FINAL MARK

GIRRAWEEN HIGH SCHOOL
Mathematics Extension 1
HSC ASSESSMENT
Task 2 – 2009
Half yearly examination
ANSWERS COVER SHEET

Name:______________________________Teacher:___________.

QUESTION	MARK	E2	E3	E4	E5	E6	E7
1	/15						√
2	/16						√
3	/16						√
4a	/4	√					√
b,c	/13						√
	/17						
5a	/4						√
b	/9		√				√
	/13						
6	/23						√
TOTAL	/100	/4	/9				/100

HSC Outcomes **Mathematics Extension 1**

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics.

HE2 uses inductive reasoning in the construction of proofs.

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion and exponential growth and decay.

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE6 determines integrals by reduction to a standard form through a given substitution.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

2009

MATHEMATICS

EXTENSION 1

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

Total Marks – 100

Attempt all questions 1-6

All questions are NOT of equal value.

Answer each question clearly ON A SEPARATE PAGE!

Question 1 (15 Marks) Use a separate piece of paper. **Marks**

(a) Solve for $x: \dfrac{5x}{2x-1} \leq 3$ **5**

(b) Divide the interval between the points $(2,1)$ and $(5,-3)$ externally in the ratio $4:3$. **2**

(c) Find the acute angle angle between the straight lines $x-2y=4$ and $3x-y-1=0$. **3**

(d) Sketch the graph of $y=2\sin 3x$ for $0 \leq x \leq 2\pi$. **3**

(e) Find $\underset{x\to 0}{limit} \dfrac{\sin 2x}{3x}$. **2**

Question 2 (16 Marks) Use a separate piece of paper.

(a) Differentiate: **9**

(i) $y=x^2e^{3x}$

(ii) $y=\dfrac{e^{2x}}{\cos x}$

(iii) $y=e^{\sin x}$

(iv) $y=\ln\left[\dfrac{\sqrt{x^2+1}}{(3x-2)^5}\right]$

(b) Find the equations of the tangent and normal to the curve $y=3e^{2x}$ at the point where $x=1$. **7**

Question 3 (16 Marks) Use a separate piece of paper. **Marks**

(a) Find (i) $\int x^2 e^{x^3} .dx$ **9**

(ii) $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\sin 2x - 3\cos 2x.dx$

(iii) $\int \frac{6x^2+8}{x^3+4x-1}.dx$

(iv) $\int \cot x.dx$

(b) (i) Show that $(\cos x + \sin x)^2 = 1 + \sin 2x$. **7**

(ii) Hence (or otherwise) find the volume of the solid of revolution formed when the curve $y = \cos x + \sin x$ is rotated about the x axis between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

Question 4 (17 Marks) Use a separate piece of paper.

(a) Prove by mathematical induction: **4**

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(b) How many different arrangements of the letters of the word MELBOURNE are possible: **6**

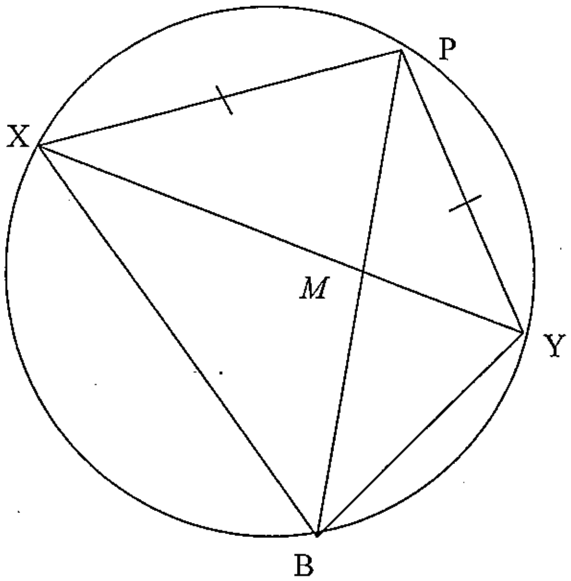
(i) Altogether?

