GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATIONS

2014

MATHEMATICS

EXTENSION 1

Time Allowed: Two hours

(Plus 5 minutes reading time)

Instructions To Students

- Attempt all questions.
- All necessary working must be shown for Questions 6 – 10.
- Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- For Questions 1 -5, write the letter corresponding to the correct answer on your answer sheet.
- For Questions 6 – 10, start each question on a new sheet of paper. Each question should be clearly labelled.

Questions 1 -5(5 marks)

Write the letter corresponding to the correct answer on your answer sheet.

1 What is the solution to the equation $\log_e(x+2)-\log_e x=\log_e 4$?

(A) $\frac{2}{5}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{2}$

2 What is the derivative of $\log_e\left(\frac{x+1}{x-1}\right)$ with respect to *x*?

(A) $\frac{x+1}{x-1}$

(B) $\frac{-2}{x-1}$

(C) $\frac{-2}{x^2-1}$

(D) $\frac{2}{x^2-1}$

3 What is the value of $\int_0^1 (e^{2x}+1)dx$?

(A) $\frac{1}{2}e^2$

(B) $\frac{1}{2}(e^2+1)$

(C) e^2

(D) e^2+1

4 Let α, β and γ be the roots of $4x^3 - 2x^2 + 3x - 2 = 0$.

What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) $-\frac{3}{2}$

(B) $-\frac{2}{3}$

(C) $\frac{2}{3}$

(D) $\frac{3}{2}$

5 In a class of 23 students, there are 12 boys and 11 girls. The class needs to elect two boys and two girls for the student council. How many different representatives are possible?

(A) 121

(B) 3630

(C) 8855

(D) 14 520

Question 6(23 marks)

a. Solve: $\frac{x^2 - 6}{x} \le 1$ **5**

b. *A* has coordinates (-2, 5) and *B* has coordinates (4, -3). Find the length of *PQ* if *P* divides *AB* internally in the ratio 3:2 and *Q* divides *AB* externally in the ratio 3:2 **5**

c. The acute angle between straight lines $2x - 3y = 0$ and $ax + 4y = 9$ is $32^{o}51'$. Find the value of a correct to 2 significant figures. **5**

d. Find $\lim_{x \to 0} \frac{\sin 4x}{3x}$ **2**

e. Find the exact value of $\sin 75°$ **2**

f. If $3\cot\theta = 2$, find the exact value of $\frac{5\sin\theta - 7\cos\theta}{\cos ec\theta + \sec\theta}$, $0° \le \theta \le 90°$ **4**

Question 7(21 marks)

a. Differentiate:

(i) $y = x\log_e x$ (ii) $y = \dfrac{e^x}{\sin x}$ (iii) $y = \ln(\sin^2 x)$ **7**

b. The line $y = mx$ is tangent to the curve $y = e^{3x}$.

(i) Find the point of intersection of the tangent and the curve. **3**

(ii) Find the value of m. **1**

c. Find:

(i) $\displaystyle\int \frac{x}{x^2 - 4}dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{2 - \cos x}dx$ **4**

d. (i) Differentiate $y = \tan^3 x$ and express your answer in terms of $\sec x$ only. **2**

(ii) Hence show that $\displaystyle\int \sec^4 x dx = \frac{1}{3}[\tan^3 x + 3\tan x] + c$. **3**

(iii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x dx$. **1**

Question 8(14 marks)

a. Use Mathematical Induction to prove that if n is positive, **4**
$6^n + 7^n$ is divisible by 13 for all odd n.

b. Find the term independent of x in the expansion of $\left(2x + \dfrac{3}{x^2}\right)^9$. **3**

c. The probability of rain on any particular day is 0.4 during a 7 day period.
Find the probability of

