GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATION

2012

MATHEMATICS

EXTENSION 1

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

PART A (5 marks)

For questions 1-5 circle the best response from the following:

1. The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots α , β and γ .The value of $\alpha + \beta + \gamma$ is equal to

A) 1 B) 2 C) -24 D) -1

2. A four person team is to be chosen at random from nine women and seven men.

In how many ways can this team be chosen?

A) 126 B)2046 C) 1820 D)32

3.Let $f(x) = \ln(x-3)$ .What is the domain of $f(x)$?

A) $x \geq 3$ B) $x \leq 3$ C) $x = 3$ D) $x > 3$

4.The derivative of e^{2x+1} is

A) e^{2x+1} B) $2e^{2x+1}$ C) $2\left(e^{2x+1}\right)^2$ D) $4e^{2x+1}$

5. $\int_0^1 \frac{dx}{2x+1} =$

A) $\frac{1}{2}\ln 3$ B) $\frac{1}{2}\ln 4$ C) $\ln 3$ D) $\ln 4$

Part B

Total Marks – 85

Attempt all questions 1-6

All questions are NOT of equal value.

Answer each question clearly ON A SEPARATE PAGE!

Question 1 (14 Marks) Use a separate piece of paper. **Marks**

(a) Solve for $x : \frac{2}{x-1} \leq 1$ **4**

(b) Divide the interval between the points $(-1,2)$ and $(3,5)$ *externally* in the ratio $3:1$. **2**

(c) Find the acute angle angle between the straight lines $-2x+3y-8=0$ and $y-5x+9=0$. **3**

(d) Write down the exact value of 135^0 in radians **2**

(e) Solve the equation $2\ln x = \ln(5x+4)$ **3**

Question 2 (28 Marks) Use a separate piece of paper.

(a) Differentiate:

(i) $y = \frac{e^x}{x+e^x}$ **3**

(ii) $y = x^2 \ln x$ **2**

(iii) $y = \frac{x^2}{e^x}$ **3**

(iv) $y = \ln\left(\frac{x-5}{x+5}\right)$ **3**

(b) Find the equations of the tangent to the curve $y = 3\ln x + 2$ at the point where $x = 1$ **3**

Question 2 continued **Marks**

(c) Find (i) $\int_{1}^{2}\frac{3}{5-2x}dx$ **2**

(ii) $\int_{3}^{6}\frac{4x-5}{2x^2-5x}dx$ **2**

(d) Find (i) $\int\frac{3x^2-2x}{x^2}dx$ **2**

(iv) $\int\sqrt{e^x}\,dx$ **2**

(e) (i) Differentiate (i) xe^x **3**

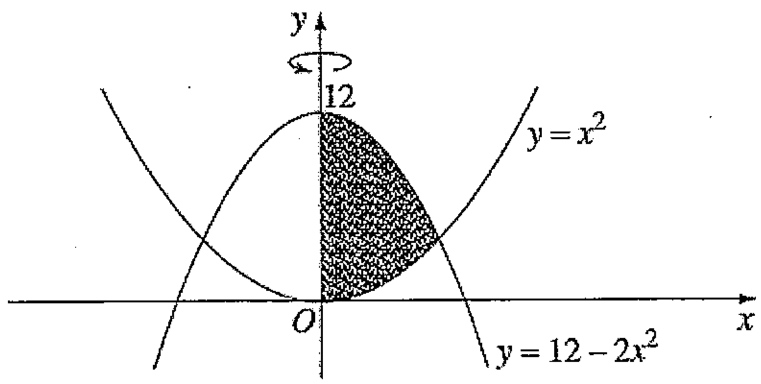
(ii) Hence find $\int_{0}^{2}xe^x dx$ **3**

Question 3(11 Marks) Use a separate piece of paper.

(a) Use mathematical induction to prove that for $n\geq 1$,

$$1\times 5+2\times 6+\ldots\ldots\ldots\ldots+n(n+4)=\frac{1}{6}n(n+1)(2n+13)$$ **5**

(b)



The graphs of curves $y=x^2$ and $y=12-2x^2$ are shown in the diagram.