(ii) With M at the front?

(iii) With vowels and consonants alternating?

Marks

(4)(c) 7



In the diagram above, $PX = PY$.

(i) Draw this diagram neatly on your answer page.

(ii) State why $\angle BXY = \angle BPY$.

(iii) Prove that PB bisects $\angle XBY$.

(iv) Prove that $\angle XMB = \angle PYB$.

Question 5 (13 Marks) Use a separate piece of paper.

(a) Find the co-efficient of x^4 in the expansion of 4

$$\left(3x^2 - \frac{2}{x}\right)^{14}.$$

(b) The probability that a person aged 70 will survive the next year is calculated to be 0.87. The probability that a person aged 75 will survive the next year is calculated to be 0.79. If five 70 year olds and four 75 year olds are picked at random, what is the probability that: 9

(i) All of the 70 year olds survive?

(ii) At least 3 of the 75 year olds survive?

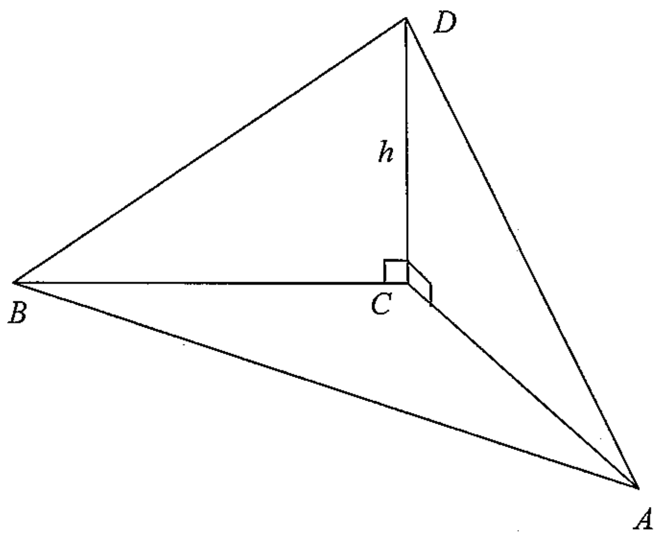
(iii) There is at most only one death in the entire group?

Question 6 (23 Marks) Use a separate piece of paper. **Marks**

(a) A mountain is spotted due south of a ship which is out to sea at point A. The angle of elevation of the top of the mountain (at D)from the ship is 2^o. After the ship has sailed for $20km$ to B, the top of the mountain is now spotted due west of the ship at an angle of elevation of 3^o. (see diagram). If the height of the mountain is h metres above sea level **6**

(i) Show that $\dfrac{h^2}{\tan^2 2^o}+\dfrac{h^2}{\tan^2 3^o}=20\,000^2$.

(ii) Find the height of the mountain above sea level to the nearest 10 metres.



(b) (i) Write the formula for $\sin(A+B)$ **5**

(ii) Hence (or otherwise) find the exact value of $\sin 75^o$.

(c) Solve for $\theta, 0\le\theta\le 360^o$ using the t formulae: **6**

$2\cos\theta+3\sin\theta=1$.

(d) (i) Express $2\sqrt{3}\cos x+2\sin x$ in the form $R\cos(x-\alpha)$ where α is in radians. **6**

(ii) Hence or otherwise sketch the graph of $y=2\sqrt{3}\cos x+2\sin x$ for $0\le x\le 2\pi$

Q.(1)(c) m of $x-2y=4$:

$x-4=2y$

$\frac{1}{2}x-2=y \Rightarrow m_2=\frac{1}{2}$.

m of $3x-y-1=0$

$3x-1=y$

$\Rightarrow m_1=3$.

By $\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$

$\tan\theta = \frac{3-\frac{1}{2}}{1+3\times\frac{1}{2}}$ 1

$\tan\theta = 1$.