(i) rain on exactly 3 days **2**

(ii) rain on 3 consecutive days and fine on all the other days **2**

(iii) rain on at least 2 days. **3**

Question 9(18 marks)

a. Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

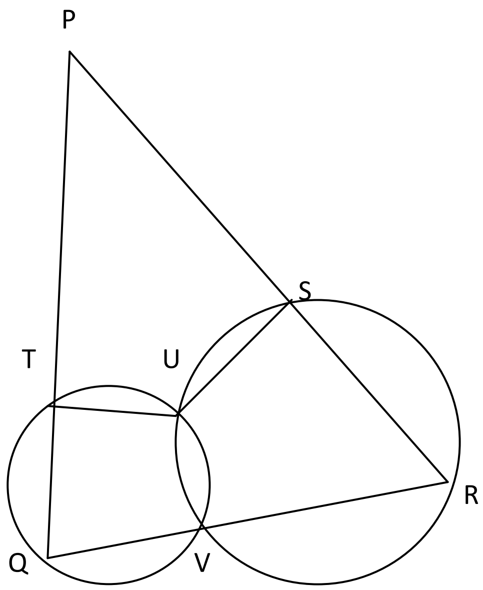
(i) Show that the equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$. **3**

(ii) If PQ passes through the point $(4a, -2a)$, show that $pq = 2p + 2q + 2$. **2**

(iii) Find the locus of M, the midpoint of PQ. **4**

b. T, V and S are points on the lines PQ, QR and RP respectively. **5**

Prove that $PTUS$ is a cyclic quadrilateral.

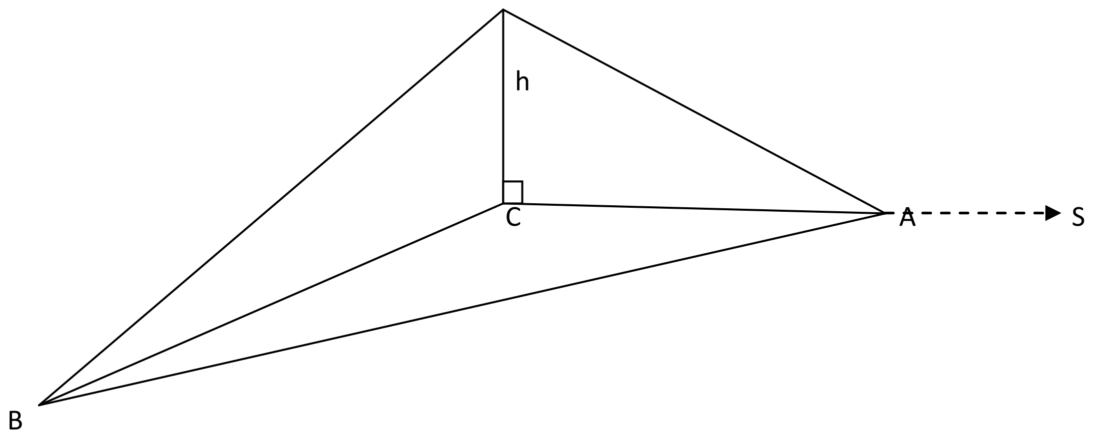


c. Solve $\sin\theta - 2\cos\theta = 2$ for $0 \leq \theta \leq 2\pi$ using the substitution $t = \tan\frac{\theta}{2}$. **4**

Question 10(10 marks)

a. Two roots of the equation $6x^3+7x^2-84x+27=0$ are reciprocals of each other. Find all its roots. **4**

b. From a point A due south of a hill a surveyor measures the elevation of the top of the hill to be 33°. From another point, B, on a bearing of 188°T from the hill, she measures the elevation to be 41°. The distance AB is 200m.



(i) Explain why $\angle BCA = 8°$. **1**

(ii) Show that the height of the hill, h is given by

$$h=\frac{200}{\sqrt{\cot^2 33°+\cot^2 41°-2\cot 33°\cot 41°\cos 8°}}$$ **4**

(iii) Find the height of the hill to 3 significant figures. **1**

END OF EXAMINATION

Ext 1 2014 HY SOLUTIONS Year 12

MC

1. B 2. C 3. B 4. D 5. B

/5

1. $\log_e(x+2) - \log_e x = \log_e 4$

$$\log_e\left(\frac{x+2}{x}\right) = \log_e 4$$

$$\therefore \frac{x+2}{x} = 4$$

$$x = 2/3$$

2. $y = \log_e\left(\frac{x+1}{x-1}\right)$

$$\frac{dy}{dx} = \frac{1}{\frac{x+1}{x-1}} \cdot \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$= \frac{x-1}{x+1} \times \frac{-2}{(x-1)^2}$$

