FINAL MARK

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 1

HSC ASSESSMENT

Task 2 – 2010

Half yearly examination

ANSWERS COVER SHEET

Name:______________________________ Teacher:__________.

QUESTION	MARK	E2	E3	E4	E5	E6	E7
1	/16						√
2	/15						√
3	/16						√
4a	/5	√					√
b	/4		√				√
c	/6						√
	/15						
5	/15						√
6	/23						√
TOTAL	/100	/4	/9				/100

HSC Outcomes **Mathematics Extension 1**

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics.

HE2 uses inductive reasoning in the construction of proofs.

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion and exponential growth and decay.

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE6 determines integrals by reduction to a standard form through a given substitution.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

2010

MATHEMATICS

EXTENSION 1

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

Total Marks – 100

Attempt all questions 1-6

All questions are NOT of equal value.

Answer each question clearly ON A SEPARATE PAGE!

Question 1 (16 Marks) Use a separate piece of paper. **Marks**

(a) Solve for x: $\frac{4}{5-x} \geq 1$ **5**

(b) The point P(–3, 8) divides the interval AB externally in the ratio k : 1. If A is the point (6, –4) and B is the point (0, 4), find the value of k. **3**

(c) The acute angle between the lines $y = 3x + 5$ and $y = mx + 4$ is 45^0. Find the two possible values of m. **3**

(d) The polynomial $P(x) = x^3 + ax + 12$ has a factor $(x+3)$ Find the value of a. **1**

(e)

(i) Find the domain of the function $y = \ln(x-3)$ **1**

(ii) Sketch the graph of $y = \ln(x-3)$, showing any asymptotes and any intercepts on the co-ordinate axis. **3**.

Question 2 (15 Marks) Use a separate piece of paper.

(a) Differentiate:

i) $\dfrac{\log_e x}{x}$ 2

ii) $x^3 e^{-2x}$ 2

iii) 5^{3x-2} 3

iv) $\log_e\left[\dfrac{x+4}{x-3}\right]$ 3

(b) Find the equation of the normal to the curve y = $\ln x$ at the point where $x = 1$ 4

Question 3 (16 Marks) Use a separate piece of paper.

(a) Evaluate i) $\int_0^2 e^{5-2x}dx$ 3

ii) $\int_1^3 \dfrac{x}{x^2+1}dx$ 3

(b) Find i) $\int \dfrac{x+1}{\sqrt{x}}dx$. 2

ii) $\int \left(1+\dfrac{3}{x-2}\right)dx$ 3

(c) (i) Find $\dfrac{d}{dx}e^{3x^2}$ 2

(ii) Hence find $\int xe^{3x^2}dx$ 3

Question 4 (15 Marks) Use a separate piece of paper.

(a) Use mathematical induction to prove that, for every positive integer n,

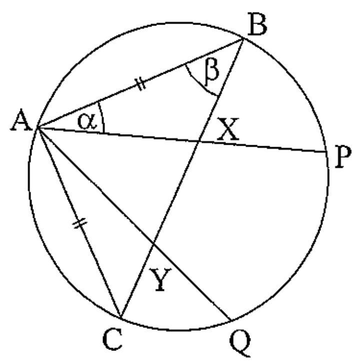
$13 \times 6^n + 2$ is divisible by 5 **5**

(b)

(i) Explain why the probability of obtaining 2 heads and a tail when three coins are tossed is $\frac{3}{8}$. **1**

(ii) Sian tosses three coins 10 times in a row. Calculate the probability of obtaining 2 heads and a tail at least 2 times. Give your answer correct to 3 significant figures. **3**

(c) **6**



Let ABPQC be a circle such that AB = AC, AP meets BC at X, and AQ meets BC at Y, as in the diagram. Let $\angle BAP = \alpha$ and $\angle ABC = \beta$.

i. Copy the diagram into your Writing Booklet and state why $\angle AXC = \alpha + \beta$.

