**FINAL MARK**

# GIRRAWEEN HIGH SCHOOL
## Mathematics Extension 1
## HSC ASSESSMENT Task 2, 2016
## ANSWERS COVER SHEET

**Name:** ______________________________________________

| QUESTION | MARK | HE2 | HE3 | HE4 | HE5 | HE6 | HE7 |
|---|---|---|---|---|---|---|---|
| Q1 - Q5 | /5 | | | | | | ✓ |
| | | | | | | | |
| Q6 | /19 | | | | | | ✓ |
| | | | | | | | |
| Q7 | /23 | | | | | | ✓ |
| | | | | | | | |
| Q8a | /4 | ✓ | | | | | ✓ |
| Q8bc | /11 | | ✓ | | | | ✓ |
| Q8 Total | /15 | | | | | | ✓ |
| | | | | | | | |
| Q9 | /19 | | | | | | ✓ |
| | | | | | | | |
| Q10 | /12 | | | | | | ✓ |
| | | | | | | | |
| **TOTAL** | | | | | | | |
| | /93 | /4 | /11 | | | | /93 |

**HSC Outcomes** **Mathematics Extension 1**

HE2 uses inductive reasoning in the construction of proofs.

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion and exponential growth and decay.

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE6 determines integrals by reduction to a standard form through a given substitution.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

# GIRRAWEEN HIGH SCHOOL

YEAR 12 HALF YEARLY EXAMINATION

2016

**MATHEMATICS EXTENSION 1**

*Time Allowed: Two hours*

*(Plus 5 minutes reading time)*

**Instructions To Students**

- **Attempt all questions.**
- **All necessary working must be shown for questions 6-10.**
- **Marks may be deducted for careless or badly arranged work.**
- **Board approved calculators may be used.**
- **For Questions 1-5, write the letter corresponding to the correct answer on your answer sheet.**
- **For Questions 6-10, start each question on a new sheet of paper. Each question should be clearly labelled.**
- **Write 'End of Solutions' on your answer paper when you finish answering all questions.**

**For Questions 1-5, write the letter corresponding to the correct answer on your answer sheet (5 marks)**

**1.** What is the solution to the inequality $3 - x \geq \frac{2}{x}$.

(A) $x < 0$ or $1 \leq x \leq 2$

(B) $x \geq 2$ or $0 < x \leq 1$

(C) $x > 0$ or $-2 \leq x \leq -1$

(D) $x \leq -2$ or $-1 \leq x < 0$

**2.** What is the acute angle between the lines $x - y + 2 = 0$ and $2x - y - 1 = 0$.

**(A)** 18°26' **(B)** 19°28' **(C)** 70°32' **(D)** 71°32'

**3.** Twelve people sit at a round table. The number of arrangements possible if two particular persons are seated together is

**(A)** 24 **(B)** 3 628 800 **(C)** 725 760 **(D)** 7 257 600

**4.** The coefficient of $x^6$ in the expansion of $\left(\frac{1}{x^2} - x\right)^{18}$.

**(A)** 18 564 **(B)** 3060 **(C)** 43 758 **(D)** -3060

**5.** A curve has parametric equations $x = \frac{2}{t}$ and $y = 2t^2$. What is the Cartesian equation of this curve.

**(A)** $y = \frac{4}{x}$ **(B)** $y = \frac{8}{x}$ **(C)** $y = \frac{4}{x^2}$ **(D)** $y = \frac{8}{x^2}$

**Question 6 (19 marks)**

**(a)** Solve $\dfrac{2x}{x+1} \le 1$ 3

**(b)** Find the coordinates of the point $P$ which divides the interval joining $A(-4,-6)$ and $B(6,-1)$ externally in the ratio 3:2. 3

**(c)** Solve: $\log_e(10x+24) = 2\log_e x$ 4

**(d)** **(i)** Express $\sin x + \sqrt{3}\cos x$ as $A\sin(x+\alpha)$. 3

**(ii)** Hence solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$, $0 < \alpha < \dfrac{\pi}{2}$, $0 \le x \le 2\pi$. 2