(i) Find the points of intersection of the two curves. **2**

(ii) The shaded region between the curves and the y-axis is rotated about the y-axix. By splitting the shaded region into two parts, or otherwise, find the volume of the solid formed. **4**

Question4 (14 Marks) Use a separate piece of paper.

(a) Find the term independent of x in the expansion of $\left(2x+\frac{1}{x}\right)^{10}$ **3**

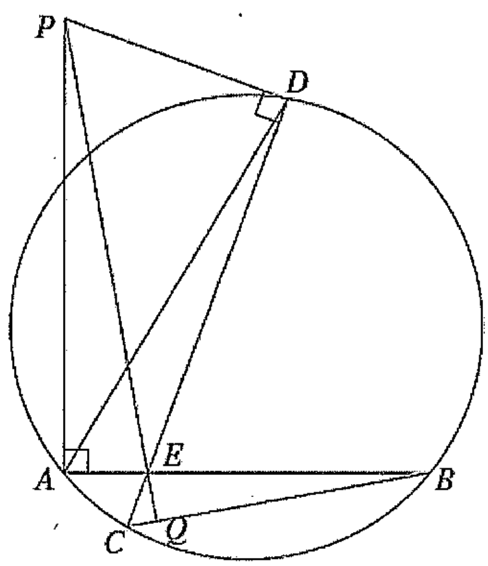
(b) If 2% of a population is colour blind, what is the probability that a random sample of 10 people could contain:

(i) No colour blind person? **1**

(ii) Exactly two colour blind people?. **2**

(iii) Three or more colour blind people? **3**

(c) Two chords of a circle, *AB* and *CD*, intersect at *E*. The perpendiculars to *AB* at *A* and *CD* at *D* intersect at *P*. The line *PE* meets *BC* at *Q*, as shown in the diagram



(i) Explain why DPAE is a cyclic quadrilateral. **1**

(ii) Prove that $\angle APE = \angle ABC$ **2**

(iii) Deduce that PQ is perpendicular to BC. **2**

Question 5 (18 Marks) Use a separate piece of paper. **Marks**

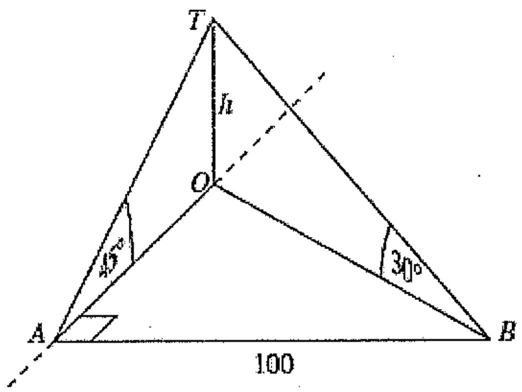
(a) The polynomial $p(x) = x^3 - ax + b$ has a remainder of 2 when divided by $(x-1)$ and a remainder of 5 when divided by $(x+2)$. Find the values of a and b. **3**

(b) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$ **3**

(c) (i) Show that $\dfrac{1+\cos 2A}{\sin 2A} = \cot A$ **3**

(ii) Hence find the exact value of $\cot 15^0$ **3**

(d)



A surveyor stands at a point A, which is due south of a tower OT of height h metres. The angle of elevation of the top of the tower from A is 45^0. The surveyor then walks 100m due east to point B, from where measures the angle of elevation of the top of the tower to be 30^0.

(i) Express the length of OB in terms of h. **2**

(ii) Show that $h = 50\sqrt{2}$ metres. **2**

(iii) Calculate the bearing of B from the base of the tower. **2**

END OF TEST

Year 12 Half-Yearly 2012 Extension 1 Solutions

Part A

1) B 2) C 3) D 4) B 5) A

Part B

Question 1. $\frac{2}{x-1} \le 1$

a) By critical points.

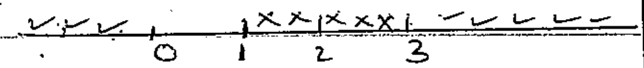
equality. $\frac{2}{x-1} = 1$

$2 = x - 1$

$x = 3.$

Discontinuity: $x \neq 1$

Testing.