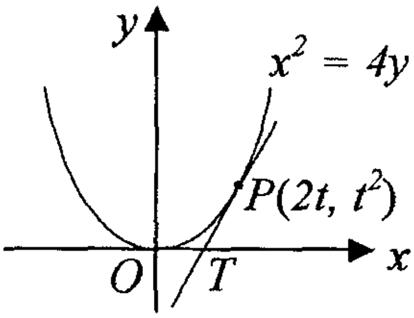
ii. Prove that $\angle BQP = \alpha$.

iii. Prove that $\angle BQA = \beta$.

iv. Prove that PQYX is a cyclic quadrilateral.

Question 5 (15 Marks) Use a separate piece of paper.

a)



$P(2t, t^2)$ is a variable point which moves on the parabola $x^2 = 4y$.
The tangent to the parabola at P cuts the x axis at T. M is the midpoint of PT

i. Show that the tangent PT has equation $tx - y - t^2 = 0$ **2**

ii. Show that M has coordinates $(\frac{3t}{2}, \frac{t^2}{2})$. **2**

iii. Hence find the Cartesian equation of the locus of M as P moves on the parabola. **4**

(b) Find the value of the term that is independent of x in the expansion of $\left(x^2 + \frac{3}{x}\right)^6$ **3**

(c) The area enclosed between the curves y = e^x, y = $e^{\frac{1}{2}x}$ and the line $x = 2$ is rotated about the x axis. Find the volume of the solid generated. **4**

Question 6 (23 Marks) Use a separate piece of paper.

(a)

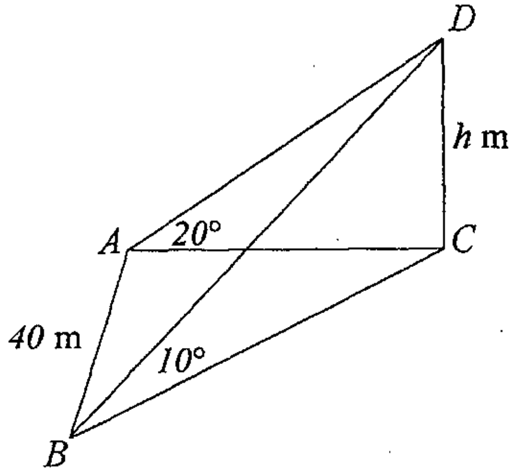


FIGURE NOT TO SCALE

A vertical flagpole CD of height **h** metres stands with its base C on horizontal ground. A is a point on the ground due West of C and B is a point on the ground 40 metres due South of A. From A and B the angles of elevation of the top D of the flagpole are 20^0 and 10^0 respectively. Find the height of the flagpole correct to the nearest metre. **6**

(b) (i) Factorize $a^3 + b^3$. **1**

(ii). Hence, or otherwise, show that $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$,

if $\sin A + \cos A \neq 0$ **4**

(c) Solve $7\sin x - 4\cos x = 4$, for $0^0 \leq x \leq 360^0$, by using the t formulae **6**

(d) (i) Express $\cos x - \sin x$ in the form $R\cos(x + \alpha)$ where α is in degrees. **3**

(ii) Hence solve the equation $\cos x - \sin x = -1$ for $0^0 \leq x \leq 360^0$ **3**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Year 12 Half yearly exam. Task 2 - 2010

Mathematics Extension 1. Solutions.

Question 1

(a) By critical points: (5)

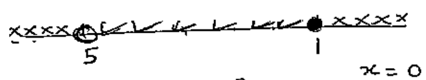
Equality $\frac{4}{5-x}=1$

$4 = 5-x$

$x = 1$

Discontinuity: $x \neq 5$

Testing.



$x=6$, $\frac{4}{5-6}=-4\geq 1$ False

$x=3$, $\frac{4}{5-3}=2\geq 1$ True

$x=0$, $\frac{4}{5-0}=\frac{4}{5}\geq 1$ False

$\therefore\ 1\leq x<5$

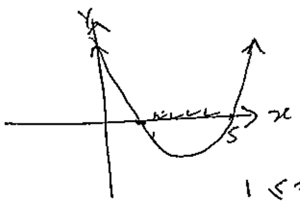
OR.

$\frac{(5-x)^2\times 4}{5-x}\geq(5-x)^2$; $x\neq 5$.