**(e)** The gradient of the tangent at any point on a curve is given by $\dfrac{dy}{dx} = e^{-2x}$. If the curve passes through the point (0,1)

**(i)** Find the equation of the curve. 3

**(ii)** The point on the curve with $x$ – coordinate $-\dfrac{1}{2}\log_e 3$. 1

**Question 7(23 marks)**

**(a)** Differentiate:

**(i)** $y = x^2\log_e(x^2)$ **(ii)** $y = \dfrac{e^x}{x^3}$ **(iii)** $y = \log_e\left(\dfrac{x}{\sqrt{x^2+1}}\right)$ 10

**(b)** Find:

**(i)** $\int x^2 e^{x^3+1}dx$ **(ii)** $\int \dfrac{3x}{x^2+11}dx$ **(iii)** $\displaystyle\int_1^{\log_e 2} \dfrac{e^{3x}+1}{e^x}dx$ 9

**(c)(i)** Find the derivative of $\log_e(x^2-5x+7)$. 2

**(ii)** Hence evaluate $\displaystyle\int_3^4 \dfrac{(2x-5)}{x^2-5x+7}dx$ correct to 4 significant figures. 2

**Question 8 (15 marks)**

**(a)** Use Mathematical Induction to prove that $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$ for $n \geq 1$. 4

**(b)** Expand $\left(2 - \frac{1}{x}\right)^3$ and hence determine the term independent of $x$ in the

expansion of $\left(2 + 3x - 4x^2\right)\left(2 - \frac{1}{x}\right)^3$. 3

**(c)** It is known that at noon the sun is hidden by clouds on an average of two days out of every three. If 5 consecutive days are taken, find the probability of the sun shining at noon on

**(i)** each day

**(ii)** the first 4 days only

**(iii)** 4 of the days

**(iv)** at least 4 days 8

**Question 9 (19 marks)**

**(a)**The points $T(2at, at^2)$ and $P(2ap, ap^2)$ lie on the parabola $x^2 = 4ay$. The equation of the normal to the parabola at $T$ is $x + ty = 2at + at^3$.

**(i)** Show that the normal at $T$ and $P$ intersect at the point $W$ with co-ordinates $\left(-apt(p+t), a(t^2 + tp + p^2 + 2)\right)$. 4

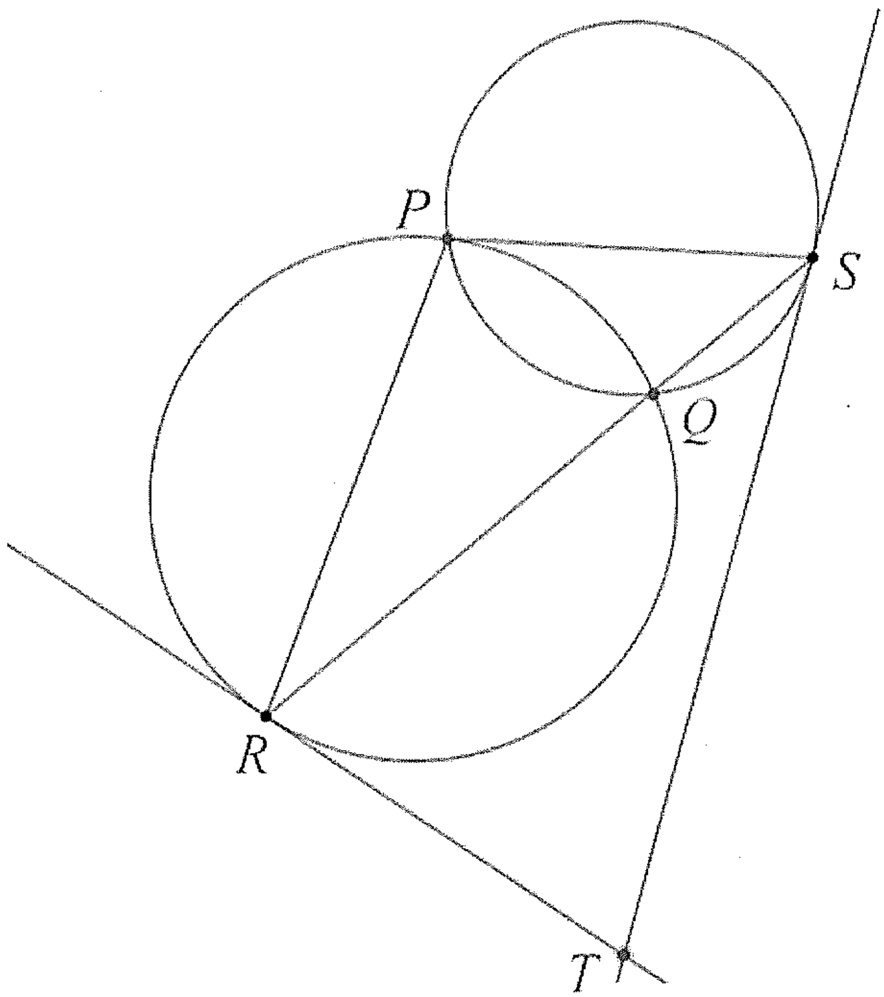
**(ii)** The equation of the chord $TP$ is $y = \frac{(p+t)x}{2} - apt$. If the chord $PT$ passes through $(0, a)$, show that $pt = -1$. 2