$$= \frac{-2}{x^2-1}$$

3. $\int_0^1 (e^{2x}+1)\,dx$

$$= \left[\tfrac{1}{2}e^{2x} + x\right]_0^1$$

$$= \left(\tfrac{1}{2}e^2 + 1\right) - \left(\tfrac{1}{2}e^0 + 0\right)$$

$$= \tfrac{1}{2}e^2 + 1 - \tfrac{1}{2}$$

$$= \tfrac{1}{2}(e^2+1)$$

4. $4x^3 - 2x^2 + 3x - 2 = 0$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{3/4}{1/2} = 3/2$$

5. ${}^{12}C_2 \times {}^{11}C_2$

$$= 3630$$

Question 6 (23 marks)

a) $\frac{x^2-6}{x} \le 1$

$x \ne 0$

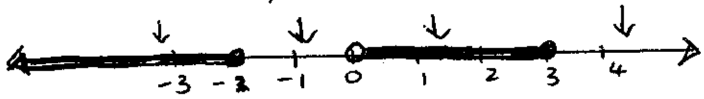
solve $\frac{x^2-6}{x} = 1$

$$x^2 - 6 = x$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = -2, 3$$



Test $x = -3$ $x = -1$ $x = 1$ $x = 4$

$\frac{9-6}{-3} \le 1$ ✓ $\frac{1-6}{-1} \le 1$ ✗ $\frac{1-6}{1} \le 1$ ✓ $\frac{16-6}{4} \le 1$ ✗

/5

$x \le -2, \quad 0 < x \le 3$

b) A(-2, 5) B(4, -3)

3 : 2

$$x_p = \frac{-2\times 2 + 4\times 3}{5} = 8/5$$

$$y_p = \frac{-3(3) + 5(2)}{5} = 1/5$$

$$P\left(\tfrac{8}{5}, \tfrac{1}{5}\right)$$

A(-2, 5) B(4, -3)

-3 2

$$x_Q = \frac{4(-3) + 2(-2)}{-1} = 16$$

$$y_Q = \frac{-3(-3) + 2(5)}{-1} = -19$$

$$Q(16, -19)$$

$$PQ = \sqrt{\left(16 - \tfrac{8}{5}\right)^2 + \left(-19 - \tfrac{1}{5}\right)^2}$$

$$= 24 \text{ units}$$

/5

6 c) $2x - 3y = 0$; $ax + 4y = 9$

$m_1 = \frac{2}{3}$ $\quad$ $m_2 = -\frac{a}{4}$; $\theta = 32^\circ 51'$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{2}{3} + \frac{a}{4}}{1 - \frac{a}{6}}\right|$$

$$= \left|\frac{8 + 3a}{12 - 2a}\right|$$

$0.6457 = \frac{8+3a}{12-2a}$ or $\frac{8+3a}{12-2a} = -0.6457$

$8 + 3a = -7.7484 + 1.2914a$

$7.7484 - 1.2914a = 8 + 3a$

$a = -0.059$ or $a = -9.2$

/5

d) $\lim\limits_{x \to 0} \frac{\sin 4x}{3x}$

$= \lim\limits_{x \to 0} \frac{\sin 4x}{\left(\frac{3}{4}\right) 4x}$

$= \frac{4}{3} \lim\limits_{x \to 0} \frac{\sin 4x}{4x}$ /2

$= \frac{4}{3}$

e) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

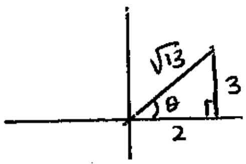
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$ /2

f) $3\cot\theta = 2$

$\cot\theta = \frac{2}{3}$



$\sin\theta = \frac{3}{\sqrt{13}}$

$\cos\theta = \frac{2}{\sqrt{13}}$

$$\frac{5\sin\theta - 7\cos\theta}{\operatorname{cosec}\theta + \sec\theta}$$

$$= \frac{5\left(\frac{3}{\sqrt{13}}\right) - 7\left(\frac{2}{\sqrt{13}}\right)}{\frac{\sqrt{13}}{3} + \frac{\sqrt{13}}{2}} = \frac{1}{\sqrt{13}} \times \frac{6}{5\sqrt{13}} = \frac{6}{65}$$