$\theta = 45^\circ$. 1

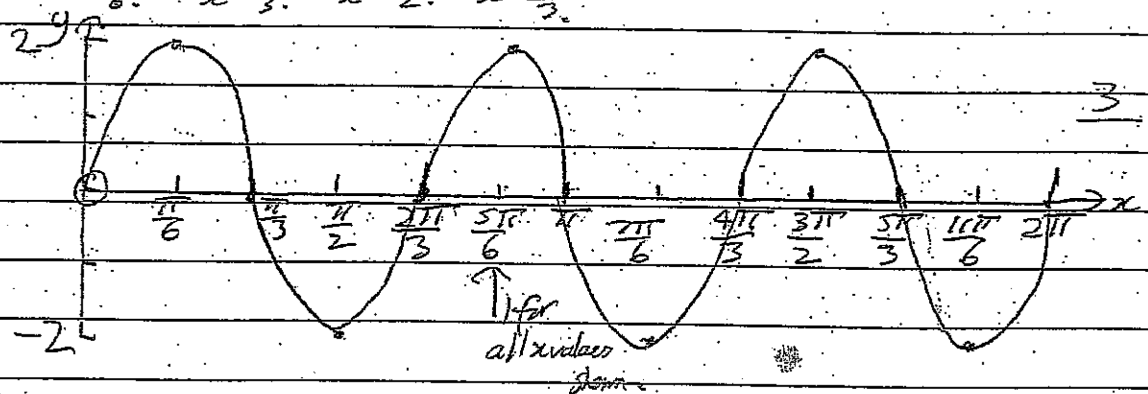
2

(d) $y=2\sin 3x$.

Period $=\frac{2\pi}{3}$ 1 Amplitude $=2$. 1

$3x=\frac{\pi}{2}$ $\quad 3x=\pi$ $\quad 3x=\frac{3\pi}{2}$ $\quad 3x=2\pi$

$x=\frac{\pi}{6}$. $\quad x=\frac{\pi}{3}$. $\quad x=\frac{\pi}{2}$. $\quad x=\frac{2\pi}{3}$.



3

(e) $\lim_{x\to 0} \frac{\sin 2x}{3x}$

$= \frac{2}{3} \times \lim_{x\to 0} \frac{\sin 2x}{2x}$

$= \frac{2}{3}$.

2

Q.(2)(a)(i) $y = x^2 e^{3x}$

Full marks here.

$y' = 2xe^{3x} + 3x^2 e^{3x}$ by product rule 2

$y' = xe^{3x}(2 + 3x)$

(ii) $y = \dfrac{e^{2x}}{\cos x}$

$y' = \dfrac{2e^{2x}\cos x - e^{2x} \times -\sin x}{\cos^2 x}$ 1

$= \dfrac{2e^{2x}\cos x + e^{2x}\sin x}{\cos^2 x}$ 1 → Full marks take. 2

$= \dfrac{e^{2x}(2\cos x + \sin x)}{\cos^2 x}$

(iii) $y = e^{\sin x}$

$y' = \cos x \, e^{\sin x}$ [by chain rule]. 1

(iv) $y = \ln\left[\dfrac{\sqrt{x^2+1}}{(3x-2)^5}\right]$

$y = \frac{1}{2}\ln(x^2+1) - 5\ln(3x-2)$

$y' = \dfrac{x}{x^2+1} - \dfrac{15}{3x-2}$ 4

(b) $y = 3e^{2x}$

$y' = 6e^{2x}$ 1 2

Where $x = 1$, $y = 3e^2$ 1

$y' = 6e^2$ 1

Tangent: By $y - y_1 = m(x - x_1)$

$y - 3e^2 = 6e^2(x-1)$ 1

$y - 3e^2 = 6e^2 x - 6e^2$

$y = 6e^2 x - 3e^2$ 1

OR $6e^2 x - y - 3e^2 = 0$.