**(iii)**

$(\alpha)$ If the chord passes through , $(0, a)$, show that the equation of the locus of $W$ is a parabola. 4

$(\beta)$ What is the focal length of this parabola? 1

**(b)** The circles intersect at $P$ and $Q$. $RQS$ is a straight line. $TR$ and $TS$ are tangents.




**(i)** Copy or trace the diagram into your writing booklet, including a construction line from $P$ to $Q$. 1

**(ii)** Prove that $PRTS$ is a cyclic quadrilateral. 4

**(c)** Solve $3\sin\theta + \cos\theta = 2$ by using $t = \tan\dfrac{\theta}{2}$, $0 \le \theta \le 2\pi$. 3

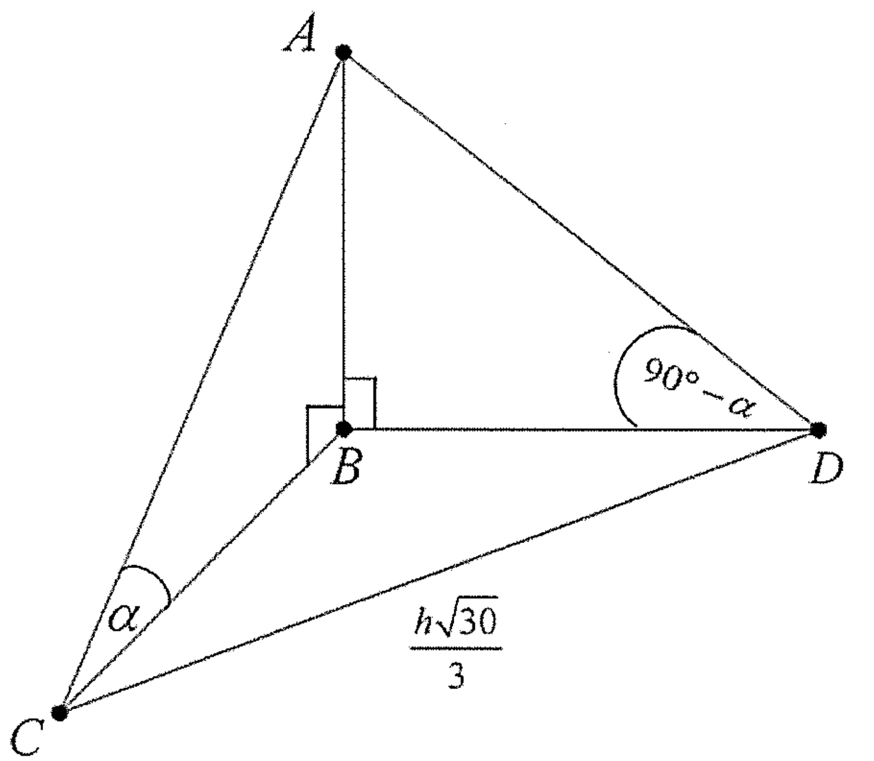
**Question 10 (12 marks)**

**(a)** When the polynomial $P(x)$ is divided by $x+1$, the remainder is 6 and when it is divided by $x-3$, the remainder is -2. Find the remainder when $P(x)$ is divided by $x^2-2x-3$. 4

**(b)** Charles is at point $C$ south of a tower $AB$ of height $h$ metres. His friend Daniel is at a point $D$, which is closer to the tower and east of it. The angles of elevation of the top $A$ of the tower from Charles and Daniel's positions are $\alpha$ and $90°-\alpha$ respectively. The distance $CD$ between Charles and Daniel is $\frac{h\sqrt{30}}{3}$ metres.

**(i)** Show that $3\tan^4\alpha - 10\tan^2\alpha + 3 = 0$. 5

**(ii)** Find $\alpha$ by solving the equation given in (i) 3




***End of Examination***

*Please remember to write 'End of Solutions' on your answer paper.*