$4(5-x)\geq(5-x)^2$

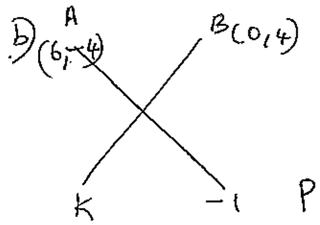
$(5-x)^2-4(5-x)\leq 0$

$(5-x)(5-x-4)\leq 0 \Rightarrow (5-x)(1-x)\leq 0$



$1\leq x<5$

b) (3)



$P=(-3,8)$

$-3=\frac{-1\times 6+k\times 0}{k-1}=\frac{-6}{k-1}$

$\therefore -3k+3=-6 \Rightarrow 3k=9$

$\therefore k=3$

c) $y=3x+5 \Rightarrow m_1=3$

$y=mx+4 \Rightarrow m_2=m.$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\tan 45^\circ=\left|\frac{3-m}{1+3m}\right|=1$

$\therefore \frac{3-m}{1+3m}=1$ or $\frac{-(3-m)}{1+3m}=1$

$\Rightarrow 3-m=1+3m$ or $-3+m=1+3m$

$\therefore m=\frac{1}{2}$ $\therefore m=-2$ (3)

d) $P(x)=x^3+ax+12$ (2)

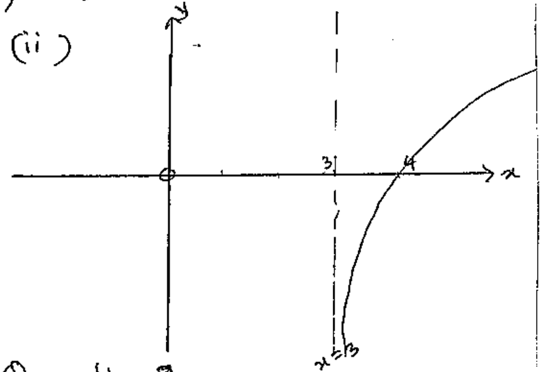
$P(-3)=-3^3+3a+12=0$

$\Rightarrow -27+3a+12=0 \Rightarrow 3a=-15$

$\therefore a=-5.$

e) Domain. $x>3$

(ii)



Question 2

a) (i) $y=\frac{\log_e x}{x}$

$\therefore \frac{dy}{dx}=\frac{x\cdot\frac{1}{x}-\log_e x\cdot 1}{x^2}$

$=\frac{1-\log_e x}{x^2}$ (2)

(ii) $y=x^3e^{-2x}$

$\frac{dy}{dx}=3x^2\cdot e^{-2x}+(e^{-2x}\cdot -2)x^3$

$=x^2e^{-2x}(3-2x)$ (2)

(iii) $y=5^{3x-2}$

$y'=5^{3x-2}\cdot\log 5\times 3$

$=3\log 5\cdot 5^{3x-2}$ (3)

(iv) $y=\log_e\left(\frac{x+4}{x-3}\right)$

$=\log_e(x+4)-\log_e(x-3)$

$\therefore y'=\frac{1}{x+4}-\frac{1}{x-3}$ (3)

b) $y=\ln x.$

$y'=\frac{1}{x}$

at $x=1$, $y'=\frac{1}{1}=1$

$\therefore y'$ normal $=-1$

at $x=1$, $y=\ln 1=0$.

$\therefore$ equation is

$y-0=-1(x-1)=-x+1$

$y+x-1=0.$ (5)

Question 5

(b) $T_{r+1} = {}^6C_r (x^2)^{6-r} \cdot \left(\frac{3}{x}\right)^r$

$= {}^6C_r \cdot x^{12-2r} \cdot 3^r \cdot x^{-r}$

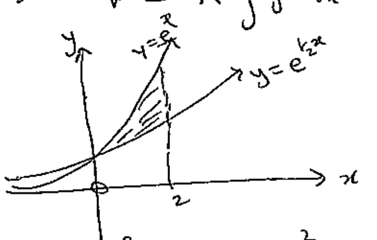
$= {}^6C_r \cdot 3^r \cdot x^{12-3r}$.

term independent of x

$\Rightarrow 12 - 3r = 0 \Rightarrow r = 4$

$\therefore T_5 = {}^6C_4 \cdot 3^4 = 1215$ (3)

c). $V = \pi \int y^2 \, dx$.