Normal:

By $y - y_1 = m(x - x_1)$

$y - 3e^2 = \dfrac{-1}{6e^2}(x-1)$ 1

$-6e^2 y + 18e^4 = x - 1$

$0 = x + 6e^2 y - 18e^4 - 1$ 1

$\therefore x + 6e^2 y - 18e^4 - 1 = 0$

Q.(3)(a)(i) $\int x^2 e^{x^3}\,dx$

$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$ 1 $\underline{2}$

$= \frac{1}{3}e^{x^3} + C$ 1

(ii) $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\sin 2x - 3\cos 2x\,dx$

$= \left[-\frac{1}{4}\cos 2x - \frac{3}{2}\sin 2x\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$ 1 $\underline{3}$

$= \left(-\frac{1}{4}\cos\frac{\pi}{2} - \frac{3}{2}\sin\frac{\pi}{2}\right) - \left(-\frac{1}{4}\cos\frac{\pi}{4} - \frac{3}{2}\sin\frac{\pi}{4}\right)$ 1

$= \quad -\frac{3}{2} \quad - \quad -\frac{7\sqrt{2}}{8}$

$= \frac{7\sqrt{2}-12}{8}$ 1

(iii) $\int \frac{6x^2+8}{x^3+4x-1}\,dx$

$= 2\int \frac{3x^2+4}{x^3+4x-1}\,dx$ 1 $\underline{2}$

$= 2\ln(x^3+4x-1) + C$ 1

(iv) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$ $\underline{2}$

$= \ln(\sin x) + C$

Q.(3)(b)(i) $(\cos x + \sin x)^2$

$= \cos^2 x + 2\sin x\cos x + \sin^2 x$ 1

$= 1 + \sin 2x$ as $\cos^2 x + \sin^2 x = 1$ 1 $\underline{2}$

& $\sin 2x = 2\sin x \cos x$

(ii) Volume of solid of revolution:

$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x + \sin x)^2 \, dx$ 1

$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + \sin 2x \, dx$ 1 [from Part (i)]

$= \pi \left[x - \frac{1}{2}\cos 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ 1

$= \pi \left[\left(\frac{\pi}{2} - \frac{1}{2}\cos \pi \right) - \left(\frac{\pi}{4} - \frac{1}{2}\cos\frac{\pi}{2} \right) \right]$ 1 $\underline{5}$

$= \pi \left[\left(\frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{4} + 0 \right) \right]$

$= \frac{\pi^2}{2} + \frac{\pi}{2} - \frac{\pi^2}{4}$

$= \left(\frac{\pi^2}{4} + \frac{\pi}{2} \right)$ 1 cubic units.

or $\frac{\pi}{4}(\pi + 2)$ cubic units.

Q.(4)(a) Step 1: Show true for $n=1$

$LHS = \frac{1}{1\times2}$ $\quad RHS = \frac{1}{2}$.

$= \frac{1}{2}$ $\Rightarrow$ True for $n=1$.

Extra tests: For $n=2$:

LHS $\quad$ RHS

$= \frac{1}{1\times2} + \frac{1}{2\times3}$ $\quad = \frac{2}{3}$.

$= \frac{1}{2} + \frac{1}{6}$

$= \frac{2}{3}$

$\Rightarrow$ True for $n=2$

Step 2: Assume true for $n=k$: i.e.

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Step 3: Prove true for $n=k+1$ i.e.

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}.$$

LHS:

$$\frac{1}{1\times2} + \frac{1}{2\times3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{[using Step 2]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$= RHS$.

Step 4: By the principle of math. induction as it is true for $k=1$ it will be true for $k=2,3,4\ldots$ & all positive integers k.

Q.(4)(b)(i) MELBOURNE ⇒ 9 letters, 2 E's.

Total arrangements $= \frac{9!}{2!}$

$= 181\,440$ ways. 1

(ii) M at the front $= 1 \times \frac{8!}{2!}$ 1

$= 20\,160$ ways.