/4

Question 7 (21 marks)

a) i) $y = x \log_e x$

$\frac{dy}{dx} = vu' + uv'$

$= \log_e x + x \cdot \frac{1}{x}$

$= 1 + \log_e x$

$u = x$
$u' = 1$
$v = \log_e x$
$v' = \frac{1}{x}$

/2

ii) $y = \frac{e^x}{\sin x}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$u = e^x$
$u' = e^x$
$v = \sin x$
$v' = \cos x$

$= \frac{e^x \sin x - e^x \cos x}{\sin^2 x}$

$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}$ /3

iii) $y = \ln(\sin^2 x)$

$\frac{dy}{dx} = \frac{1}{\sin^2 x} \cdot 2\sin x \cos x$

$= \frac{2\cos x}{\sin x}$ /2

$= 2\cot x$

b) $y = e^{3x}$; $E_{tangent}$: $y = mx$

i) $\frac{dy}{dx} = 3e^{3x}$

ie, $m = 3e^{3x}$

$\therefore 3e^{3x} \times x = e^{3x}$

$3x = 1$

$x = \frac{1}{3}$ $\quad$ when $x = \frac{1}{3}$, $y = e^{3\left(\frac{1}{3}\right)}$
$= e$

Point of Intersection: $\left(\frac{1}{3}, e\right)$ /3

ii) $\left(\frac{1}{3}, e\right)$ lies of $y = mx$

$\therefore e = m\left(\frac{1}{3}\right)$

$m = 3e$ /1

7c) i) $\int \frac{x}{x^2-4}\,dx$

$= \frac{1}{2}\ln(x^2-4) + C$ $\quad /1$

ii) $\int_0^{\pi/3} \frac{\sin x}{2-\cos x}\,dx$

$= \left[\ln(2-\cos x)\right]_0^{\pi/3}$

$= \ln(2-\cos\frac{\pi}{3}) - \ln(2-\cos 0))$

$= \ln\frac{3}{2} - \ln 1$

$= \ln\frac{3}{2}$ $\quad /3$

d) i) $y = \tan^3 x = (\tan x)^3$

$\frac{dy}{dx} = 3\tan^2 x \cdot \sec^2 x$

$= 3(\sec^2 x - 1)\cdot\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$ $\quad /2$

ii) $3\int(\sec^4 x - \sec^2 x)\,dx = \tan^3 x$

$3\int\sec^4 x\,dx - 3\int\sec^2 x\,dx = \tan^3 x$

$3\int\sec^4 x\,dx = \tan^3 x + 3\int\sec^2 x\,dx$

$3\int\sec^4 x\,dx = \tan^3 x + 3\tan x$

$\therefore \int\sec^4 x\,dx = \frac{1}{3}(\tan^3 x + 3\tan x) + C$ $\quad /3$

iii) $\int_0^{\pi/4}\sec^4 x\,dx = \frac{1}{3}\left[(\tan\frac{\pi}{4})^3 + 3\tan\frac{\pi}{4}) - (\tan 0)^3 + \tan 0)\right]$

$= \frac{1}{3}[1 + 3 - 0]$

$= \frac{4}{3}$ $\quad /1$

Question 8 (14 marks)

a) $6^n + 7^n$ is divisible by 15 for odd n

Step 1: Show true for $n=1$

$6^1 + 7^1 = 13$ which is divisible by 13

Step 2: Assume true for $n = 2k-1$

ie. $6^{2k-1} + 7^{2k-1} = 13p$ (p an integer)