# 12 Extension 1 Half Yearly 2016 Solutions

1 A  2 A  3 D  4 B  5 D

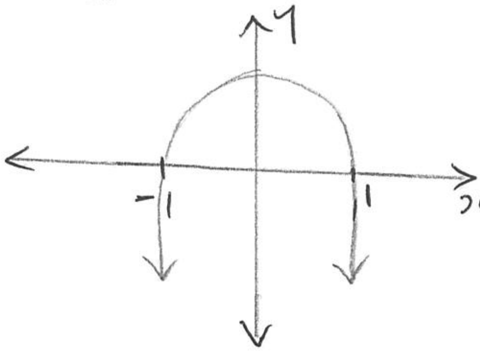
## Question 6 (19 marks)

(a) $\frac{2x}{x+1} \le 1$

$(x+1)^2 \times \frac{2x}{x+1} \le (x+1)^2, \ x \ne -1$

$(x+1)2x \le (x+1)^2$

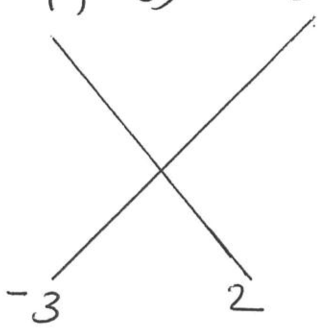
$(x+1)(1-x) \ge 0$




3

$\underline{\underline{-1 < x \le 1}}$

(b) $A(-4,-6)$  $B(6,-1)$




3

$x = \frac{(-3\times 6)+(2\times -4)}{-3+2} = 26$

$y = \frac{(-3\times -1)+(2\times -6)}{-3+2} = 9$

$\underline{\underline{P(26,9)}}$

(c) $\log_e(10x+24) = 2\log_e x$

$\log_e(10x+24) = \log_e x^2$

$e^{\log_e(10x+24)} = e^{\log_e x^2}$

1A

$10x+24 = x^2$ 2A

$x^2 - 10x - 24 = 0$ 3D

4B

$(x-12)(x+2) = 0$ 5D

4

$x = 12$ or $-2$

$\underline{\underline{x = 12 \quad (\because \log_e x > 0)}}$

(d)(i)

Let $\sin x + \sqrt{3}\cos x = A\sin(x+\alpha)$

$= A\sin x\cos\alpha + A\cos x\sin\alpha$

Equating coefficients of $\sin x$ and $\cos x$

$1 = A\cos\alpha$

3

$\sqrt{3} = A\sin\alpha$

$A^2 = 4$  $A = 2 \ (\because A > 0)$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\cos\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{3}$