(iii) 5 consonants, 4 vowels so must go: $\frac{6}{4}$

[C = consonant, V = vowels]

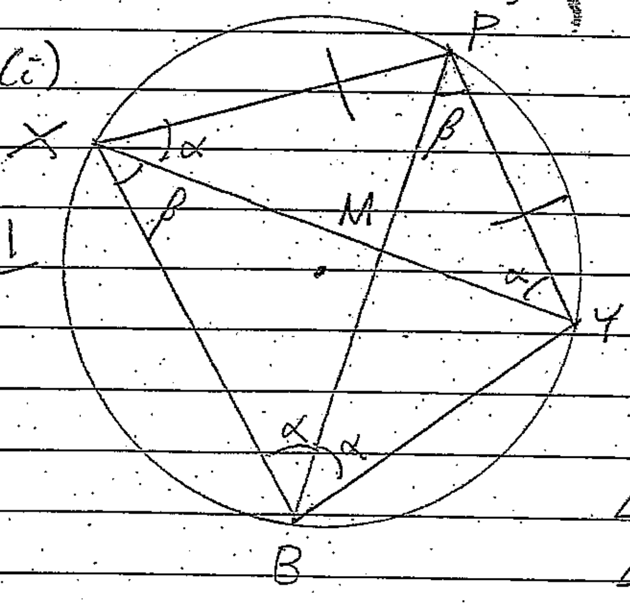
C V C V C V C V C 1

Ways of arranging consonants $= 5!$ 1

vowels $= \frac{4!}{2!}$ 1

Total ways $= 5! \times \frac{4!}{2!} = 1440$ ways. 1

(c) (i)



1

(ii) $\angle BXY = \angle BPY$ 1

[∠'s on same arc =]

(iii) Let $\angle PXY = \alpha$

$\angle PYX = \alpha$

[∠'s opposite = sides in isosceles $\triangle PXY$ =]. 1 $\frac{3}{}$

$\angle PBY = \angle PXY = \alpha$ [∠'s on same arc =] 1

$\angle PBX = \angle PYX = \alpha$ [" "]. 1

∴ PB bisects $\angle XBY$

Note: $\angle PBX = \angle PBY$ [∠'s standing on equal chords are equal] is also correct.

(iv) Let $\angle BXY = \beta$

$\angle BPY = \beta$ [from (ii)]

$\angle XMB = 180^\circ - \alpha - \beta$ [∠ sum $\triangle XMB$] 1

$\angle PYB = 180^\circ - \alpha - \beta$ [∠ sum $\triangle PYB$]. 1 $\frac{2}{}$

∴ $\angle XMB = \angle PYB$.

Q.(5)(a) Co-efficient of x^4:

General [k+1]th term

$$= {}^{14}C_k \times (3x^2)^{14-k} \times (-1)^k \times \left(\frac{2}{x}\right)^k \quad 1$$

To find k for co-efficient of x^4:

$$(x^2)^{14-k} \times x^{-k} = x^4 \quad 1$$

$$x^{28-2k} \times x^{-k} = x^4$$

$$x^{28-3k} = x^4$$

$$28 - 3k = 4$$

$$24 = 3k$$

$$8 = k \quad 1$$

$$\text{Co-efficient of } x^4 = {}^{14}C_8 \times 3^6 \times (-1)^8 \times 2^8 \quad 1$$

$$= 560\ 431\ 872$$

4

(b) (i) All 70 year olds survive:

$$= {}^5C_5 \times 0.87^5 \quad 1$$

$$= 0.4984 \text{ [4DP]} \quad 1$$

2

(ii) At least 3 75 year olds

$$= {}^4C_4 \times 0.79^4 + {}^4C_3 \times 0.79^3 \times 0.21$$

1 1

$$= 0.8037 \text{ [4 DP]} \quad 1$$

3

9

(iii) Only 1 death entire group [at most]