$6^{2k-1} = 13p - 7^{2k-1}$

Step 3: Prove true for $n = 2k+1$

ie. $6^{2k+1} + 7^{2k+1} = 13q$ (q an integer)

LHS $= 6^{2k+1} + 7^{2k+1}$

$= 6^2 \cdot 6^{2k-1} + 7^2 \cdot 7^{2k-1}$

$= 36(13p - 7^{2k-1}) + 49 \cdot 7^{2k-1}$

$= 36 \cdot 13p - 36 \cdot 7^{2k-1} + 49 \cdot 7^{2k-1}$

$= 36 \cdot 13p + 13 \cdot 7^{2k-1}$

$= 13(36p + 7^{2k-1})$

$= 13q$ (where $q = 36p + 7^{2k-1}$)

$=$ RHS

$\therefore$ If result is true for $n = 2k-1$, then it is also true for $n = 2k+1$

Step 4: By the principle of Mathematical Induction, the result is true for all odd n.

$/4$

8. b) $\left(2x+\frac{3}{x^2}\right)^9$

$$T_{k+1} = {}^9C_k\,(2x)^{9-k}\,(3x^{-2})^k$$
$$= {}^9C_k\, 2^{9-k}.\, x^{9-k}.\, 3^k.\, x^{-2k}$$
$$= {}^9C_k\, 2^{9-k}\, 3^k\, x^{9-3k}$$

For term independent of x,

$$9-3k=0$$
$$k=3$$
$$\therefore T_4 = {}^9C_3\, 2^6\, 3^3 \quad /3$$

c) $p=0.4$, $q=0.6$

i) $P(\text{rain 3 days}) = {}^7C_3\,(0.4)^3(0.6)^4$
$= 0.2903 \quad /2$

ii) RRR - - - -, - RRR - - -, - - RRR - -, - - - RRR - -, - - - - RRR [5]

P(rain of 3 consecutive days)
$= 5\times(0.4)^3(0.6)^4 \quad /2$
$= 0.04147$

iii) P(rain at least 2days) =
$1-P(\text{rain on 0 or 1day})$
$= 1-\left[(0.6)^7 + {}^7C_1(0.4)(0.6)^6\right] \quad /3$
$= 0.8414$

Question 9 (18 marks)

9a) $P(2ap, ap^2)$; $Q(2aq, aq^2)$

i) $m_{\text{chord}} = \dfrac{ap^2-aq^2}{2ap-2aq}$
$= \dfrac{a(p^2-q^2)}{2a(p-q)}$
$= \frac{1}{2}(p+q)$

E_{chord} : $y-y_1 = m(x-x_1)$

$$y-ap^2 = \tfrac{1}{2}(p+q)(x-2ap)$$
$$y-ap^2 = \tfrac{1}{2}(p+q)x - \tfrac{1}{2}(p+q)(2ap)$$
$$y-ap^2 = \tfrac{1}{2}(p+q)x - ap^2 - apq$$
$$y = \tfrac{1}{2}(p+q)x - apq \quad /3$$

ii) PQ passes through $(4a,-2a)$

$$\therefore -2a = \tfrac{1}{2}(p+q)4a - apq$$
$$-2a = 2ap+2aq-apq$$
$$apq = 2ap+2aq+2a$$
$$pq = 2p+2q+2 \quad /2$$

iii) $M\left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right)$

$$M = \left(ap+aq, \frac{ap^2+aq^2}{2}\right)$$

$x = ap+aq$ $\qquad$ $y = \dfrac{a(p^2+q^2)}{2}$

$x = a(p+q)$

$p+q = \dfrac{x}{a}$ — ① $\qquad$ $p^2+q^2 = \dfrac{2y}{a}$ — ②

$$(p+q)^2 = \left(\frac{x}{a}\right)^2 = p^2+q^2+2pq$$
$$\frac{x^2}{a^2} = \frac{2y}{a} + 2pq$$
$$pq = \frac{1}{2}\left[\frac{x^2}{a^2} - \frac{2y}{a}\right]$$