$\sin x + \sqrt{3}\cos x = 2\sin\left(x+\frac{\pi}{3}\right)$

(ii) $2\sin\left(x+\frac{\pi}{3}\right) = \sqrt{2}$

$\sin\left(x+\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$x = \frac{5\pi}{12}, \frac{23\pi}{12}$ (2)

(e)(i) $\frac{dy}{dx} = e^{-2x}$

$y = \int e^{-2x}\,dx$

$= \frac{e^{-2x}}{-2} + c$

when $x=0, y=1$

$1 = -\frac{1}{2} + c$

$c = \frac{3}{2}$ (3)

$\therefore y = \frac{e^{-2x}}{-2} + \frac{3}{2}$

(ii) when $x = -\frac{1}{2}\log_e 3$

$y = \frac{e^{-2\times -\frac{1}{2}\log_e 3}}{-2} + \frac{3}{2}$

$= -\frac{3}{2} + \frac{3}{2} = 0$

Point $\left(-\frac{1}{2}\log_e 3, 0\right)$

## Question 7 (23 marks)

(a)(i) $y = x^2 \log_e x^2$

$y' = x^2 \times \frac{1}{x^2} \times 2x + \log_e(x^2) \times 2x$

$= 2x + 2x\log_e(x^2)$ (3)

$= 2x(1 + \log_e x^2)$

(ii) $y = \frac{e^x}{x^3}$

$y' = \frac{x^3 \times e^x - e^x \times 3x^2}{x^6}$ (3)

$= \frac{x^2 e^x(x-3)}{x^6} = \frac{e^x(x-3)}{x^4}$

(iii) $y = \log_e\left(\frac{x}{\sqrt{x^2+1}}\right)$

$= \log_e x - \log_e \sqrt{x^2+1}$

$= \log_e x - \log_e (x^2+1)^{\frac{1}{2}}$

$= \log_e x - \frac{1}{2}\log_e(x^2+1)$

$y' = \frac{1}{x} - \frac{1}{2} \times \frac{1}{x^2+1} \times 2x$ (4)

$= \frac{1}{x} - \frac{x}{x^2+1} = \frac{1}{x(x^2+1)}$

(b) (i) $\int x^2 e^{x^3+1}\,dx$

$= \frac{1}{3}\int 3x^2 e^{x^3+1}\,dx$ 3

$= \frac{1}{3} e^{x^3+1} + C$ 2

(ii) $\int \frac{3x}{x^2+11}\,dx$

$= 3\int \frac{x\,dx}{x^2+11}$

$= \frac{3}{2}\int \frac{2x\,dx}{x^2+11}$ 3

$= \frac{3}{2}\log_e(x^2+11) + C$

(iii) $\int_1^{\log_e 2} \frac{e^{3x}+1}{e^x}\,dx$

$= \int_1^{\log_e 2} (e^{2x} + e^{-x})\,dx$

$= \left[\frac{e^{2x}}{2} + \frac{e^{-x}}{-1}\right]_1^{\log_e 2}$

$= \left[\frac{e^{2x}}{2} - \frac{1}{e^x}\right]_1^{\log_e 2}$

$= \left(\frac{e^{2\times\log_e 2}}{2} - \frac{1}{e^{\log_e 2}}\right) - \left(\frac{e^2}{2} - \frac{1}{e}\right)$

$= \frac{4}{2} - \frac{1}{2} - \frac{e^2}{2} + \frac{1}{e}$ 3 4

$= \frac{3}{2} + \frac{1}{e} - \frac{e^2}{2}$

(c) (i)

$\frac{d}{dx}\left[\log_e(x^2-5x+7)\right] = \frac{2x-5}{x^2-5x+7}$ 2

$\int_3^4 \frac{2x-5}{x^2-5x+7}\,dx = \left[\log_e|x^2-5x+7|\right]_3^4$

$= \log_e|16-20+7| - \log_e|9-15+7|$

$= \log_e|3| - \log_e|1|$

$= \log_e 3 = 1.099$ 2

Question 8 (15 marks)