$V = \pi \int_0^2 (e^x)^2 dx - \int_0^2 (e^{\frac{1}{2}x})^2 \, dx$

$= \pi \int_0^2 (e^{2x} - e^x) \, dx$

$= \pi \left[\frac{e^{2x}}{2} - e^x \right]_0^2$

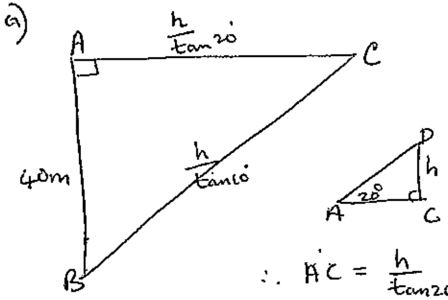
$= \pi \left[\frac{e^4}{2} - e^2 - \left(\frac{1}{2} - 1\right) \right]$

$= \pi \left[\frac{e^4}{2} - e^2 + \frac{1}{2} \right]$

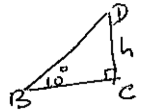
$= \frac{\pi}{2} \left[e^4 - 2e^2 + 1 \right]$. (4)

Question 6

a)



$\therefore AC = \frac{h}{\tan 20^\circ}$



$BC = \frac{h}{\tan 10^\circ}$

$\therefore \left(\frac{h}{\tan 10^\circ}\right)^2 - \left(\frac{h}{\tan 20^\circ}\right)^2 = 40^2$

$h^2 \left[\left(\frac{1}{\tan 10}\right)^2 - \left(\frac{1}{\tan 20}\right)^2 \right] = 40^2$

$\therefore h^2 = 65$

$h \doteqdot 8m$ (6)

b) (i) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ (1)

(ii) $\frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$

LHS

$\frac{2(\sin^3 A + \cos^3 A)}{\sin A + \cos A} =$

$= \frac{2(\sin A + \cos A)(\sin^2 A - \sin A \cos A + \cos^2 A)}{\sin A + \cos A}$ $(\sin A + \cos A \neq 0)$

$= 2(1 - \frac{1}{2}\sin 2A) = 2 - \sin 2A$.

Question 6 $\qquad \tan\frac{x}{2} = t$

(c)

$7\sin x - 4\cos x = 4$

$7 \cdot \frac{2t}{1+t^2} - \frac{4(1-t^2)}{1+t^2} = 4$

$14t - 4 + 4t^2 = 4 + 4t^2$

$\therefore 14t = 8$

$t = \frac{8}{14} = \frac{4}{7}$

$\tan\left(\frac{x}{2}\right) = \frac{4}{7}$

$\therefore \frac{x}{2} \doteqdot 29^\circ 44'$

$\therefore x \doteqdot 59^\circ 29'$

Check $\theta = 180^\circ$ in $7\sin x - 4\cos x = 4$

LHS. $7 \times 0 - 4 \times -1 = 4 = RHS$

$\therefore x = 180^\circ$ is also a solution.

$\therefore x = 180^\circ$ or $x \doteqdot 59^\circ 29'$ (6)

(d) (i) $\cos x - \sin x = R\cos(x + \alpha)$

Expanding,

$\cos x - \sin x = R\cos x \cos\alpha - R\sin x \sin\alpha$

$\therefore R = \sqrt{1+1} = \sqrt{2}$

As $\cos x = R\cos x \cos\alpha$,

$\cos\alpha = \frac{1}{R} = \frac{1}{\sqrt{2}}$

As $\sin x = R\sin x \sin\alpha$,

$\sin\alpha = \frac{1}{\sqrt{2}}$

$\therefore \alpha = 45^\circ$.

$\therefore \cos x - \sin x = \sqrt{2}\cos(x + 45^\circ)$ (2)

(ii) $\sqrt{2}\cos(x + 45^\circ) = -1$

$\therefore \cos(x + 45^\circ) = -\frac{1}{\sqrt{2}}$

$\therefore$ related angles are 135°, 225°

$\therefore x + 45^\circ = 135^\circ$ or $x + 45^\circ = 225^\circ$

$\therefore x = 90^\circ$ or $x = 180^\circ$ (4)