From (ii) $pq = 2p+2q+2$

$$\therefore 2(p+q)+2 = \frac{1}{2}\left[\frac{x^2}{a^2}-\frac{2y}{a}\right]$$
$$2\left(\frac{x}{a}\right)+2 = \frac{1}{2}\left[\frac{x^2}{a^2}-\frac{2y}{a}\right]$$
$$4ax + 4a^2 = x^2 - 2ay$$

/4

9 b) Join UV; let $\angle QVU = x^\circ$

$\therefore \angle RVU = 180 - x$ (straight $\angle$)

$\angle QTU = 180 - x$ (exterior $\angle$ of cyclic quadrilateral)

$\angle RSU = x$ (opposite $\angle$s of cyclic quadrilateral)

$\therefore \angle PTU = x$ (straight $\angle$)

and $\angle PSU = 180 - x$ (straight $\angle$)

$\therefore$ PTUS is a cyclic quadrilateral (opposite angles are supplementary)

/5

c) $\sin\theta - 2\cos\theta = 2$

$$\frac{2t}{1+t^2} - \frac{2(1-t^2)}{1+t^2} = 2$$

$$2t - 2(1-t^2) = 2(1+t^2)$$

$$2t - 2 + 2t^2 = 2 + 2t^2$$

$$2t = 4$$

$$t = 2$$

ie. $\tan\frac{\theta}{2} = 2$

$\therefore \theta = 2.214$

Test $\theta = \pi$

LHS $= \sin\pi - 2\cos\pi$

$= 2 = $ RHS

$\therefore \theta = \pi, 2.214$

/4

<u>Question 10</u> (10 marks)

$6x^3 + 7x^2 - 84x + 27 = 0$

a) roots : $\alpha, \frac{1}{\alpha}, \beta$

product $= \beta = -\frac{27}{6}$

$\beta = -\frac{9}{2}$

$\therefore (2x+9)$ is a factor

$$\begin{array}{r} 3x^2 - 10x + 3 \\ 2x+9 \overline{) 6x^3 + 7x^2 - 84x + 27} \\ -\ 6x^3 + 27x^2 \\ \hline -20x^2 - 84x \\ -\ -20x^2 - 90x \\ \hline 6x + 27 \\ 6x + 27 \\ \hline - \end{array}$$

$\therefore 6x^3 + 7x^2 - 84x + 27 = (2x+9)(3x-1)(x-3)$

$\therefore$ Roots : $-\frac{9}{2}, \frac{1}{3}, 3$.

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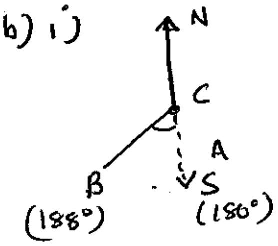
b) i)

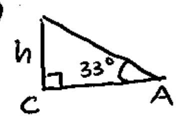


$\angle BCA = 188^\circ - 180^\circ$

$= 8^\circ$

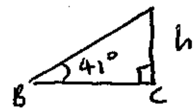
/1

ii)



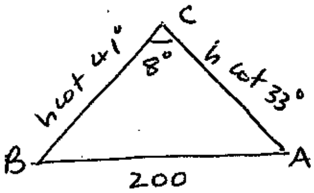
$\cot 33^\circ = \frac{AC}{h}$

$AC = h\cot 33^\circ$



$\cot 41^\circ = \frac{BC}{h}$

$BC = h\cot 41^\circ$



Using cosine rule,

$$200^2 = AC^2 + BC^2 - 2\,AC.BC\cos 8^\circ$$

$$= h^2\cot^2 33^\circ + h^2\cot^2 41^\circ - 2h^2\cot 33^\circ\cot 41^\circ\cos 8^\circ$$

$$200^2 = h^2\left[\cot^2 33^\circ + \cot^2 41^\circ - 2\cot 33^\circ\cot 41^\circ\cos 8^\circ\right]$$

$$h = \frac{200}{\sqrt{\cot^2 33^\circ + \cot^2 41^\circ - 2\cot 33^\circ\cot 41^\circ\cos 8^\circ}}$$

/4

iii) $h = 464$ m

/1