$x = 0,$ $\frac{2}{-1} \le 1$ True.

$x = 2$ $\frac{2}{1} \le 1$ False.

$x = 4$ $\frac{2}{3} \le 1$ True.

$x < 1, \ x \geq 3$

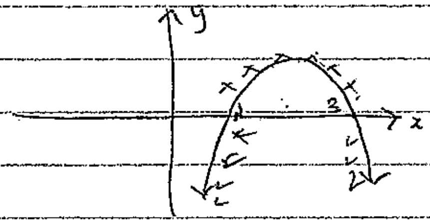
By multiplying by the square of the denominator:

$2(x-1) \le (x-1)^2$

$2(x-1) - (x-1)^2 \le 0$

$(x-1)(2 - x + 1) \le 0$

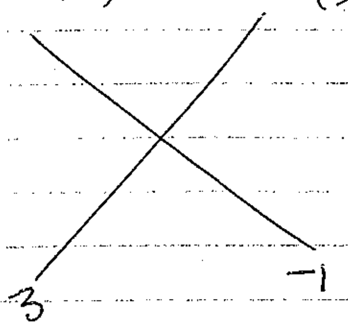
$(x-1)(-x+3) \le 0$



$x < 1$ and $x \geq 3$

④

b) $A(-1, 2)$ $B(3, 5)$



$C = \left(\frac{1+9}{2}, \frac{-2+15}{2}\right) =$

$= (5, 6\frac{1}{2})$

②

c) $3y = 2x + 8 \Rightarrow m_1 = \frac{2}{3}$

$y = 5x - 9 \Rightarrow m_2 = 5$

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{\frac{2}{3} - 5}{1 + \frac{2}{3} \cdot 5} \right| = 1$$

$$\therefore \theta = 45^\circ$$

d) $135^\circ = \frac{\pi}{180} \times 135 = \frac{3\pi}{4}$

e) $\cancel{2 \ln x = \ln(5 + 4x)}$

$\cancel{\ln x^2 = \ln(5 + 4x)}$

$\cancel{x^2 = 5 + 4x}$

$\cancel{x^2 - 4x - 5 = 0}$

$\cancel{(x - 5)(x + 1) = 0}$

$\cancel{x = 5}$ $\cancel{(x \neq -1}$

$2 \ln x = \ln(5x + 4)$

$\ln x^2 = \ln(5x + 4)$

$x^2 - 5x - 4 = 0$

$x = \frac{5 \pm \sqrt{25 + 16}}{2} = \frac{5 + \sqrt{41}}{2}$ as $x > 0$.

Question 2

a) (i) $y = \dfrac{e^x}{x+e^x}$

$$y' = \frac{(x+e^x)\,e^x - e^x(1+e^x)}{(x+e^x)^2} = \frac{xe^x - e^x}{(x+e^x)^2}$$ (3)

$$= \frac{e^x(x-1)}{(x+e^x)^2}$$

(ii) $y = x^2 \ln x$

$$y' = \ln x \cdot 2x + \frac{1}{x} \cdot x^2$$

$$= 2x\ln x + x$$ (2)

$$= x[2\ln x + 1]$$

(iii) $y = \dfrac{x^2}{e^x}$

$$y' = \frac{2x \cdot e^x - e^x \cdot x^2}{e^{2x}} = \frac{e^x \cdot x[2-x]}{e^{2x}}$$

$$= \frac{x(2-x)}{e^x}$$ (3)

(iv) $y = \ln\left(\dfrac{x-5}{x+5}\right) = \ln(x-5) - \ln(x+5)$

$$y' = \frac{1}{x-5} - \frac{1}{x+5} = \frac{x+5-x+5}{x^2-25}$$

$$= \frac{10}{x^2-25}$$ (3)

(b) $y = 3\ln x + 2$

$$y' = \frac{3}{x}$$

at $x = 1$, $y' = \frac{3}{1} = 3$

when $x = 1$, $y = 3\ln 1 + 2 = 2$

$\therefore$ eq^n. of tangent: $y - 2 = 3(x-1)$

$y = 3x - 1$ (3)