= All 70 & all 75 survive OR All 70 & 3 75 OR 4 70 & all 75

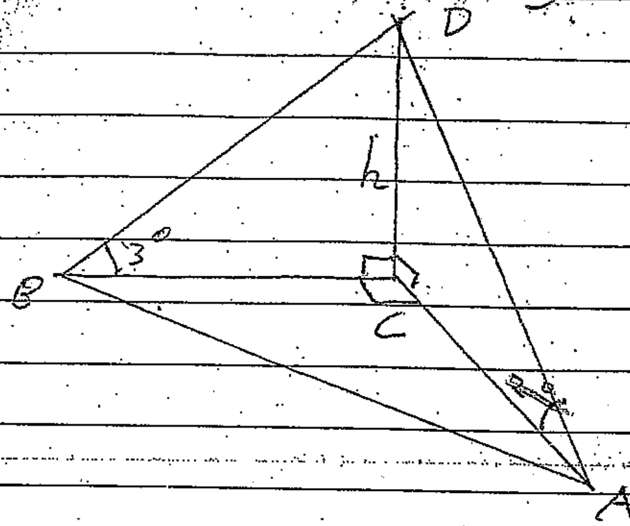
$$= 0.87^5 \times 0.79^4 + 0.87^5 \times {}^4C_3 \times 0.79^3 \times 0.21 + {}^5C_4 \times 0.87^4 \times 0.13 \times 0.79$$

1 1 1

$$= 0.5456 \text{ [4 DP]} \quad 1$$

4

Q. (6)(a)(i)



Length AC. $\tan 2^\circ = \frac{h}{AC}$

$\therefore AC = \frac{h}{\tan 2^\circ}$ 1

Length BC $\tan 3^\circ = \frac{h}{BC}$

$BC = \frac{h}{\tan 3^\circ}$ 1

By Pythagoras' theorem,

$$(AB)^2 = (BC)^2 + (AC)^2$$

$$20\,000^2 = \frac{h^2}{\tan^2 3^\circ} + \frac{h^2}{\tan^2 2^\circ} \quad 1$$

3

(ii) $20\,000^2 = h^2\left(\frac{1}{\tan^2 3^\circ} + \frac{1}{\tan^2 2^\circ}\right)$ 1

$$\frac{20\,000^2}{\left(\frac{1}{\tan^2 3^\circ} + \frac{1}{\tan^2 2^\circ}\right)} = h^2$$

$337\,802 \ldots = h^2$ 1

3

$581.2 \ldots = h$ 1

* The mountain is 580m high [nearest 10m].

Q. (6)(b)(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$ 1

(ii) Hence $\sin 75^\circ = \sin(45^\circ + 30^\circ)$ 1

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$ 1

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$ 1

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ 1 → Full marks to here. 4

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

5

(c) $2\cos\theta + 3\sin\theta = 1$

$\frac{2(1-t^2) + 3\times 2t}{1+t^2} = 1$ where $t = \tan\left(\frac{\theta}{2}\right)$ 1

$2 - 2t^2 + 6t = 1 + t^2$

$0 = 3t^2 - 6t - 1$ 1

$t = \frac{6 \pm \sqrt{6^2 - 4\times 3\times -1}}{2\times 3}$

$= \frac{6 \pm \sqrt{48}}{6}$

$= \frac{3 \pm 2\sqrt{3}}{3}$ 1

$\therefore \tan\left(\frac{\theta}{2}\right) = \frac{3+2\sqrt{3}}{3}$ or $\tan\left(\frac{\theta}{2}\right) = \frac{3-2\sqrt{3}}{3}$

$\frac{\theta}{2} = 65^\circ 6'$ ← 1 → $\frac{\theta}{2} = 171^\circ 12'$

$\theta = 130^\circ 12'$ 1 or $\theta = 342^\circ 25'$

Check: $\theta = 180^\circ$ 1 — 1 for check.

$2\cos 180^\circ + 3\sin 180^\circ = -2$ → not a solution

6

Q.(6)(d)(i) $2\sqrt{3}\cos x + 2\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

$R = \sqrt{(2\sqrt{3})^2 + 2^2}$ 1

$= 4$

As $2\sqrt{3}\cos x = 4\cos x\cos\alpha$ As $2\sin x = 4\sin x\sin\alpha$

$\frac{\sqrt{3}}{2} = \cos\alpha$. (either) 1 $\frac{1}{2} = \sin\alpha$.