$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$

$n = 1$

LHS $= 1 \times 1! = 1$

RHS $= 2! - 1 = 1$

LHS $=$ RHS $\therefore$ true for $n = 1$

Assume the result is true for $n = k$

$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! = (k+1)! - 1$ —— (1)

To prove for $n = k+1$

$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)!$

$= (k+2)! - 1$ —— (2)

LHS of (2)

$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)!$

$= (k+1)! - 1 + (k+1) \times (k+1)!$

4

$= (k+1)! + (k+1)(k+1)! - 1$

$= (k+1)!\,[1 + k + 1] - 1 = (k+1)!\,(k+2) - 1$

$= (k+2)! - 1 =$ RHS of (2)

By the parinciple of Mathematical Induction the result is true for $n \geq 1$.

(b) $\left(2-\frac{1}{x}\right)^3 = 8 - \frac{12}{x} + \frac{6}{x^2} - \frac{1}{x^3}$

$\left(2+3x-4x^2\right)\left(8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}\right)$

Terms independent of $x$

$16 + 3x \times -\frac{12}{x} - 4x^2 \times \frac{6}{x^2}$ (3)

$= 16 - 36 - 24 \quad = \underline{\underline{-44}}$

(c) $p = \frac{1}{3} \qquad q = \frac{2}{3} \qquad n = 5$

(a) $P(x=5) = {}^5C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^0 = \frac{1}{243} = 0.0041$

(b) P(sun shining at noon on the first 4 days) $= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{243} = 0.0082$

(c) $P(x=4) = {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 = \frac{10}{243} = 0.0412$ (8)

(d) $P(x=4) + P(x=5) = \frac{10}{243} + \frac{1}{243} = \frac{11}{243} = 0.0453$

<u>Question 9 (19 marks)</u>

(a)(i) $x + ty = 2at + at^3$ —— (1)

$x + py = 2ap + ap^3$ —— (2)

(1) − (2) $\quad ty - py = 2a(t-p) + a(t^3 - p^3)$

$y(t-p) = 2a(t-p) + a(t-p)(t^2 + tp + p^2)$

$$y = 2a + a(t^2 + tp + p^2)$$
$$= a(t^2 + tp + p^2 + 2)$$

$$x = 2at + at^3 - ty$$
$$= 2at + at^3 - at(t^2 + tp + p^2 + 2)$$
$$= 2at + at^3 - at^3 - apt^2 - ap^2t - 2at$$
$$= -apt^2 - ap^2t = -apt(t+p) \quad 4$$
$$\therefore W\left(-apt(t+p),\ a(t^2 + tp + p^2 + 2)\right)$$

