4C_0 \frac{1}{3}^4 + {}^4C_1 \frac{1}{3}^3 \frac{2}{3} + {}^4C_2 \frac{1}{3}^2 \frac{2}{3}^2 \ldots$

4 BLUE 3 BLUE 2 BLUE

P(at least two) = $^{11}/_{27}$ (3)

(ii) $\left(\frac{16}{27}+\frac{11}{27}\right)^4 = \ldots {}^4C_1 \left(\frac{16}{27}\right)^3\left(\frac{11}{27}\right)$

$= 0\cdot 339$ (3).

Question 4.

(a) (i) IN QUAD ACEM.

$\angle CAM = 90^\circ$ (DATA)

$\angle MEN = 90^\circ$ (STAND ON DIAMETRE)

$\therefore$ ACEM IS A CYCLIC QUAD

(EXT $\angle$ = INT OPP $\angle$)

(ii) IN $\triangle$ CAN AND $\triangle$ MEN

$\angle CAN = \angle MEN$ (90°)

$\angle CNA$ IS COMMON

$\therefore \angle ACN = \angle EMN$ (3^{RD} $\angle$ OF $\triangle$)

BUT $\angle NDE = \angle EMN$ (SAME ARC)

$\therefore \angle ACN = \angle NDE$

$\therefore$ QUAD BCED IS CONCYCLIC

(OPP INT $\angle$ = EXT $\angle$). (3)

(b) $\left(2x^2 - \frac{3}{x}\right)^9 = \ldots {}^9C_6\, 8x^6 \cdot \left(\frac{-3}{x}\right)^6$

TERM $= {}^9C_6\; 8 \times 729$

$= 489888.$

(c) $5\sin\theta - 2\cos\theta = 3$

$5\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 3$

$10t - 2 + 2t^2 = 3 + 3t^2$

$0 = t^2 - 10t + 5.$

$t = \frac{10 \pm \sqrt{100-20}}{2}$

$t = 5 \pm 2\sqrt{5}$

$\tan\theta/_2 = 5 \pm 2\sqrt{5}$, $\theta/_2 = 1\cdot 4656, 0\cdot 4856$

$\theta = 0\cdot 97^c\ 2\cdot 93^c$ (4)

(d) (i) $y = 2^x$

$y = e^{(\ln 2)x}$

$\frac{dy}{dx} = \ln 2 \cdot e^{\ln 2 x}$

$= \ln 2\,(2^x)$ (2)

(ii) $\int_1^3 2^x\,dx$

$= \left[\frac{2^x}{\ln 2}\right]_1^3$

$= {}^6/_{\ln 2}.$

(e) (i) # $= \frac{9!}{2!\,2!} = 90720$ (2)

(ii) 2R's + 2E's + 1 other letter — Selections $= 1\times1\times5 = 5$

2R's or 2E's + 3 other letters $= 2\times{}^6C_3 = 40$

no doubles $\quad {}^7C_5 = 21$

$\therefore$ 66 SELECTIONS (2)

Arrangements

① $\frac{5!}{2!\,2!} = 30$, $30\times 5 = 150$

② $\frac{5!}{2!} = 60$, $60\times 40 = 2400$

③ $5! = 120$, $120\times 21 = 2520$

TOTAL ARRANGEMENTS = 5070 (2)

Question 5

(a) $f'(x) = \lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim_{h \to 0} \dfrac{x - (x+h)}{h(x+h)(x)}$

$= \lim_{h \to 0} \dfrac{-h}{h(x+h)(x)}$

$= \lim_{h \to 0} \dfrac{-1}{(x+h)(x)}$

$= \dfrac{-1}{x^2}$ (3)

(b) (i) $f(x) = \sec x$

$= \dfrac{1}{\cos x}$

$f(x) = (\cos x)^{-1}$

$f'(x) = -1(\cos x)^{-2} \cdot -\sin x$

$= \dfrac{\sin x}{\cos^2 x}$

$= \dfrac{\tan x}{\cos x}$

$= \tan x \sec x$ (2)

(ii) $f''(x) = v\dfrac{du}{dx} + u \cdot \dfrac{dv}{dx}$

$= \sec x \sec^2 x + \tan^2 x \sec x$

$= \sec x(\sec^2 x + \tan^2 x)$ (2)

(iii) $f''(x) = \sec x(2\tan^2 x + 1)$

for cc up $f''(x) > 0$

$f''(x) > 0 \quad \sec x > 0$

$0 < \theta < \pi/2 \qquad 3\pi/2 < \theta \leq 2\pi$ (2

(c) $3\tan^5\theta - 10\tan^3\theta + 3\tan\theta = 0$

$\tan\theta(3\tan^4\theta - 10\tan^2\theta + 3) = 0$

$\therefore \tan\theta = 0 \qquad \theta = 0, 180, 36$

$\theta = 0, \pi, 2\pi$

OR $3\tan^4\theta - 10\tan^2\theta + 3 = 0$

$(3\tan^2\theta - 1)(\tan^2\theta - 3) = 0$

$\tan^2\theta = \frac{1}{3} \qquad \tan^2\theta = 3$

$\tan\theta = \pm\frac{1}{\sqrt{3}} \qquad \tan\theta = \pm\sqrt{3}$

$\theta = \pi/6, \pi/3, \frac{2\pi}{3}, \frac{5\pi}{6}, 7\pi/6,$

$4\pi/3, 5\pi/3, 11\pi/6.$

$+ \; 0, \pi, 2\pi$ (4)

(d) (i) $\dfrac{1}{x-2} - \dfrac{1}{x+3} = \dfrac{x+3-(x-2)}{(x-2)(x+3)}$

$= \dfrac{5}{(x-2)(x+3)}$ (1)

(ii) $\displaystyle\int_3^5 \frac{1}{(x-2)(x+3)}\,dx$

$= \frac{1}{5}\displaystyle\int_3^5 \frac{5}{(x-2)(x+3)}\,dx$

$= \frac{1}{5}\displaystyle\int_3^5 \frac{1}{x-2} - \frac{1}{x+3}\,dx$

$= \frac{1}{5}\Big[\ln|x-2| - \ln|x+3|\Big]_3^5$

Question 5 cont.

$= \frac{1}{5}\left\{(\ln 3 - \ln 8) - (\ln 1 - \ln 6)\right\}$

$= \frac{1}{5} \ln \frac{3 \times 6}{8}$

$= \frac{1}{5} \ln \frac{18}{8} \quad = \frac{1}{5} \ln \frac{9}{4}$

$= \frac{1}{5} \ln\left(\frac{3}{2}\right)^2$

(3) $= \frac{2}{5} \ln \frac{3}{2} \; u^2$

(e) (i) $2\sin\theta + 2\sqrt{3}\cos\theta$

$= R\sin(\theta + \alpha)$

$\therefore R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$= 2\sin\theta + 2\sqrt{3}\cos\theta$

$R\cos\alpha = 2 \quad R\sin\alpha = 2\sqrt{3}$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}^c$

and $R^2\cos^2\alpha = 4$

$R^2\sin^2\alpha = 12$

$R^2(\cos^2\alpha + \sin^2\alpha) = 16$

$R^2 = 16. \quad R = 4$

$2\sin\theta + 2\sqrt{3}\cos\theta$

$= 4\sin(\theta + \frac{\pi}{3})$.

(3)

5 millimetre Grid