$\therefore \alpha = \frac{\pi}{6}$ radians. 1 $\underline{3}$

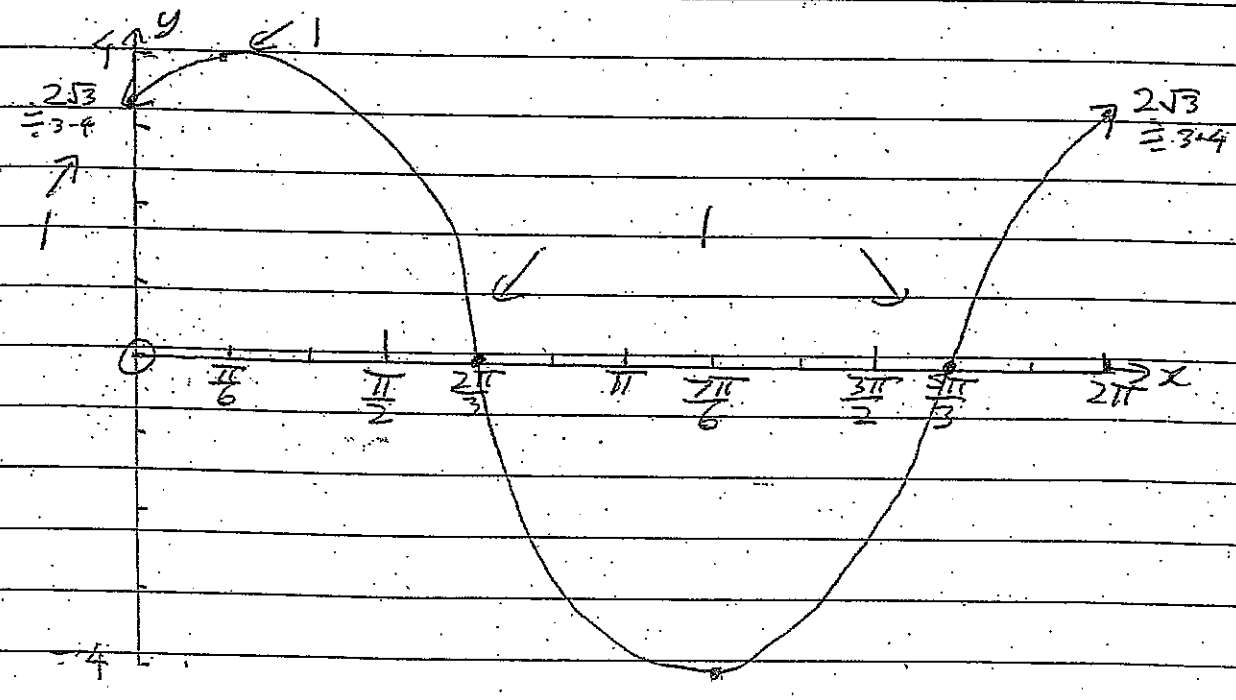
$\therefore 2\sqrt{3}\cos x + 2\sin x = 4\cos\left(x - \frac{\pi}{6}\right)$

(ii) Key values:

$x - \frac{\pi}{6} = 0$	$x - \frac{\pi}{6} = \frac{\pi}{2}$	$x - \frac{\pi}{6} = \pi$	$x - \frac{\pi}{6} = \frac{3\pi}{2}$
$x = \frac{\pi}{6}$	$x = \frac{2\pi}{3}$	$x = \frac{7\pi}{6}$	$x = \frac{5\pi}{3}$
$y = 4.$	$y = 0$	$y = -4$	$y = 0.$

At $x = 0$, $y = 4\cos\left(-\frac{\pi}{6}\right)$ At $x = 2\pi$, $y = 2\sqrt{3}$.

$= 2\sqrt{3}$.



3 marks in total
1 off per mistake.