Question 3 continued

(c) (i) $\int_1^2 \frac{3}{5-2x}\,dx = \frac{3}{-2}\left[\ln 5-2x\right]_1^2$

$= -\frac{3}{2}\left[\ln 1 - \ln 3\right]$

$= \frac{3}{2}\ln 3$ (2)

(ii) $\int_3^6 \frac{4x-5}{2x^2-5x}\,dx = \left[\ln(2x^2-5x)\right]_3^6$

$= \ln(42) - \ln(3)$

$= \ln\left(\frac{42}{3}\right) = \ln 14$ (2)

d) (i) $\int \frac{3x^2-2x}{x^2}\,dx = \int \left(3 - \frac{2}{x}\right)dx$

$= 3x - 2\ln x + c$ (2)

(ii) $\int e^{\sqrt{x}}\,dx = \int e^{\frac{1}{2}x}\,dx = 2e^{\frac{1}{2}x} + c$ (2)

e) (i) $y = xe^x$ (u v)

$y' = e^x + xe^x$

$\therefore \int_0^2 e^x\,dx + \int_0^2 xe^x\,dx = \left[xe^x\right]_0^2$ (3)

$\int_0^2 xe^x\,dx = \left[xe^x\right]_0^2 - \left[e^x\right]_0^2$

$= 2e^2 - 0 - e^2 + 1$ (3)

$= e^2 + 1$

Question 3

$1\times5 + 2\times6 + \cdots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$

a) Step 1 Prove true for $n=1$.

LHS $1\times5 = 5$

RHS $\frac{1}{6}\cdot1\cdot(2)(15) = 5$

$\therefore$ true for $n=1$

Step 2

assume true for $n=k$

ie $1\times5 + 2\times6 + \cdots + k(k+4) = \frac{1}{6}k(k+1)(2k+13)$

Step 3 prove true for $n=k+1$

ie $\underbrace{1\times5 + 2\times6 + \cdots + k(k+4)} + (k+1)(k+5) = \frac{1}{6}(k+1)(k+2)(2k+15)$

LHS $\frac{k}{6}(k+1)(2k+13) + (k+1)(k+5)$

$= (k+1)\left[\frac{k(2k+13)}{6} + k+5\right] = (k+1)\left[\frac{2k^2+13k+6k+30}{6}\right]$

$= \frac{(k+1)(2k^2+19k+30)}{6} = \frac{(k+1)(k+2)(2k+15)}{6}$

$=$ RHS.

$\therefore$ by the principle of mathematical induction, it is true for all integers $n \geq 1$.

(b) (i) $x^2 = 12-2x^2 \Rightarrow 3x^2 = 12$

$x = \pm 2$

$x=2, y=4$ and $x=-2, y=4$

$\therefore$ points of intersection $(2,4)$ and $(-2,4)$

(ii) $V = \pi\int_0^4 y\,dy + \int_4^{12}\left(6-\frac{y}{2}\right)dy = \pi\left\{\left[\frac{y^2}{2}\right]_0^4 + \left[6y-\frac{y^2}{4}\right]_4^{12}\right\}$

$= \pi\left[8 + (72-36-24+4)\right] = 8+16 = \ldots \quad 24\pi u^3$

Question 4

$\left(2x+\frac{1}{x}\right)^{10}$

a) $T_{r+1} = 10C_r \cdot (2x)^{10-r} \cdot \left(-\frac{1}{x}\right)^r$

$= 10C_r \cdot 2^{10-r} \cdot x^{10-r} \cdot x^{-r}$

$= 10C_r \cdot 2^{10-r} \cdot x^{10-2r}$

term independent of $x \Rightarrow 10-2r=0$

$\therefore r=5$

$\therefore T_6 = {}^{10}C_5 \cdot 2^5 = 8064$ (3)

b) $n=10$

p = probability of a colour blind person

$p = 2\% = 0.02$

$q = 1 - 0.02 = 0.98$

Let x = no. of colour blind people.