(ii) substitute $(0, a)$ in $y = \frac{1}{2}(p+t)x - apt$

$$a = 0 - apt \quad 2$$
$$1 = -pt \qquad \therefore pt = -1$$

(iii) (a) $x = -apt(p+t)$

$$y = a(t^2 + tp + p^2 + 2)$$

substitute $tp = -1$

$$x = a(p+t) \text{ --- } (1)$$
$$y = a(t^2 + p^2 + 1) \text{ --- } (2)$$

squaring (1) $(p+t)^2 = \frac{x^2}{a^2}$

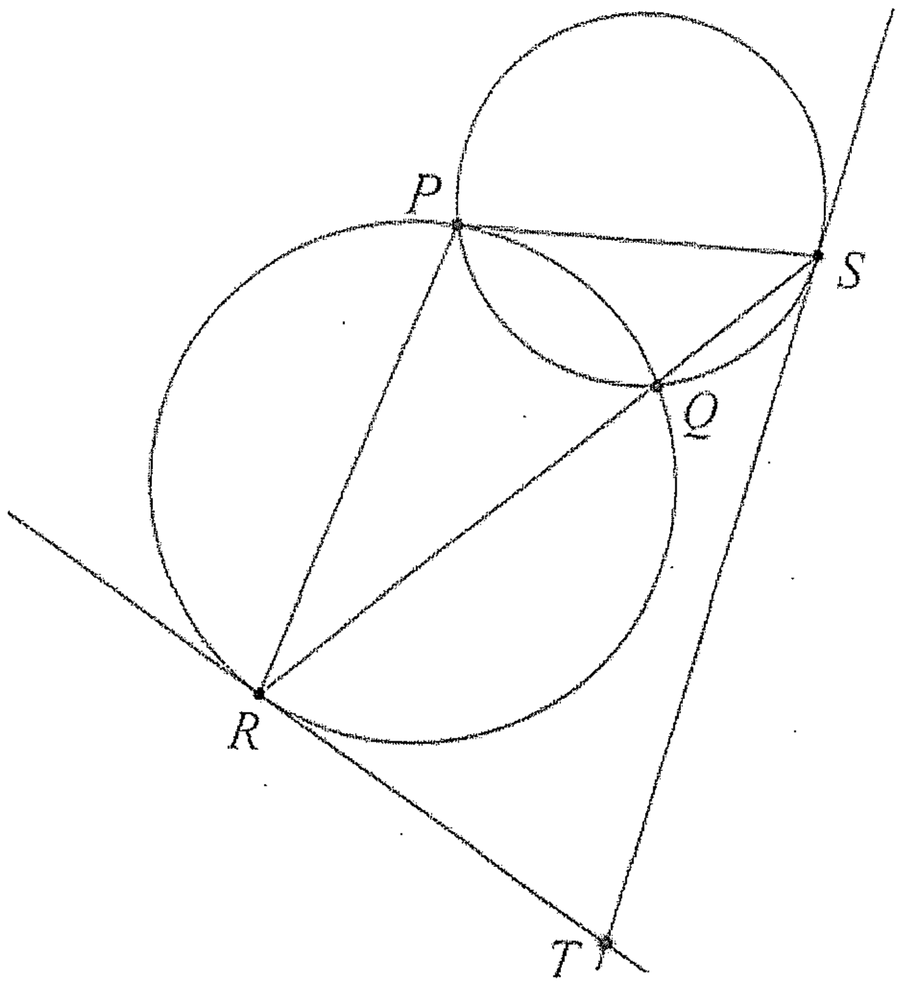
$$y = a(t^2+p^2+2pt-2pt+1)$$
$$= a\left[(t+p)^2-2pt+1\right]$$
$$= a\left[(t+p)^2+3\right]$$
$$= a\left[\frac{x^2}{a^2}+3\right] = \frac{x^2}{a}+3a$$

$$\frac{x^2}{a} = y-3a \qquad (4)$$

$$\underline{\underline{x^2 = a(y-3a)}}$$

($\beta$) $x^2 = 4 \times \frac{a}{4}(y-3a)$

Focal length $= \underline{\underline{\frac{a}{4}}}$ (1)




(1)

Let $\angle TSQ = x$

$\angle SPQ = x$ (angle between tangent and chord is equal to angle in the alternate segment.

Let $\angle TRQ = y$

$\angle RPQ = y$ (alternate segment theorem)

$\angle RTS = 180 - (x+y)$ (angle sum of $\Delta RTS$)

$\angle RTS + \angle RPS = 180 - (x+y) + x + y = 180$

(4)

PRTS is a cyclic quadrilateral

(opposite angles of a quadrilateral are supplementary)

(c) $3\sin\theta + \cos\theta = 2$

substitute $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\frac{3\times 2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 2$$

$$\frac{6t+1-t^2}{1+t^2} = 2$$

$$3t^2 - 6t + 1 = 0$$

$$t = \frac{6 \pm \sqrt{24}}{6}$$

3

$$\tan\frac{\theta}{2} = \frac{6 \pm \sqrt{24}}{2}$$

$$\theta = 0.3630^c, 2.1351^c$$

Question 10 (12 marks)

(a) Let $ax+b$ be the remainder

$$p(x) = (x-3)(x+1)Q(x) + ax+b$$

$$p(-1) = 6, \quad p(3) = -2$$

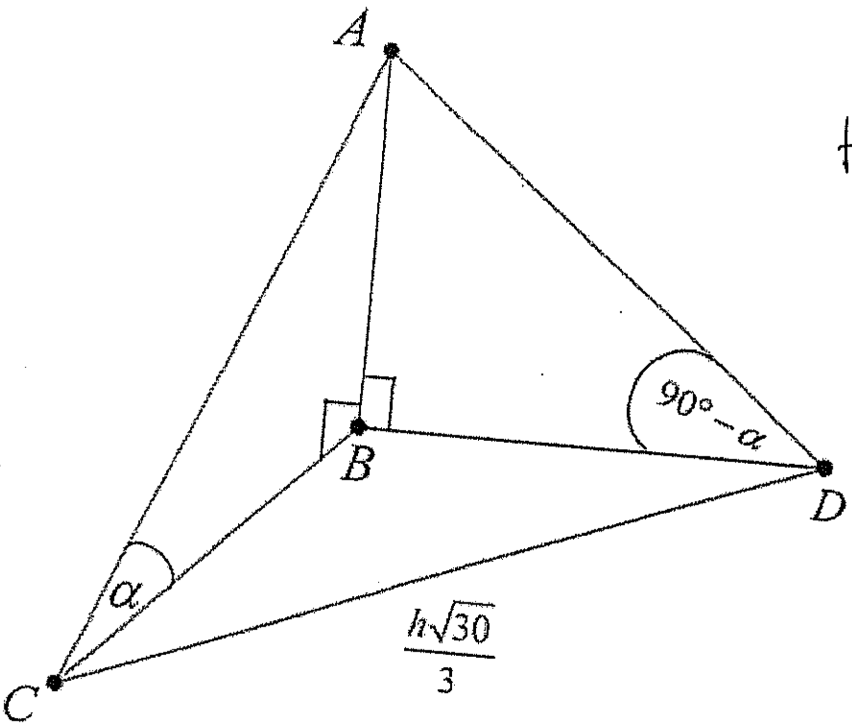
$$3a+b = -2$$
$$-a+b = 6$$
$$4a = -8$$
$$a = -2$$

$$b = 6+a$$
$$= 4$$

remainder $= -2x+4$

4

(b)




$$\tan\alpha = \frac{h}{BC}$$

$$BC = \frac{h}{\tan\alpha}$$

$$\tan(90-\alpha) = \frac{h}{BD}$$

$$BD = \frac{h}{\tan(90-\alpha)}$$

Apply Pythagoras' theorem in $\Delta BCD$

$$\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2(90-\alpha)} = \frac{h^2 \times 30}{9}$$

$$h^2\left[\frac{1}{\tan^2\alpha} + \frac{1}{\tan^2(90-\alpha)}\right] = \frac{30h^2}{9}$$

$$\frac{1}{\tan^2\alpha} + \frac{1}{\tan^2(90-\alpha)} = \frac{10}{3}$$

$$\frac{1}{\tan^2\alpha} + \frac{1}{\frac{1}{\tan^2\alpha}} = \frac{10}{3} \quad \left(\because \tan(90-\alpha) = \frac{1}{\tan\alpha}\right)$$

$$\frac{1}{\tan^2\alpha} + \tan^2\alpha = \frac{10}{3}$$

$$1 + \tan^4\alpha = \frac{10\tan^2\alpha}{3}$$

$$\therefore 3 + 3\tan^4\alpha = 10\tan^2\alpha$$

$$3\tan^4\alpha - 10\tan^2\alpha + 3 = 0 \quad (5)$$

Let $m = \tan^2\alpha$

$$3m^2 - 10m + 3 = 0$$

$$(m-3)(3m-1) = 0$$

$$m = 3 \text{ or } m = \frac{1}{3}$$

$$\tan^2\alpha = 3 \text{ or } \tan^2\alpha = \frac{1}{3}$$

$$\tan\alpha = \pm\sqrt{3} \text{ or } \tan\alpha = \pm\frac{1}{\sqrt{3}} \quad (3)$$

$$\alpha = 30^\circ \text{ or } 60^\circ$$