a) $P(x=0) = (0.98)^{10} = 0.82$

b) $P(x=2) = {}^{10}C_2 \cdot q^8 p^2 = 45\,(0.98)^8(0.02)^2 = 0.02$

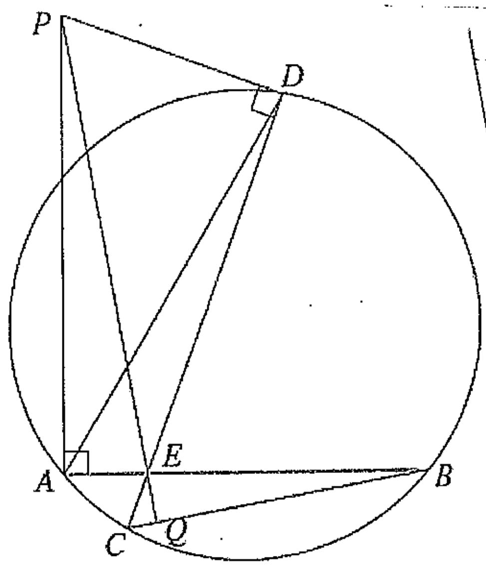
c) $P(x \geq 3) = 1 - P(x=0) - P(x=1) - P(x=2)$

$= 1 - (0.98)^{10} - {}^{10}C_1 (0.98)^9 (0.02) - {}^{10}C_2 (0.98)^8 (0.02)^2$

$= 0.00086$

Question 4(c) continued.

(c)



(i) $\angle PDE = 90^\circ$ (given).

$\angle PAE = 90^\circ$ (given)

$\therefore \angle PDE + \angle PAE = 180^\circ$

opposite angles in the quadrilateral are supplementary.

$\therefore$ DPAE is a cyclic quadrilateral.

(ii) $\angle APE = \angle ADE$ ($\angle$'s standing on the same arc in the quad. DPAE

but $\angle ADE = \angle ABC$ ($\angle$'s standing on the same arc in cyclic quad.)

$\therefore \angle APE = \angle ABC$

(iii) $\angle APE = \angle ABC$ (Proven in ii)

$\angle AEP = \angle BEQ$ (vertically opposite $\angle$'s).

$\angle APE + \angle AEP = 90^\circ$ ($\angle$ sum of a ~~quad~~ Δ).

$\therefore \angle ABC + \angle BEQ = 90^\circ$.

$\therefore$ In Δ BEQ, $\angle BEQ = 90^\circ$ ($\angle$ sum of a Δ)

$\therefore PQ \perp BC$

Question 5

a) $P(x) = x^3 - ax + b$

$P(1) = 1 - a + b = 2 \quad \therefore a - b = -1$ — (1)

$P(-2) = -8 + 2a + b = 5 \quad \therefore 2a + b = 13$ — (2)

(1) + (2) $\quad 3a = 12$

$\therefore a = 4$

$\therefore b = \cancel{4+1}\ a+1 = 5 \quad$ (sub in (1))

$a = 4;\ b = 5$ (3)

b) $2\sin^2\theta = \sin 2\theta \quad$ for $0 \le \theta \le 2\pi$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$2\sin^2\theta - 2\sin\theta\cos\theta = 0$

$2\sin\theta(\sin\theta - \cos\theta) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

$\sin\theta = 0 \Rightarrow \theta = 0, 2\pi$

$\sin\theta - \cos\theta = 0 \Rightarrow \sin\theta = \cos\theta$

$(\div\cos\theta) \therefore \tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore \theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi.$ (3)

c) (i) $\frac{1+\cos 2A}{\sin 2A} = \cot A \qquad \left\{ \begin{array}{l} \cos 2A = 2\cos^2 A - 1 \\ \sin 2A = 2\sin A\cos A \end{array} \right.$

LHS $\frac{1 + 2\cos^2 A - 1}{2\sin A\cos A} = \frac{\cancel{2}\cos^{\cancel{2}} A}{\cancel{2}\sin A\cancel{\cos A}} = \cot A = \text{RHS}.$ (3)

(ii) $\cot 15^\circ = \frac{1+\cos 30^\circ}{\sin 30^\circ} = \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2+\sqrt{3}$ (3)